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Collapsible subgraphs of a 4-edge-connected graph

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ABSTRACT

Jaeger in 1979 showed that every 4-edge-connected graph is supereulerian, graphs that have spanning eulerian subgraphs. Catlin in 1988 sharpened Jaeger's result by showing that every 4-edge-connected graph is collapsible, graphs that are contractible configurations of supereulerian graphs. To further study collapsible subgraphs of a 4-edge-connected graph, in Catlin et al. (2009), it is shown that every 4-edge-connected graph remains collapsible after removing any two edges. We prove the following.

(i) Every 4-edge-connected G contains two vertices x, y such that one of x and y has minimum degree in G and both $G - x$ and $G - y$ are collapsible.
(ii) Let G be a 4-edge-connected graph and let $X \subset E(G)$ be an edge subset with $|X| \le 3$. Then $G - X$ is collapsible if and only if X is not contained in a 4-edge-cut of G.
(iii) Let G be a 4-edge-connected graph and let $X \subset E(G)$ be an edge subset with $|X| \le 4$. Then $G - X$ is collapsible if and only if $G - X$ is not contractible to a member in $\{K_2^c, K_2, K_{2,2}, K_{2,3}, K_{2,4}\}$.

These extend former results of Jaeger (1979) and Catlin (1988).

1. The problem

We follow [2] for terminology and notation not defined here, and consider loopless finite graphs in which multiple edges are allowed. In particular, $\kappa'(G)$ and $\delta(G)$ denote the edge-connectivity and the minimum degree of a graph G, respectively. We define $O(G)$ to be the set of odd degree vertices of G and $\tau(G)$ to be the maximum number of edge-disjoint spanning tress of G. As in [2], a graph G is **eulerian** if G is connected with $O(G) = \emptyset$. A graph G is **supereulerian** if G contains a spanning eulerian subgraph.

Boesch, Suffel and Tindell [1] proposed the problem to characterize supereulerian graphs. It is indicated in [1] that this problem would be difficult. Pulleyblank [16] proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been lots of researches on this topic. Catlin [4] presented the first survey on supereulerian graphs. Later Chen et al. [7] gave an update in 1995, specifically on the reduction method associated with the supereulerian problem. A recent survey on supereulerian graphs can be found in [10].

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Catlin introduced collapsible graphs as a tool to study supereulerian graphs. A graph H is **collapsible** if for any subset $R \subseteq V(H)$ with $|R| \equiv 0 \pmod 2$, H has a spanning connected subgraph Γ_R with $O(\Gamma_R) = R$. As $R = \emptyset$ is possible, it follows that collapsible graphs are supereulerian. Furthermore, in [3], Catlin proved that if H is a collapsible subgraph of a graph G, then G is supereulerian if and only if G/H, the graph obtained from G by contracting the edges of H, is supereulerian. This is known as Catlin's reduction method. As revealed by the theorems and examples in [7,10], successful applications of Catlin's reduction method often depend on the knowledge of collapsible graphs and the so called reduced graphs, graphs that do not contain nontrivial collapsible subgraph. Thus determining which graphs are collapsible becomes an interesting and important problem in this area. The following two classical results are well known.

Theorem 1.1 (*Jaeger [9]*)**.** *Every graph G with $\kappa'(G) \geq 4$ is supereulerian.*

Theorem 1.2 (*Catlin [3]*)**.** *Every graph G with $\kappa'(G) \geq 4$ is collapsible.*

Catlin in [3] showed that every graph G with $\tau(G) \geq 2$ is collapsible, and every graph with a spanning collapsible subgraph is collapsible. Both Theorems 1.1 and 1.2 utilized the well known theorem of Nash-Williams [18] and Tutte [19] on spanning tree packing of graphs, showing that every 4-edge-connected graph contains 2 edge-disjoint spanning trees. As identifying collapsible graphs is of particular interest and importance in the studies of supereulerian graphs, or more generally, the studies of eulerian subgraphs, there have been lots of efforts to find collapsible subgraphs of a 4-edge-connected graph, or of a graph with 2-edge-disjoint spanning trees, as presented in Section 2.2 of [10]. Following the notation in [5], we define $F(G)$ to be the minimum number of additional edges that must be added to G to result in a graph with 2-edge-disjoint spanning trees. Theorem 1.3 presents some of the former efforts to identify collapsible subgraphs of a 4-edge-connected graph, and extends Theorems 1.1 and 1.2.

Theorem 1.3. *Let G be a connected graph and $k \geq 1$ be an integer. Each of the following holds.*
(i) (Theorem 7 of [3]) Suppose that $F(G) \leq 1$. Then G is collapsible if and only if $\kappa'(G) \geq 2$.
(ii) (Theorem 1.3 of [5]) Suppose that $F(G) \leq 2$. Then G is collapsible if and only if G cannot be contracted to $\{K_2\} \cup \{K_{1,t} : t \geq 1\}$.
(iii) (Theorems 1.1 and 1.3 of [6]) $\kappa'(G) \geq 2k$ if and only if $\forall X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$.
(iv) (Theorems 1.1 and 1.3 of [6]) $\kappa'(G) \geq 2k + 1$ if and only if $\forall X \subseteq E(G)$ with $|X| \leq k + 1$, $\tau(G - X) \geq k$.

The current research is motivated by the results above. Our main results are the following.

Theorem 1.4. *Every 4-edge-connected graph G contains two distinct vertices $x, y \in V(G)$ such that both $G - x$ and $G - y$ are collapsible.*

The vertices x and y in Theorem 1.4 cannot be chosen arbitrarily, as seen in the following example. Let Q be a 3-connected cubic non-hamiltonian graph (for example, a snark), and let G be obtained from Q by adding a new vertex x and joining x to every vertex of Q. Then G is 4-edge-connected but $G - x$ is not collapsible. Since $\delta(G) \geq \kappa'(G)$, for the vertex x (or y) in Theorem 1.4, $G/(G-x)$ is a graph with 2 vertices and at least 4 edges, and so G itself must also be collapsible. Thus Theorem 1.4 extends Theorems 1.1 and 1.2. The next result extends Theorem 1.3 (iii) when $k = 2$. An edge cut of size k is also called a k-edge-cut of G.

Theorem 1.5. *Let G be a 4-edge-connected graph and let $X \subset E(G)$ be an edge subset with $|X| \leq 3$. Then $G - X$ is collapsible if and only if X is not contained in a 4-edge-cut of G.*

Theorem 1.6. *Let K_2^c be the edgeless graph on 2 vertices, G be a 4-edge-connected graph and $X \subset E(G)$ be an edge subset with $|X| \leq 4$. Then $G - X$ is collapsible if and only if $G - X$ is not contractible to a member in $\{K_2^c, K_2, K_{2,2}, K_{2,3}, K_{2,4}\}$.*

We organize the paper as follows. In Section 2, the needed mechanism which will be used in the proof of the main results. The proofs of the main results are in the sections following Section 2.

2. Tools

We start with some notation and terminologies to be used in the arguments. Let K_n^l denote the graph with n vertices in which there are exactly l edges connecting any two vertices. For a vertex $v \in V(G)$ and a subgraph K of G, define $N_K(v) = \{u \in V(K) : uv \in E(G)\}$, $E_K(v) = \{e \in E(K) : e \text{ is incident with } v\}$, and $d_K(v) = |E_K(v)|$.

For an integer $i > 0$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$. Let $D_\delta(G) = \{v \in V(G) : d_G(v) = \delta(G)\}$ and $K(G)$ be the set of all cut vertices of G. For argument convenience, throughout this paper, we define $\kappa'(K_1) = \infty$, as K_1 is connected and has no edge-cut of finite size; and $\tau(K_1) = \infty$, in accordance to Theorem 1.3.

For a vertex $u \in D_4(G)$ with $N_G(u) = \{v_1, v_2, v_3, v_4\}$, let $\pi = \langle \{v_{i_1}, v_{i_2}\}, \{v_{i_3}, v_{i_4}\} \rangle$ be a 2-partition of $N_G(u)$ into a pair of 2-subsets. Define $G_\pi = (G - u) + \{v_{i_1} v_{i_2}, v_{i_3} v_{i_4}\}$.

Theorem 2.1 (*Fleischer [8], Mader [14] and [15]*)**.** *If $u \in D_4(G)$ with $|N_G(u)| = 4$, then for some 2-partition π of $N_G(u)$, $\kappa'(G_\pi) = \kappa'(G)$.*

By Theorem 1.3, every graph with two edge-disjoint spanning trees is collapsible. Our arguments will make use of this property and examine graphs with more than one edge-disjoint spanning trees.

Theorem 2.2 (*Brualdi, Theorem 2 of [17]*)**.** *Let T_1 and T_2 be any two spanning trees of G. For any edge $e_1 \in T_1 - T_2$, there exists an edge $e_2 \in T_2 - T_1$ such that both $T_1 - x + y$ and $T_2 - y + x$ are two spanning trees of G.*

Lemma 2.3. *Let G be a graph and v be a vertex of degree 4. If $\kappa'(G) \geq 4$, then $\tau(G - v) \geq 2$.*

Proof. Let $E_G(v) = \{e_1, e_2, e_3, e_4\}$. By Theorem 1.3(iii) with $k = 2$, $G - \{e_3, e_4\}$ has two edge-disjoint spanning trees, denoted by T_1' and T_2'. As there are only two edges e_1 and e_2 incident with v in $G - \{e_3, e_4\}$, e_1 and e_2 cannot be in the same spanning tree. We may assume that $e_i \in T_i'$ for $i = 1, 2$. Then $T_1 = T_1' - e_1$ and $T_2 = T_2' - e_2$ are two edge-disjoint spanning trees of $G - v$, and so $\tau(G - v) \geq 2$. ■

Lemma 2.4 (*Li et al. [11–13]*)**.** *Let $k > 0$ be an integer and $\mathcal{T}_k$ be the graph family consisting of graphs with at least k edge-disjoint spanning trees. Then each of the following holds.*

(C1) $K_1 \in \mathcal{T}_k$.

(C2) If $G \in \mathcal{T}_k$ and if $e \in E(G)$, then $G/e \in \mathcal{T}_k$.

(C3) Suppose that H is a subgraph of G. If $H \in \mathcal{T}_k$ and $G/H \in \mathcal{T}_k$, then $G \in \mathcal{T}_k$.

Lemma 2.5. *Let G be a graph with $v_0 \in D_4(G)$ and $N_G(v_0) = \{v_1, v_2, v_3, v_4\}$, and let $G' = G - v + \{v_1v_2, v_3v_4\}$. If for some $u \in V(G')$, $\tau(G' - u) \geq 2$, then $\tau(G - u) \geq 2$.*

Proof. For each $i \in \{1, 2, 3, 4\}$, let $e_i = v_0v_i$. Let $e_1' = v_1v_2$, $e_3' = v_3v_4$. By assumption, $\tau(G' - u) \geq 2$. Without loss of generality, we assume $u \notin \{v_1, v_2, v_3\}$. Choose two edge-disjoint spanning trees T_1' and T_2' of $G' - u$ so that

$$(|\{e_1', e_3'\} \cap E(T_1')| + 1)^2 + (|\{e_1', e_3'\} \cap E(T_2')| + 1)^2 \text{ is minimized.} \tag{1}$$

Suppose that $e_1', e_3' \in E(T_2')$. Then $e_1' \notin E(T_1')$ and so by Theorem 2.2, there exists an edge $e'' \in T_1'$ such that both $T_1'' = T_1' - e'' + e_1'$ and $T_2'' = T_2' + e'' - e_1'$ are spanning trees of $G' - u$. Since T_1' and T_2' are edge-disjoint, it follows that T_1'' and T_2'' are also edge-disjoint. Direct computation yields that the existence of T_1'' and T_2'' violates (1). Therefore and by symmetry, we conclude that

$$|\{e_1', e_3'\} \cap E(T_1')| \leq 1 \text{ and } |\{e_1', e_3'\} \cap E(T_2')| \leq 1. \tag{2}$$

To show that $\tau(G - u) \geq 2$, we are to construct two edge-disjoint spanning trees of $G - u$ from T_1' and T_2' as follows.

(A) Suppose that $|\{e_1', e_3'\} \cap (E(T_1') \cup E(T_2'))| = 0$. Then let $T_1 = T_1' + e_1$ and $T_2 = T_2' + e_3$.

(B) Suppose that $|\{e_1', e_3'\} \cap (E(T_1') \cup E(T_2'))| = 1$. Without loss of generality, we assume that $e_1' \in E(T_1')$ and $e_3' \notin E(T_1') \cup E(T_2')$. Let $T_1 = T_1' + e_1 + e_2 - e_1'$ and $T_2 = T_2' + e_3$.

(C) Suppose that $|\{e_1', e_3'\} \cap (E(T_1') \cup E(T_2'))| = 2$. By (2), we may assume that $e_1' \in E(T_1')$ and $e_3' \in E(T_2')$. Let $T_1 = T_1' + e_1 + e_2 - e_1'$ and $T_2 = T_2' + e_3 + e_4 - e_2'$.

It is routine to show that in each case, (A), (B) or (C) defines two edge disjoint spanning trees T_1 and T_2 of $G - u$, and so the lemma follows. ■

Following [2], an edge cut X of G is essential if each side of $G - X$ contains an edge and we use $ess'(G)$ to denote the **essential edge connectivity** of G.

Lemma 2.6. *Let G be a graph with $\kappa'(G) \geq 4$ and $\delta(G) \geq 5$. If $ess'(G) \geq 5$, then for any vertex $u \in V(G)$, there exists an edge $e \in E_G(u)$ such that $\kappa'(G - e) \geq 4$.*

Proof. By contradiction, there exists a vertex $u \in V(G)$ such that for any $e \in E_G(u)$, $\kappa'(G - e) \leq 3$. Pick an edge subset $X \subseteq E(G - e)$ with $|X| \leq 3$ such that X is a minimum edge cut of $G - e$. Thus $X \cup e$ is an edge-cut of G. Since $\delta(G) \geq 5$, $X \cup e$ must be an essential edge-cut of G, contrary to $ess'(G) \geq 5$. ■

Lemma 2.7. *Let G be a graph with $\kappa'(G) \geq 4$ and $\delta(G) \geq 6$. Then for any $x \in V(G)$, there exists an edge $e \in E_G(x)$ such that $\kappa'(G - e) \geq 4$.*

Proof. By Lemma 2.6, G must have an essential 4-edge-cut. Let X be a minimum essential 4-edge-cut of G with G_1 and G_2 being the two components of $G - X$ such that $x \in V(G_1)$ and such that the choice of X minimizes $|V(G_1)|$.

Let $e \in E_{G_1}(x)$. If $\kappa'(G - e) \geq 4$, then done. Assume that $\kappa'(G - e) \leq 3$ and $Y \subseteq E(G - e)$ is an edge cut of $G - e$ with $|Y| \leq 3$. Then $Y \cup \{e\}$ is also a 4-edge-cut of G. Let H_1 and H_2 be the two components of $G - (Y \cup \{e\})$. Define

$$V_1 = V(G_1) \cap V(H_1), V_2 = V(G_1) \cap V(H_2), V_3 = V(G_2) \cap V(H_1), V_4 = V(G_2) \cap V(H_2).$$

Without loss of generality, we assume that $x \in V_1$. Let α_{ij} be the number of edges in G joining V_i and V_j. Then we have

$$\begin{cases} \alpha_{12}+\alpha_{14}+\alpha_{23}+\alpha_{34} &= |Y \cup \{e\}| = 4, \\ \alpha_{13}+\alpha_{14}+\alpha_{23}+\alpha_{24} &= |X| = 4, \\ \alpha_{12}+\alpha_{13}+\alpha_{14} &\geq \kappa'(G) \geq 4, \\ \alpha_{12}+\alpha_{23}+\alpha_{24} &\geq \kappa'(G) \geq 4, \\ \alpha_{13}+\alpha_{23}+\alpha_{34} &\geq \kappa'(G) \geq 4, \\ \alpha_{14}+\alpha_{24}+\alpha_{34} &\geq \kappa'(G) \geq 4. \end{cases}$$

It follows from the last 4 inequalities above that $\alpha_{12}+\alpha_{13}+\alpha_{24}+\alpha_{34} \geq 8$, and so either $\alpha_{12}+\alpha_{34} \geq 4$ or $\alpha_{13}+\alpha_{24} \geq 4$, forcing $\alpha_{14} = \alpha_{23} = 0$. As $\kappa'(G) \geq 4$, this implies that $\alpha_{12} = \alpha_{13} = \alpha_{24} = \alpha_{34} = 2$, and so the $\alpha_{12}+\alpha_{13} = 4$ edges with exactly one end in V_1 is also an essential 4-edge-cut of G, violating the choice of X. This contradiction shows that for any $e \in E_{G_1}(x)$, $\kappa'(G-e) \geq 4$. ■

3. Proof of Theorem 1.4

We introduce a few more terms. If $X \subseteq E(G)$, then $G[X]$ is the subgraph of G induced by the edge subset X. For two vertices $u \neq v$, $E(u, v)$ is the set of edges joining u and v. For an edge subset $X \subset E(G)$, a contraction of G, denoted by G/X, is the graph obtained first from G by identifying the two ends of each edge in X, and then deleting all the resulting loops. If H is a subgraph of G, then we also use G/H for $G/E(H)$.

By Theorem 1.3(i), every graph G with $\tau(G) \geq 2$ is collapsible. To prove Theorem 1.4, we shall show a slightly stronger result, as stated below.

Theorem 3.1. *If G is a graph with $\kappa'(G) \geq 4$ and $|V(G)| \geq 3$, then for any $x \in D_4(G) \cup D_5(G)$ if $\delta(G) \leq 5$ or $x \in D_\delta(G) - K(G)$ if $\delta(G) \geq 6$, there exists a vertex $y \in V(G) - \{x\}$ such that*
(i) both $\tau(G-x) \geq 2$ and $\tau(G-y) \geq 2$, and
(ii) $\{x, y\} \cap D_\delta(G) \neq \emptyset$.

Proof. We argue by induction on $|V(G)|+|E(G)|$. Assume first that $|V(G)| = 3$. Since $\kappa'(G) \geq 4$, we have $|E(G)| \geq 6$. It follows that G is spanned by a path of length at most 2. If for any pair of distinct vertices $u, v \in V(G)$, $|E(u, v)| \geq 2$, then for any distinct $x \in V(G)$, $G-x$ is a K_2^ℓ with $\ell \geq 2$, and so $\tau(G-x) \geq 2$. Therefore the theorem holds in this case. Assume that there exists a pair of distinct vertices $u, v \in V(G)$ such that $|E(u, v)| \leq 1$. Let $w \in V(G) - \{u, v\}$. Then as $\kappa'(G) \geq 4$, we have $D_\delta(G) \subseteq \{u, v\}$ and both $|E(u, w)| \geq 3$ and $|E(v, w)| \geq 3$. Hence Theorem 3.1 holds with $\{x, y\} = \{u, v\}$.

Therefore, we assume that $|V(G)| \geq 4$, and Theorem 3.1 holds for graphs with smaller values of $|V(G)|+|E(G)|$. Let $x \in V(G)$ be a vertex such that either $\delta(G) \leq 5$ and $x \in D_4(G) \cup D_5(G) - K(G)$, or $\delta(G) \geq 6$ and $x \in D_\delta(G) - K(G)$. We have the following claims.

Claim 1. *Either Theorem 3.1 holds or $\delta(G) \geq 5$.*

Pick a vertex $v_0 \in V(G)$ with $d = d_G(v_0) = \delta(G)$ and let $E_G(v) = \{e_1, e_2, \ldots, e_d\}$. By contradiction, we assume that Theorem 3.1 does not hold and $d = 4$. By Lemma 2.3, $\tau(G - v_0) \geq 2$.

(1A) If $x = v_0$, then we need to show that there exists a vertex $u \in V(G) - \{v_0\}$ such that $\tau(G-u) \geq 2$.

If $G[E_G(v_0)] = K_2^4$, then $|N_G(v_0)| = 1$. Let v_0' be the only neighbor of v_0 in G. Then as $|V(G)| \geq 4$, v_0' is a cut vertex of G and so $\kappa'(G-v_0) \geq 4$. If follows by induction that $G-v_0$ has a vertex $y \in V(G) - \{v_0, v_0'\}$ such that $\tau(G - \{v_0, y\}) \geq 2$. Since $G[E_G(v_0)] = K_2^4$, $G - v_0 = G/E_G(v_0)$, and so by Lemma 2.4(C3), $\tau(G-y) \geq 2$ as well. Hence the theorem holds in this case.

Therefore we assume that $G[E_G(v_0)] \neq K_2^4$. It follows that $N_G(v_0)$ contains at least two vertices. Let $N_G(v_0) = \{v_1, v_2, \ldots\}$ be so labeled that $|E(v_0, v_1)| = \max\{|E(v_0, v)| : v \in N_G(v_0)\}$. As $d = 4$ and as $G[E_G(v_0)] \neq K_2^4$, we have $|E(v_0, v_1)| \leq 3$.

Suppose first that $|E(v_0, v_1)| \geq 2$. Define $G' = G/E(v_0, v_1)$ and let v^* denote the vertex in G' onto which the edges in $E(v_0, v_1)$ are contracted. Since $\kappa'(G') \geq \kappa'(G) \geq 4$ and $|V(G')| < |V(G)|$, by induction, there exists a vertex $u \in V(G') - v^*$ such that $\tau(G'-u) \geq 2$. By Lemma 2.4, and as $\tau(G[E(v_0, v_1)]) \geq 2$, we have $\tau(G-u) \geq 2$. Hence Theorem 3.1 holds.

Thus we may assume that $N_G(v_0) = \{v_1, v_2, v_3, v_4\}$. Denote $e_i = vv_i$ for $1 \leq i \leq 4$. By Theorem 2.1, we may assume that $G' = G - v + \{v_1v_2, v_3v_4\}$ is 4-edge-connected. By induction, there exists a vertex $u \in V(G')$ such that $\tau(G'-u) \geq 2$. It follows from Lemma 2.5 that $\tau(G-u) \geq 2$.

(1B) Assume that $x \neq v_0$. We need to show that $\tau(G-x) \geq 2$. As in the arguments to show (1A), we may assume that $N_G(v_0) = \{v_1, v_2, v_3, v_4\}$, and $G' = G - v + \{v_1v_2, v_3v_4\}$ is 4-edge-connected. Since $x \neq v_0$, by the definition of G', we have $x \in V(G')$ and $d_{G'}(x) = d_G(x)$. Hence by $\delta(G) = 4$, we must have $x \in D_4(G') \cup D_5(G')$. By induction, $\tau(G'-x) \geq 2$. By Lemma 2.5, $\tau(G-x) \geq 2$. This completes the proof for Claim 1.

By Claim 1, either $x \in D_5(G)$ or $\delta(G) \geq 6$ and $x \in D_\delta(G)$. If $\delta(G) \geq 6$, then by Lemma 2.7, there must be an edge $e \in E_G(x)$ such that $\kappa'(G-e) \geq 4$. As $e \in E_G(x)$ and $x \in D_\delta(G)$, we also have $x \in D_\delta(G-e)$. By induction, there must be a $y \in V(G-e) - \{x\}$ such that both $\tau(G-x) \geq 2$ and $\tau(G-y) \geq 2$. Hence we assume that $x \in D_5(G)$.

Claim 2. *Either Theorem 3.1 holds or $ess'(G) \geq 5$.*

Assume that Theorem 3.1 does not hold and $ess'(G) \le 4$. Since $\kappa'(G) \ge 4$, $ess'(G) = 4$. Suppose G has a minimum essential edge-cut X with $|X| = 4$. Let G_1 and G_2 be the two components of $G-X$. Without loss of generality, we assume that $x \in V(G_1)$. For $i \in \{1, 2\}$, let $G^i = G/G_i$, and v^i be the vertex of G^i onto which G_i is contracted. By the definition of contraction, we have $\kappa'(G_i) \ge \kappa'(G) \ge 4$. As $x \in D_5(G)$, we also have $x \in D_5(G^2)$. By induction, there exists a vertex $u_1 \in V(G^1 - v^1)$ such that $\tau(G^1 - u_1) \ge 2$, and $\tau(G^2 - x) \ge 2$. Let $u_2 = x$ and fix $i \in \{1, 2\}$. Since $u_i \ne v^i$, $G^i - u_i = (G - u_i)/G_i$. As $\tau(G_i) \ge 2$ and $\tau(G - u_i)/G_i \ge 2$, it follows by Lemma 2.4 (C3) that $\tau(G - u_i) \ge 2$. Hence the theorem holds, and so Claim 2 is justified.

By Claims 1 and 2, we may assume that $\delta(G) \ge 5$ and $ess'(G) \ge 5$. By Lemma 2.6, there must be an edge $e \in E_G(x)$ such that $\kappa'(G - e) \ge 4$. By induction, there exists a vertex $y \in V(G - e) - \{x\}$ such that both $\tau(G - x) \ge \tau((G - e) - x) \ge 2$ and $\tau(G - y) \ge \tau((G - e) - y) \ge 2$. This completes the proof of the theorem. ■

Theorem 3.1 indicates that every 4-edge-connected graph G contains a vertex x of minimum degree such that $\tau(G-x) \ge 2$. This property can be applied to show a slight extension of Theorem 1.3 (iii), as seen in Corollary 3.2.

Corollary 3.2. *Let G be a graph with $\kappa'(G) \ge 4$. Then each of the following holds.*

(i) For any vertex $v \in V(G)$ with $d_G(v) = 4$, $\tau(G - v) \ge 2$.

(ii) G has at most one vertex of degree 4 and there exists an edge subset $X \subseteq E(G)$ with $|X| = 3$ such that $\tau(G - X) \ge 2$.

Proof. If G has a vertex v of degree 4, then by Lemma 2.3, $\tau(G - v) \ge 2$ and so Corollary 3.2(i) holds. Now assume that G has at most one vertex of degree 4. By Theorem 3.1, there exists a vertex $u \in V(G)$ of degree at least 5 in G such that $\tau(G - u) \ge 2$. Let $E_G(u) = \{e_1, e_2, \ldots, e_d\}$ with $d = d_G(u) \ge 5$, and let $X = \{e_1, e_2, e_3\}$. Define $G' = (G - X)/(G - u)$. Then G' has two vertices and $d - 3 \ge 2$ edges. It follows that $\tau(G') \ge 2$, and so by Lemma 2.4 and by $\tau(G - u) \ge 2$, we conclude that $\tau(G - X) \ge 2$. ■

4. Proof of Theorems 1.5 and 1.6

We start with an observation that every collapsible graph, being supereulerian, must be 2-edge-connected. Throughout this section, we assume that G is a graph with $\kappa'(G) \ge 4$. Let $X \subseteq E(G)$ be an edge subset with $|X| = 3$. If X is a subset of an edge 4-cut, then $G - X$ has a cut edge, and so $G - X$ cannot be collapsible. Therefore, to prove Theorem 1.5, it suffices to prove the following.

Lemma 4.1. *Let G be a graph with $\kappa'(G) \ge 4$ and let $X \subseteq E(G)$ be an edge subset with $|X| \le 3$. Then $G - X$ is collapsible if and only if X is not lying in an edge-cut of size 4 in G.*

Proof. Let $X_1 \subseteq X$ be such that $|X_1| \le 2$ and $|X - X_1| \le 1$. Then by Theorem 1.3(iii), $\tau(G - X_1) \ge 2$. Let $H = G - X$. Since $|X - X_1| \le 1$ and since $\tau(G - X_1) \ge 2$, it follows that $F(H) \le 1$. By Theorem 1.3(i), H is collapsible if and only if $\kappa'(H) \ge 2$. Since $H = G - X$ and $\kappa'(G) \ge 4$, it follows that $\kappa'(H) \ge 2$ if and only if X is not lying in a 4-edge-cut of G. ■

With an identical argument, we also have the following slightly strengthening proposition, whose proof is omitted.

Proposition 4.2. *Let G be a graph with $k = \kappa'(G) \ge 4$ and let $X \subseteq E(G)$ be an edge subset with $|X| \le k - 1$. Then $G - X$ is collapsible if and only if X is not lying in an edge-cut of size k in G.*

To prove Theorem 1.6, we need one more result of Catlin.

Theorem 4.3 (*Caltin, Lemma 3 of [3]*). *If G is collapsible, then any contraction of G is also collapsible.*

Let K_2^c denote the edgeless graph on 2 vertices. We first make some observations. If $G - X$ is contractible to a member in $\{K_2^c, K_2, K_{2,2}, K_{2,3}, K_{2,4}\}$, then since it is known that none of the graphs in $\{K_2^c, K_2, K_{2,2}, K_{2,3}, K_{2,4}\}$ is collapsible (see, for example, [5]), it follows by Theorem 4.3 that $G - X$, being contractible to a member in that family, cannot be collapsible. Therefore, to prove Theorem 1.6, it remains to prove the following lemma.

Lemma 4.4. *Let G be a graph with $\kappa'(G) \ge 4$ and let $X \subseteq E(G)$ be an edge subset with $|X| \le 4$. If $G - X$ is not contractible to a member in $\{K_2^c, K_2, K_{2,2}, K_{2,3}, K_{2,4}\}$, then $G - X$ is collapsible.*

Proof. Let $X_2 \subseteq X$ be such that $|X_2| \le 2$ and $|X - X_2| \le 2$. Then by Theorem 1.3(iii), $\tau(G - X_2) \ge 2$. Let $H = G - X$. Since $|X - X_2| \le 2$ and since $\tau(G - X_1) \ge 2$, it follows that $F(H) \le 2$. By Theorem 1.3(ii), H is collapsible if and only if H is disconnected, or H is connected but not contractible to a member in $\{K_2\} \cup \{K_{2,t} : t \ge 1\}$. Since $\kappa'(G) \ge 4$ and $|X| \le 4$, it follows that $t \le 4$ and so H is collapsible if and only if H is connected but not contractible to a member in $\{K_2^c, K_2, K_{2,2}, K_{2,3}, K_{2,4}\}$. ■

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