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Bounds for the matching number and cyclomatic number of a signed graph in terms of rank

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ABSTRACT

A signed graph (G, σ) is a graph with a sign attached to each of its edges, where G is the underlying graph of (G, σ). Let $m(G)$, $c(G)$ and $r(G, \sigma)$ be the matching number, the cyclomatic number and the rank of the adjacency matrix of (G, σ), respectively. In this paper, we investigate the relation among the rank, the matching number and the cyclomatic number of a signed graph, and prove that $2m(G) - 2c(G)) \leq r(G, \sigma) \leq 2m(G) + c(G)$. Furthermore, signed graphs reaching the lower bound or the upper bound are respectively characterized.

1. Introduction

In this paper, we consider only graphs without multiedges and loops. An *undirected graph* G is denoted by $G = (V, E)$, where $V = \{v_1, v_2, \cdots, v_n\}$ is the vertex set and E is the edge set. The *adjacency matrix* $A(G)$ of G is an $n \times n$ matrix whose (i, j)-entry equals

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to 1 if vertices v_i and v_j are adjacent and 0 otherwise. We refer to [24] for undefined terminologies and notation.

A *signed graph* (G, σ) consists of a simple graph $G = (V, E)$, referred to as its underlying graph, and a mapping $\sigma : E \to \{+, -\}$, its edge labelling. To avoid confusion, we often write $V(G)$ and $E(G)$ for $V(G, \sigma)$ and $E(G, \sigma)$, respectively. The adjacency matrix of (G, σ) is $A((G, \sigma)) = (a^{\sigma}_{ij})$ with $a^{\sigma}_{ij} = \sigma(v_i v_j)a_{ij}$, where (a_{ij}) is the adjacent matrix of the underlying graph G. Let (G, σ) be a signed graph. An edge e' is said to be positive or negative if $\sigma(e') = +$ or $\sigma(e') = -$, respectively. In the case of $\sigma = +$, which is an all-positive edge labelling, $A(G, +)$ is exactly the classical adjacency matrix of G. Thus a simple graph can always be viewed as a singed graph with all positive edges. Let C be a cycle of (G, σ). The *sign* of C is defined by $\sigma(C) = \prod_{e \in C} \sigma(e)$. A cycle C is said to be positive or negative if $\sigma(C) = +$ or $\sigma(C) = -$, respectively. Denote by C_n and P_n a signed cycle and a signed path on n vertices, respectively. By definition, a cycle C is positive if and only if it has even number of negative edges. The *rank* of a signed graph (G, σ), written as $r(G, \sigma)$, is defined to be the rank of its adjacency matrix $A(G, \sigma)$. The *nullity* of (G, σ) is the multiplicity of the zero eigenvalues of $A(G, \sigma)$.

If G is a graph which the cycles (if any) of G are pairwise vertex-disjoint, contracting each cycle of G into a vertex (called a *cyclic vertex*), we obtain a forest denoted by T_G. Moreover, denoted by $[T_G]$ the subgraph of T_G induced by all non-cyclic vertices.

Let $c(G)$ be the *cyclomatic number* of a graph G, that is $c(G) = |E(G)| - |V(G)| + \omega(G)$, where $\omega(G)$ is the number of connected components of G. Two distinct edges in a graph G are *independent* if they do not have common end-vertex in G. A set of pairwise independent edges of G is called a *matching*, while a matching with the maximum cardinality is a *maximum matching* of G. The *matching number* of G, denoted by $m(G)$, is the cardinality of a maximum matching of G. For a signed graph (G, σ), the cyclomatic number and matching number of (G, σ) are defined to be the cyclomatic number and matching number of its underlying graph, respectively.

The rank and nullity of graphs have been studied intensively by many researchers. In [9] Gutman and Sciriha investigated the nullity of line graphs of trees. Bevis et al. [3] researched the rank of a graph after vertex addition and obtained some results examining several cases of vertex addition. In [15,16], Ma et al. studied the skew-rank of an oriented graph in terms of matching number and the nullity of a undirected graph in terms of the dimension of cycle space and the number of pendant vertices. Rula et al. investigated the nullity of a undirected graph in terms of the dimension of cycle space and the number of maximum matching in [18]. Liu and Li [13] researched the cospectral problems among mixed graphs with its underlying graph. Mohar et al. [10] characterized all the mixed graphs with rank equal to 2. Wang et al. [23] investigated the graphs with H-rank 3. Chen et al. studied the relation between the H-rank of a mixed graph and the matching number of its underlying graph in [4]. Wang and Wong [20] researched the relation between the rank and the chromatic number of a simple graph.

In recent years, the study of the rank and nullity of signed graphs received increased attention. In [14], Liu and You investigated the nullity of signed graphs. Fan et al.

researched the nullity of unicyclic signed graphs and bicyclic signed graphs in [7] and [6], respectively. Tian et al. [19] characterized the signed planar graphs with rank at most 4. Belardo et al. studied the spectral characterizations and the Laplacian coefficients of signed graphs in [1] and [2], respectively. Hou et al. investigated the Laplacian eigenvalues of signed graphs in [11]. Wang [4] studied the relation between the rank of a signed graph and the rank of its underlying graph. In [22], Wong et al. characterized the signed graphs with cut points whose positive inertia indexes are two. For other research of the rank of a graph one may be referred to those in [8,24,12,17,5].

In this paper, we studied the relation among the rank of a signed graph (G, σ) and the matching number and the cyclomatic number of its underlying graph. For any connected signed graph (G, σ), we proved that $2m(G) - 2c(G) \leq r(G, \sigma) \leq 2m(G) + c(G)$. Moreover, we characterized the extremal graphs which attended the upper and lower bounds, respectively. Our main results are the following Theorems 1.1, 1.2 and 1.3.

Theorem 1.1. *Let (G, σ) be a connected signed graph. Then*

$$2m(G) - 2c(G) \leq r(G, \sigma) \leq 2m(G) + c(G).$$

Theorem 1.2. *Let (G, σ) be a connected signed graph. Then $r(G, \sigma) = 2m(G) + c(G)$ if and only if all the following conditions hold for (G, σ):*

(i) *the cycles (if any) of (G, σ) are pairwise vertex-disjoint;*

(ii) *each cycle (if any) of (G, σ) is odd;*

(iii) $m(T_G) = m([T_G])$.

Theorem 1.3. *Let (G, σ) be a connected signed graph. Then $r(G, \sigma) = 2m(G) - 2c(G)$ if and only if all the following conditions hold for (G, σ):*

(i) *the cycles (if any) of (G, σ) are pairwise vertex-disjoint;*

(ii) *for each cycle (if any) C_q of (G, σ), either $q \equiv 0 \pmod 4$ and $\sigma(C_q) = +$ or $q \equiv 2 \pmod 4$ and $\sigma(C_q) = -$;*

(iii) $m(T_G) = m([T_G])$.

The rest of this paper is organized as follows. Prior to showing our main results, in Section 2, we establish some elementary notations and some useful lemmas. In Section 3, we give the main result of this paper. In Section 4 and 5, we characterize the extremal signed graphs which attained the upper bound and the lower bound of Theorem 1.1, respectively.

2. Preliminaries

In this section, we present and develop some useful lemmas which will be used in the proofs of our main results.

For a signed graph (G, σ), let $\mathscr{F}(G)$ be the set of edges of (G, σ) which has an endpoint on a cycle and the other endpoint outside the cycle. For an induced subgraph H and a vertex x that is not in $V(H)$, we define $H+x$ to be the induced subgraph of G with vertex set $V(H)\cup\{x\}$. A vertex of G is called a *pendant vertex* if it is a vertex of degree one in G, whereas a vertex of G is called a *quasi-pendant vertex* if it is adjacent to a pendant vertex in G unless it is a pendant vertex.

Lemma 2.1. [21] *Let (G,σ) be a signed graph.*

(i) *If (H,σ) is an induced subgraph of (G,σ), then $r(H,\sigma)\leq r(G,\sigma)$.*

(ii) *If $(G_1,\sigma),(G_2,\sigma),\cdots,(G_t,\sigma)$ are all the connected components of (G,σ), then $r(G,\sigma)=\sum_{i=1}^{t} r(G_i,\sigma)$.*

(iii) *$r(G,\sigma)\geq 0$ with equality if and only if (G,σ) is an empty graph.*

Lemma 2.2. [7] *Let y be a pendant vertex of (G,σ) and x is the neighbour of y. Then $r(G,\sigma)=r((G,\sigma)-\{x,y\})+2$.*

Lemma 2.3. [3] *Let x be a vertex of (G,σ). Then $r(G,\sigma)-2\leq r((G,\sigma)-x)\leq r(G,\sigma)$.*

Lemma 2.4. [5] *Let (F,σ) be a signed acyclic graph. Then $r(F,\sigma)=r(F)=2m(F)$.*

Lemma 2.5. [4] *Let G be a simple undirected graph. Then $m(G)-1\leq m(G-v)\leq m(G)$ for any vertex $v\in V(G)$.*

Lemma 2.6. [15] *Let G be a graph obtained by joining a vertex of an even cycle C by an edge to a vertex of a connected graph H. Then $m(G)=m(C)+m(H)$.*

Lemma 2.7. [15] *Let x be a pendant vertex of G and y be the neighbour of x. Then $m(G)=m(G-y)+1=m(G-\{x,y\})+1$.*

Lemma 2.8. [18] *Let G be a graph with at least one cycle. Suppose that all cycles of G are pairwise-disjoint and each cycle is odd, then $m(T_G)=m([T_G])$ if and only if $m(G)=\sum_{C\in\mathscr{L}(G)} m(C)+m([T_G])$, where $\mathscr{L}(G)$ denotes the set of all cycles in G.*

Lemma 2.9. [15] *Let G be a graph with $x\in V(G)$.*

(i) *$c(G)=c(G-x)$ if x lies outside any cycle of G;*

(ii) *$c(G-x)\leq c(G)-1$ if x lies on a cycle of G;*

(iii) *$c(G-x)\leq c(G)-2$ if x is a common vertex of distinct cycles of G.*

Lemma 2.10. [18] *Let G be a connected graph without pendant vertices and $c(G)\geq 2$. Suppose that for any vertex u on a cycle of G, $c(G-x)\geq c(G)-2$. Then there are at most $c(G)-1$ vertices of G which are not covered by its maximum matching.*

Lemma 2.11. [14] *Let (G,σ) be a signed graph on n vertices and*

$$P_{(G,\sigma)}(\lambda)=|\lambda I_n-A(G,\sigma)|=\lambda^n+a_1\lambda^{n-1}+\cdots+a_n$$

be the characteristic polynomial of $A(G,\sigma)$. Then for any $i \in \{1,2,\cdots,n\}$,

$$a_i = \sum_{(U,\sigma)\in\mathscr{U}_i} (-1)^{p(U)+s(U)} \cdot 2^{c(U)},$$

where $\mathscr{U}_i$ is the set of all basic subgraphs contains in (G,σ) which have exactly i vertices, $p(U)$, $c(U)$ and $s(U)$ are the number of components, the number of cycles and the number of negative edges in the cycle of (U,σ), respectively.

Lemma 2.12. [14] *Let (C_n,σ) be a signed cycle of order n. Then*

$$r(C_n,\sigma) = \begin{cases} n, & \text{if } n \text{ is odd;} \\ n, & \text{if } n \equiv 0 \pmod 4 \text{ and } \sigma(C_n) = -; \\ n, & \text{if } n \equiv 2 \pmod 4 \text{ and } \sigma(C_n) = +; \\ n-2, & \text{if } n \equiv 0 \pmod 4 \text{ and } \sigma(C_n) = +; \\ n-2, & \text{if } n \equiv 2 \pmod 4 \text{ and } \sigma(C_n) = -. \end{cases}$$

Lemma 2.13. [15] *Let T be a tree with at least one edge.*

(i) *$r(T_1) < r(T)$, where T_1 is the subtree obtained from T by deleting all the pendant vertices of T.*

(ii) *If $r(T - W) = r(T)$ for a subset W of $V(T)$. Then there is a pendant vertex v of T such that $v \notin W$.*

Lemma 2.14. [5] *Let T be a undirected tree with n vertices. Let*

$$\phi(T,\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

be the characteristic polynomial of $A(T)$. Then for any $i \in \{1,2,\cdots,n\}$,

$$a_i = \sum_{B\in\mathscr{B}_i} (-1)^{p(B)},$$

where $\mathscr{B}_i$ is the set of all basic graphs contains in T which have exactly i vertices, $p(B)$ is the number of components of B.

Let (T,σ) be a signed tree and T be the underlying graph of (T,σ). By Lemmas 2.11 and 2.14, it can be checked that the characteristic polynomial of $A(T,\sigma)$ is same as the characteristic polynomial of $A(T)$. Then, by Lemma 2.13, one has the following Lemma 2.15.

Lemma 2.15. *Let (T,σ) be a signed tree with at least one edge.*

(i) *$r(T_1,\sigma) < r(T,\sigma)$, where (T_1,σ) is the subtree obtained from (T,σ) by deleting all the pendant vertices of (T,σ).*

(ii) *If $r((T,\sigma) - W) = r(T,\sigma)$ for a subset W of $V(T)$. Then there is a pendant vertex v of (T,σ) such that $v \notin W$.*

3. Proof of Theorem 1.1

In this section, we study the relation among the rank and the matching number and the cyclomatic number of a signed graph, and give the proof for Theorem 1.1.

The proof of Theorem 1.1. First, we prove the inequality on the right of Theorem 1.1. Let $P_{(G,\sigma)}(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ be the characteristic polynomial of $A(G,\sigma)$. By Lemma 2.11 and the definition of basic subgraph, it is routine to verify that there exists no basic subgraph with j vertices if $j > 2m(G) + c(G)$. Thus, $r(G,\sigma) \leq 2m(G) + c(G)$.

Next, we argue by induction on $c(G)$ to show that $2m(G) - 2c(G) \leq r(G,\sigma)$. If $c(G) = 0$, then (G,σ) is a signed tree, and so result follows from Lemma 2.4. Hence we assume that $c(G) \geq 1$. Let u be a vertex on some cycle of (G,σ) and $(G',\sigma) = (G,\sigma) - u$. Let $(G_1,\sigma), (G_2,\sigma), \cdots, (G_l,\sigma)$ be all connected components of (G',σ). By the induction hypothesis, for each $j \in \{1, 2, \cdots, l\}$, one has

$$r(G_j,\sigma) \geq 2m(G_j) - 2c(G_j). \tag{1}$$

By Lemmas 2.5, 2.1 and 2.3, we have

$$\sum_{i=1}^{l} m(G_i) = m(G') \geq m(G) - 1, \tag{2}$$

and

$$\sum_{i=1}^{l} r(G_i,\sigma) = r(G',\sigma) \leq r(G,\sigma). \tag{3}$$

Moreover, by Lemma 2.9, we have

$$\sum_{i=1}^{l} c(G_i) = c(G') \leq c(G) - 1. \tag{4}$$

Thus the desired inequality now follows by combining (1), (2), (3) and (4),

$$\begin{aligned} r(G,\sigma) &\geq \sum_{i=1}^{l} r(G_i,\sigma) \\ &\geq \sum_{i=1}^{l} (2m(G_j) - 2c(G_j)) \\ &\geq 2(m(G) - 1) - 2(c(G) - 1) = 2m(G) - 2c(G). \end{aligned} \tag{5}$$

This completes the proof of Theorem 1.1.

A signed graph (G, σ) with $r(G, \sigma) = 2m(G) - 2c(G)$ is called a **lower-optimal** signed graph. One can utilize the arguments above to make the following observations.

Corollary 3.1. *Let u be a vertex of (G, σ) lying on a signed cycle. If $r(G, \sigma) = 2m(G) - 2c(G)$, then each of the following holds.*
(i) $r(G, \sigma) = r((G, \sigma) - u)$;
(ii) $(G, \sigma) - u$ *is lower-optimal;*
(iii) $c(G) = c(G - u) + 1$;
(iv) $m(G) = m(G - u) + 1$;
(v) *u lies on just one signed cycle of (G, σ) and u is not a quasi-pendant vertex of (G, σ).*

Proof. In the proof arguments of Theorem 1.1 that justifies $r(G, \sigma) \geq 2m(G) - 2c(G)$. If both ends of (5) are the same, then all inequalities in (5) must be equalities, and so Corollary 3.1 (i)-(iv) are observed.

To show (v). By Corollary 3.1 (iii) and Lemma 2.9, we conclude that u lies on just one signed cycle of (G, σ). Suppose to the contrary that u is a quasi-pendant vertex which adjacent to a vertex v. Then by Lemma 2.2, we have

$$r((G, \sigma) - u) = r((G, \sigma) - \{u, v\}) = r(G, \sigma) - 2,$$

which is a contradiction to (i). This completes the proof of the corollary. □

4. Proof of Theorem 1.2

A signed graph (G, σ) is said to be *upper-optimal* if $r(G, \sigma) = 2m(G) + c(G)$, or equivalently, the signed graph which attain the upper bound in Theorem 1.1. In this section, we characterize the properties of the signed graphs which are upper-optimal. Firstly, we present some lemmas which give some fundamental characterization of upper-optimal signed graphs in what following.

Lemma 4.1. *Let (G, σ) be a signed graph and $(G_1, \sigma), (G_2, \sigma), \cdots, (G_k, \sigma)$ be all connected components of (G, σ). Then (G, σ) is upper-optimal if and only if (G_j, σ) is upper-optimal for each $j \in \{1, 2, \cdots, k\}$.*

Proof. (Sufficiency.) For each $i \in \{1, 2, \cdots, k\}$, one has that

$$r(G_i, \sigma) = 2m(G_i) + c(G_i).$$

Then, one can get (G, σ) is upper-optimal immediately follows from the fact that

$$r(G, \sigma) = \sum_{j=1}^{k} r(G_j, \sigma),$$

$$m(G) = \sum_{j=1}^{k} m(G_j),$$

and

$$c(G) = \sum_{j=1}^{k} c(G_j).$$

(Necessity.) Suppose to the contrary that there is a connected component of (G, σ), say (G_1, σ), which is not upper-optimal. Then

$$r(G_1, \sigma) < 2m(G_1) + c(G_1),$$

and by Lemma 1.1, for each $j \in \{2, 3, \cdots, k\}$, we have

$$r(G_j, \sigma) \leq 2m(G_j) + c(G_j).$$

Thus, one has that

$$r(G, \sigma) = \sum_{j=1}^{k} r(G_j, \sigma) < 2m(G) + c(G),$$

a contradiction. □

Lemma 4.2. *Let u be a pendant vertex of a signed graph (G, σ) and v be the vertex which adjacent to u. Let $(G_0, \sigma) = (G, \sigma) - \{u, v\}$. Then, (G, σ) is upper-optimal if and only if v is not on any signed cycle of (G, σ) and (G_0, σ) is upper-optimal.*

Proof. (Sufficiency.) If v is not on any signed cycle, by Lemma 2.9, we have

$$c(G) = c(G_0).$$

By Lemmas 2.2 and 2.7, one has that

$$r(G, \sigma) = r(G_0, \sigma) + 2, m(G) = m(G_0) + 1.$$

Thus, one can get (G, σ) is upper-optimal by the condition that

$$r(G_0, \sigma) = 2m(G_0) + c(G_0).$$

(Necessity.) By Lemmas 2.2 and 2.7 and the condition that (G, σ) is upper-optimal, it can be checked that

$$r(G_0, \sigma) = 2m(G_0) + c(G).$$

It follows from Theorem 1.1 that one has

$$r(G_0,\sigma) \leq 2m(G_0) + c(G_0).$$

By the fact that $c(G_0) \leq c(G)$, then we have

$$c(G) = c(G_0), r(G_0,\sigma) = 2m(G_0) + c(G_0).$$

Thus (G_0,σ) is also upper-optimal and so the lemma is justified. □

Lemma 4.3. *Let (G,σ) be a signed unicyclic graph which contains the unique signed cycle C_l. Then (G,σ) is upper-optimal if and only if l is odd and $m(T_G) = m([T_G])$.*

Proof. (Sufficiency.) Let $P_{(G,\sigma)}(\lambda) = |\lambda I_n - A(G,\sigma)| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ be the characteristic polynomial of $A(G,\sigma)$ and $m = m(G)$. By Theorem 1.1, we just need to prove $a_{2m+1} \neq 0$.

Since l is odd and $m(T_G) = m([T_G])$, by Lemma 2.8, we have

$$m(G) = m(C_l) + m([T_G])$$

which is equivalent to

$$2m + 1 = l + 2m([T_G]).$$

Let M_0 be a maximum matching of $[T_G]$, then $|M_0| = m([T_G])$. Then, it can be checked that the order of $M_0 \cup C_l$ is $2m([T_G]) + l = 2m + 1$. Then, $M_0 \cup C_l \in \mathscr{U}_{2m+1}$. Thus, by Lemma 2.11, one has that

$$\begin{aligned} a_{2m+1} &= \sum_{(U,\sigma)\in\mathscr{U}_{2m+1}} (-1)^{p(U)+s(U)} \cdot 2^{c(U)} \\ &= \sum_{(U,\sigma)\in\mathscr{U}_{2m+1}} (-1)^{1+\frac{2m+1-l}{2}+s(U)} \cdot 2^1 \\ &= 2|\mathscr{U}_{2m+1}|(-1)^{\frac{2m+3-l}{2}+s(U)} \neq 0, \end{aligned}$$

where $\mathscr{U}_{2m+1}$ is the set of all basic subgraphs contains in (G,σ) which have exactly $2m+1$ vertices. Moreover, $p(U)$, $c(U)$ and $s(U)$ are the number of components, the number of cycles and the number of negative edges in the cycle of (U,σ), respectively.

(Necessity.) Since (G,σ) is upper-optimal, $r(G,\sigma) = 2m + 1$. By Lemma 2.11, there exists at least one basic subgraph of order $2m+1$. Note that $2m+1$ is odd, then each basic subgraph of order $2m+1$ must contains the unique cycle C_l as its connected component. Moreover, l is odd.

Next, we prove $m(T_G) = m([T_G])$ by induction on $|V(T_G)|$. Sine (G,σ) contains a cycle, $|V(T_G)| \geq 1$. If $|V(T_G)| = 1$, then $(G,\sigma) \cong (C_l,\sigma)$, which implies that $m(T_G) =$

$m([T_G]) = 0$. Now suppose $|V(T_G)| \geq 2$, then there exists a pendant vertex u of T_G which is also a pendant vertex of (G, σ). Let v be the unique neighbour of u and $(G', \sigma) = (G, \sigma) - \{u, v\}$. By Lemma 4.2, (G', σ) is also upper-optimal and v is not on any signed cycle of (G, σ). It is obviously that $|V(T_{G'})| < |V(T_G)|$. By induction hypothesis, one has

$$m(T_{G'}) = m([T_{G'}]).$$

By Lemma 2.7, we have $m(T_G) = m(T_{G'}) + 1 = m([T_{G'}]) + 1 = m([T_G])$. This completes the proof of the lemma. □

Lemma 4.4. *Let (G, σ) be a signed graph without pendant vertex and $c(G) \geq 2$. Then (G, σ) is not upper-optimal.*

Proof. If there exists a vertex u of (G, σ) such that $c(G-u) \leq c(G) - 3$. By Lemmas 2.5 and 2.3, one has that

$$m(G) \geq m(G-u), r(G, \sigma) \leq r((G, \sigma) - u) + 2.$$

Suppose to the contrary that (G, σ) is upper-optimal, then,

$$c(G) = r(G, \sigma) - 2m(G) \leq r((G, \sigma) - u) + 2 - 2m(G-u).$$

By Theorem 1.1, we have

$$r((G, \sigma) - u) - 2m(G-u) \leq c(G-u).$$

Then,

$$c(G) \leq c(G-u) + 2 \leq c(G) - 1,$$

a contradiction.

So one can suppose that for any vertex u, $c(G-u) \geq c(G) - 2$. By Lemma 2.10, there are at most $c(G) - 1$ vertices of (G, σ) which are not covered by its maximum matching. Then,

$$m(G) \geq \frac{|V(G)| - c(G) + 1}{2}.$$

Suppose to the contrary that (G, σ) is upper-optimal. Then one has that $r(G, \sigma) = 2m(G) + c(G) \geq |V(G)| + 1$. This contradiction completes the proof of the lemma. □

The proof of Theorem 1.2. (Sufficiency.) Let (G, σ) be a signed graph which satisfies all the conditions of (i)-(iii). Let $P_{(G,\sigma)}(\lambda) = |\lambda I_n - A(G, \sigma)| = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$ be the characteristic polynomial of $A(G,\sigma)$, $m(G)$ and $c(G)$ are simply written as m and c, respectively. By Theorem 1.1, it suffices to show that $a_{2m+c} \neq 0$.

By Lemma 2.4, we may assume that (G,σ) contains at least one cycle. By Lemma 2.8, we have $m(G) = \sum_{C\in\mathscr{L}(G,\sigma)} m(C) + m([T_G])$. Let $(O_1,\sigma),(O_2,\sigma),\cdots,(O_c,\sigma)$ be all cycles of (G,σ) and M_1 be a maximum matching of $[T_G]$. Then, it can be checked that $(\cup_{j=1}^{c}(O_j,\sigma)) \cup M_1 \in \mathscr{U}_{2m+c}$. Consequently, $\mathscr{U}_{2m+c} \neq \emptyset$. Now suppose that (U,σ) is a basic subgraph of order $2m+c$ with $(O_{i_1},\sigma),(O_{i_2},\sigma),\cdots,(O_{i_k},\sigma),K_2^1,K_2^2,\cdots,K_2^q$ as all of its connected components, where (O_{i_j},σ) ($j\in\{1,2,\cdots,k\}$) denotes an odd cycle and K_2^h ($h\in\{1,2,\cdots,q\}$) denotes a edge. Obviously,

$$|V(O_{i_1})| + |V(O_{i_2})| + \cdots + |V(O_{i_k})| + 2q = 2m + c.$$

Note that $m \geq m(U,\sigma)$. Hence, one has that

$$m \geq \frac{|V(O_{i_1})|-1}{2} + \frac{|V(O_{i_2})|-1}{2} + \cdots + \frac{|V(O_{i_k})|-1}{2} + q = \frac{2m+c-k}{2},$$

which implies that $k \geq c$, thus we have $k = c$. Therefore, each basic subgraph of (G,σ) with order $2m+c$ must contain all cycles of (G,σ). Combining with Lemma 2.11, one has

$$a_{2m+c} = \sum_{(U,\sigma)\in\mathscr{U}_{2m+c}} (-1)^{p(U)+s(U)} \cdot 2^{c(U)} = 2^c \cdot |\mathscr{U}_{2m+c}|(-1)^{p(U)+s(U)} \neq 0,$$

where $\mathscr{U}_{2m+c}$ is the set of all basic subgraphs contains in (G,σ) which have exactly $2m+c$ vertices, $p(U)$ and $s(U)$ are the number of components and the number of negative edges in the cycle of (U,σ), respectively.

(Necessity.) We proceed by induction on the order n of (G,σ) to prove (i)-(iii). If $n = 1$, then (i)-(iii) hold trivially. Suppose that (i)-(iii) hold for all upper-optimal connected signed graph of order smaller than n. Now, let (G,σ) be an upper-optimal connected signed graph of order $n \geq 2$. If $c(G) = 0$, then (G,σ) is a signed tree and (i)-(iii) hold trivially. If $c(G) = 1$, then (G,σ) is a signed unicyclic graph and (i)-(iii) follow immediately from Lemma 4.3. If $c(G) \geq 2$, then by Lemma 4.4, (G,σ) has at least one pendant vertex. Let u be a pendant vertex of (G,σ) and v be the unique neighbour of u. Denote $(G_0,\sigma) = (G,\sigma) - \{u,v\}$, then it follows from Lemma 4.2 that v does not lie on any cycle of (G,σ) and (G_0,σ) is also upper-optimal. In view of Lemma 4.1, we know that every connected components of (G_0,σ) is upper-optimal. Applying induction hypothesis to every connected component of (G_0,σ) yields each of the following:
(a) the cycles (if any) of (G_0,σ) are pairwise vertex-disjoint;
(b) each cycle of (G_0,σ) is odd;
(c) $m(T_{G_0}) = m([T_{G_0}])$.

By (a) and (b), one has that the cycles (if any) of (G,σ) are pairwise vertex disjoint and each cycle of (G,σ) is odd since all cycles of (G,σ) belong to (G_0,σ) in this case. Hence,

(i) and (ii) hold in this case. Moreover, it can be checked that u is also a pendant vertex of T_G (resp., $[T_G]$) which adjacent to v and $T_{G_0} = T_G - \{u, v\}$ (resp., $[T_{G_0}] = [T_G] - \{u, v\}$). Thus, by Lemma 2.7 and assertion (c), one has

$$m(T_G) = m(T_{G_0}) + 1 = m([T_{G_0}]) + 1 = m([T_G]).$$

This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

Recall that a signed graph (G, σ) is *lower-optimal* if $r(G, \sigma) = 2m(G) - 2c(G)$, or equivalently, the signed graph which attain the lower bound in Theorem 1.1. In this section, we introduce some lemmas firstly, and then we give the proof for Theorem 1.3. The following Lemma 5.1 is simple and useful. Its proof is rather similar as Lemma 4.1, which is omitted here.

Lemma 5.1. *Let (G, σ) be a signed graph with connected components $(G_1, \sigma), (G_2, \sigma), \cdots, (G_k, \sigma)$. Then (G, σ) is lower-optimal if and only if (G_j, σ) is lower-optimal for each $j \in \{1, 2, \cdots, k\}$.*

Lemma 5.2. *Let (G, σ) be a connected signed graph with $m(T_G) = m([T_G])$. Then every vertex lying on a signed cycle can not be a quasi-pendant vertex of (G, σ).*

Proof. Suppose to the contrary that there exists a quasi-pendant vertex u lying on a signed cycle of (G, σ). Let v be the pendant vertex which is adjacent to u and M be a maximum matching of $[T_G]$. It follows by definition that $M \cup \{uv\}$ is also a matching of T_G. Thus, $m(T_G) \geq m([T_G]) + 1$. This contradiction proves the lemma. □

Lemma 5.3. *Let (G, σ) be a connected signed unicyclic graph whose unique signed cycle is C_l. Then the following are equivalent:*

(i) $r(G, \sigma) = 2m(G) - 2$*;*

(ii) *Either* $l \equiv 0 \pmod 4$ *and* $\sigma(C_l) = +$*, and* $m(T_G) = m([T_G])$ *or* $l \equiv 2 \pmod 4$ *and* $\sigma(C_l) = -$*, and* $m(T_G) = m([T_G])$*.*

Proof. ((ii) ⇒ (i).) We prove $r(G, \sigma) = 2m(G) - 2$ by induction on the order of T_G. If $|V(T_G)| = 1$, then $(G, \sigma) \cong (C_l)$. Since either $l \equiv 0 \pmod 4$ and $\sigma(C_l) = +$ or $l \equiv 2 \pmod 4$ and $\sigma(C_l) = -$, by Lemma 2.12, we have $r(G, \sigma) = 2m(G) - 2$.

Next, one can suppose that $|V(T_G)| \geq 2$. Then there is a pendant vertex u of T_G which is also a pendant vertex of (G, σ). Let v be the unique neighbour of u. By Lemma 5.2, v is not on any cycle of (G, σ). Let $(G_0, \sigma) = (G, \sigma) - \{u, v\}$ and $(G_1, \sigma), (G_2, \sigma), \cdots, (G_k, \sigma)$ be all the connected components of (G_0, σ). Then it follows from Lemmas 2.2 and 2.7 that

$$r(G, \sigma) = r(G_0, \sigma) + 2, m(G) = m(G_0) + 1.$$

Without loss of generality, assume that (G_1, σ) is the connected component which contains the unique cycle (C_l, σ). Then (G_j, σ) is a tree for each $j \in \{2, 3, \cdots, k\}$. Since

$$T_{G_0} = T_G - \{u, v\}, [T_{G_0}] = [T_G] - \{u, v\},$$

one has $m(T_{G_0}) = m([T_{G_0}])$. Note that $m(T_{G_0}) = m([T_{G_0}])$ implies $m(T_{G_j}) = m([T_{G_j}])$ for each $j \in \{1, 2, \cdots, k\}$. Since (G_1, σ) is a connected unicyclic graph and $|V(T_{G_1})| < |V(T_G)|$, by induction hypothesis, one has

$$r(G_1) = 2m(G_1) - 2.$$

By Lemma 2.4, one has that $r(G_j, \sigma) = 2m(G_j)$ for each $j \in \{2, 3, \cdots, k\}$. Note that $r(G_0, \sigma) = \sum_{j=1}^{k} r(G_j, \sigma)$ and $m(G_0) = \sum_{j=1}^{k} m(G_j)$. Then we have

$$\begin{aligned} r(G, \sigma) &= r(G_0, \sigma) + 2 \\ &= r(G_1, \sigma) + \sum_{j=2}^{k} r(G_j, \sigma) + 2 \\ &= 2m(G_1) - 2 + 2\sum_{j=2}^{k} m(G_j) + 2 \\ &= 2m(G_0) \\ &= 2m(G) - 2. \end{aligned}$$

((i) ⇒ (ii).) Let $P_{(G,\sigma)}(\lambda) = |\lambda I_n - A(G, \sigma)| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ be the characteristic polynomial of $A(G, \sigma)$ and $m = m(G)$. By Lemma 2.11, we obtain

$$\begin{aligned} a_{2m} &= \sum_{(U,\sigma)\in\mathscr{U}'_{2m}} (-1)^{p(U)+s(U)} \cdot 2^1 + \sum_{M\in\mathbb{M}} (-1)^m \\ &= \sum_{(U,\sigma)\in\mathscr{U}'_{2m}} (-1)^{1+\frac{2m-l}{2}+s(U)} \cdot 2^1 + \sum_{M\in\mathbb{M}} (-1)^m \\ &= (-1)^m\{|\mathbb{M}| + 2|\mathscr{U}'_{2m}|(-1)^{\frac{l-2}{2}+s(U)}\}, \end{aligned}$$

where $\mathscr{U}'_{2m}$ denotes the set of all basic subgraphs of order $2m$ which contains C_l as its connected component, $\mathbb{M}$ is the set of all maximum matching of (G, σ), $p(U)$ and $s(U)$ are the number of components and the number of negative edges in the cycle of (U, σ), respectively.

By the condition $r(G, \sigma) = 2m - 2$, one has that $a_{2m} = 0$. So, $|\mathbb{M}| = 2|(U_{2m}, \sigma)|$ and $\frac{l-2}{2} + s(U)$ is odd, which implies that either $l \equiv 0 \pmod 4$ and $\sigma(C_l) = +$ or $l \equiv 2 \pmod 4$ and $\sigma(C_l) = -$.

We show $m(T_G) = m([T_G])$ by induction on $|V(T_G)|$. Since (G, σ) contains cycle, $|V(T_G)| > 0$. If $|V(T_G)| = 1$, then $(G, \sigma) \cong C_l$ and $m(T_G) = m([T_G]) = 0$. Next, one can assume that $|V(T_G)| \geq 2$. Then there exists a pendant vertex u of T_G which is also a pendant vertex of (G,σ). Let v be the unique neighbour of u. By Corollary 3.1, we have v is not on any cycle of (G,σ). Denote $(G',\sigma) = (G,\sigma) - \{u,v\}$. Let $(G'_1,\sigma),(G'_2,\sigma),\cdots,(G'_k,\sigma)$ be all connected components of (G',σ). Without loss of generality, assume that (G'_1,σ) contains the unique cycle (C_l,σ). It is obviously that $c(G) = c(G') = 1$. By Lemmas 2.2 and 2.7, it can be checked that

$$r(G',\sigma) = 2m(G') - 2.$$

By the induction hypothesis, one has $m(T_{G'_1}) = m([T_{G'_1}])$. Since (G'_j,σ) is acyclic signed graph, $T_{G'_j} \cong [T_{G'_j}]$ for each $j \in \{2,3,\cdots,k\}$. By Lemma 2.7, we have

$$\begin{aligned} m(T_{(G,\sigma)}) &= m(T_{G'}) + 1 \\ &= \sum_{j=1}^{k} m(T_{G'_j}) + 1 \\ &= \sum_{j=1}^{k} m([T_{G'_j}]) + 1 \\ &= m([T_{G'}]) + 1 \\ &= m([T_G]). \end{aligned}$$

The result follows. □

Lemma 5.4. *Let (G,σ) be a signed graph which contains a pendant vertex u with its unique neighbour v. Let $(G',\sigma) = (G,\sigma) - \{u,v\}$. If (G,σ) is lower-optimal, then (G',σ) is also lower-optimal.*

Proof. By Lemmas 2.2 and 2.7, we have

$$r(G',\sigma) = r(G,\sigma) - 2, m(G') = m(G) - 1.$$

By Lemma 2.9, one has that $c(G) = c(G')$. Since (G,σ) is lower-optimal, it follows by definition that $r(G',\sigma) = 2m(G') - 2c(G')$, and so (G',σ) is also lower-optimal. □

Lemma 5.5. *Let (G,σ) be a signed graph obtained by joining a vertex x of a signed cycle O by a signed edge to a vertex y of a signed connected graph K. If (G,σ) is lower-optimal, then the following properties hold for (G,σ).*

(i) *Every signed cycle C_l of (G,σ) is either $l \equiv 0 \pmod 4$ and $\sigma(C_l) = +$ or $l \equiv 2 \pmod 4$ and $\sigma(C_l) = -$;*

(ii) *The edge xy does not belong to any maximum matching of (G,σ);*

(iii) *Each maximum matching of K covers y;*

(iv) *$m(K + x) = m(K)$;*

(v) *(K, σ) is lower-optimal;*

(vi) *Let (G', σ) be the induced signed subgraph of (G, σ) with vertex set $V(K) \cup \{x\}$. Then (G', σ) is also lower-optimal.*

Proof. (i): We show (i) by induction on $c(G)$. If $c(G) = 1$, then (G, σ) is a signed unicyclic graph. By Lemma 5.3, (i) holds immediately. Next, one can suppose that $c(G) \geq 2$, then (K, σ) contains at least one cycle. Let u be a vertex lying on some cycle of (K, σ). Suppose (G_0, σ) is the connected component of $(G, \sigma) - u$ which contains O. By Corollary 3.1 and Lemma 2.9, we have (G_0, σ) is lower-optimal and $c(G_0) < c(G)$. By induction hypothesis, one has that each cycle in G_0, including O, satisfies (i). By a similar discussion as for $(G, \sigma) - x$, we can show that all the cycles in K satisfy (i). This completes the proof of (i).

(ii): Suppose to the contrary that there is a maximum matching M of (G, σ) containing xy. By (i), one has that O is even. Then there exists a vertex $w \in V(O)$ such that w is not covered by M. Then we have $m(G) = m(G - w)$, a contradiction to Corollary 3.1.

(iii): By Lemma 2.6, we have $m(G) = m(K) + m(O)$. Then $M_1 \cup M_2$ is a maximum matching of (G, σ) where M_1 and M_2 are maximum matchings of O and K, respectively. Suppose to the contrary that each maximum matching of K fails to cover y. Then we obtain a maximum matching $M_1' \cup M_2$ of (G, σ) which contains xy, where M_1' is obtained from M_1 by replacing the edge in M_1 which covers x with xy, a contradiction to (ii).

(iv): It is immediately follows from (iii).

(v): By Corollary 3.1, $(G, \sigma) - x$ is lower optimal. Then (v) immediately follows from Lemma 5.1.

(vi): Suppose that $O = xx_2x_3 \cdots x_{2s}x$. Since (G, σ) is lower-optimal, by Corollary 3.1, one has that $(G_1, \sigma) = (G, \sigma) - x_2$ is also lower-optimal. Obviously, x_3 and x_4 are pendant vertex and quasi-pendant vertex of (G_1, σ), respectively. By Lemma 5.4, one has that $(G_2, \sigma) = (G_1, \sigma) - \{x_3, x_4\}$ is also lower-optimal. Repeating such process (deleting a pendant vertex and a quasi-pendant vertex), after $s - 1$ steps, the result graph is $(G, \sigma) - \{x_2, x_3, \cdots, x_{2s}\} = (G', \sigma)$. By Lemma 5.4, (G', σ) is also lower-optimal. □

Lemma 5.6. *Let (G, σ) be a connected signed graph. If (G, σ) is lower-optimal, then there exists a maximum matching M of (G, σ) such that $M \cap \mathscr{F}(G) = \emptyset$, where $\mathscr{F}(G)$ denotes the set of edges of (G, σ) that each of which has one endpoint in a cycle and the other endpoint outside the cycle.*

Proof. We argue by induction on the order of T_G. If $|V(T_G)| = 1$, then (G, σ) is either a signed cycle or an isolated vertex and the conclusion holds trivially. Hence we that $|V(T_G)| \geq 2$, and so T_G has at least one pendant vertex, say u.

If u does not lie on any signed cycle of (G, σ), then u is also a pendant vertex of (G, σ). Let v be the unique neighbour of u in (G, σ) and $(G_0, \sigma) = (G, \sigma) - \{u, v\}$. By Corollary 3.1 and Lemma 5.4, one has that v is not on any signed cycle of (G, σ) and

(G_0, σ) is also lower-optimal. Let $(G_1, \sigma), (G_2, \sigma), \cdots, (G_k, \sigma)$ be all connected components of (G_0, σ). Then it follows from Lemma 5.1 that (G_j, σ) is lower-optimal for each $j \in \{1, 2 \cdots, k\}$. Applying induction hypothesis to (G_j, σ) yields that there exists a maximum matching M_j of G_j such that $M_j \cap \mathscr{F}(G) = \emptyset$ for each $j \in \{1, 2 \cdots, k\}$. Let $M = \cup_{j=1}^{k} M_j \cup \{uv\}$. Then it can be checked that M is a maximum matching of (G, σ) which satisfies $M \cap \mathscr{F}(G) = \emptyset$.

If u lies on some signed cycle of (G, σ), then (G, σ) has a pendant cycle, say C. Let $(K, \sigma) = (G, \sigma) - (C, \sigma)$. By Lemma 5.5, each cycle of (G, σ) is even and (K, σ) is lower-optimal. Applying the induction hypothesis to (K, σ) implies that there exists a maximum matching M_0 of (K, σ) such that $M_0 \cap \mathscr{F}(G) = \emptyset$. Let M_0' be a maximum matching of (C, σ). By Lemma 2.6, it is routine to verify that $M = M_0 \cup M_0'$ is a maximum matching of C satisfying $M \cap \mathscr{F}(G) = \emptyset$. This completes the proof. □

The proof of Theorem 1.3. (Sufficiency.) We proceed by induction on the order of T_G. If $|V(T_G)| = 1$, then (G, σ) is either a signed cycle or an isolated vertex and the conclusion holds by Lemma 2.12.

Therefore we assume that $|V(T_G)| \geq 2$. Since $m(T_G) = m([T_G])$, by Lemma 2.4, one has that $r(T_G) = r([T_G])$. By Lemma 2.15, (G, σ) has at least one pendant vertex, say u. Let v be the unique neighbour of u in (G, σ). By Lemma 5.2, v does not lie on any cycle of (G, σ). Let $(G_0, \sigma) = (G, \sigma) - \{u, v\}$ and $(G_1, \sigma), (G_2, \sigma), \cdots, (G_k, \sigma)$ be all connected components of (G_0, σ). Then, one has that

$$c(G) = c(G_0) = \sum_{j=1}^{k} c(G_j), m(G_0) = \sum_{j=1}^{k} m(G_j).$$

By Lemma 2.7, we have

$$m(G) = m(G_0) + 1 = \sum_{j=1}^{k} m(G_j) + 1.$$

As v does not lie on any cycle of (G, σ), it follows that v is also a vertex of T_G (resp. $[T_G]$) which is adjacent to u and $T_{G'} = T_G - \{u, v\}$ (resp., $[T_{G'}] = [T_G] - \{u, v\}$). Hence, we have

$$m(T_G) = \sum_{j=1}^{k} m(T_{G_j}) + 1, m([T_G]) = \sum_{j=1}^{k} m([T_{G_j}]) + 1.$$

If there exists some $j \in \{1, 2, \cdots, k\}$ such that $m(T_{G_j}) > m([T_{G_j}])$. Then we have $m(T_G) > m([T_G])$, a contradiction to (iii). Thus, one has that $m(T_{G_j}) = m([T_{G_j}])$ for each $j \in \{1, 2, \cdots, k\}$. Therefore, each (G_j, σ) satisfies (i)-(iii) for each $j \in \{1, 2, \cdots, k\}$.

Applying the induction hypothesis to (G_j, σ) yields that for each $j \in \{1, 2, \cdots, k\}$, we have

$$r(G_j, \sigma) = 2m(G_j) - 2c(G_j).$$

By Lemma 2.2, one has that

$$\begin{aligned} r(G, \sigma) &= r(G_0, \sigma) + 2 \\ &= \sum_{j=1}^{k} r(G_j, \sigma) + 2 \\ &= \sum_{j=1}^{k} 2m(G_j) - 2\sum_{j=1}^{k} c(G_j) + 2 \\ &= 2m(G_0) - 2c(G_0) + 2 \\ &= 2m(G) - 2c(G). \end{aligned}$$

(Necessity.) Let (G, σ) be a lower-optimal signed graph. If (G, σ) is a signed acyclic graph, then (i)-(iii) holds directly. So one can suppose that (G, σ) contains cycles. By Corollary 3.1, one has that the cycles (if any) of (G, σ) are pairwise vertex-disjoint. This completes the proof of (i).

Next, we show (ii) and (iii) by induction on the order n of (G, σ). If $n = 1$, then (ii) and (iii) hold trivially. Suppose that (ii) and (iii) hold for any lower-optimal signed graph of order smaller than n, and suppose (G, σ) is a lower-optimal signed graph with order $n \geq 2$. If $|V(T_G)| = 1$, then (G, σ) is either a signed cycle or an isolated vertex. Thus (ii) follows from Lemma 2.12 and (iii) follows from the fact that $m(T_G) = m([T_G]) = 0$. So, one can suppose that $|V(T_G)| \geq 2$, then T_G has at least one pendant vertex, say u. Therefore, it suffices to consider the following two possible cases.

Case 1. u is a pendant vertex of (G, σ).

Let v be the adjacent vertex of u and $(G', \sigma) = (G, \sigma) - \{u, v\}$. By Corollary 3.1 and Lemma 5.4, v does not lie on any cycle of (G, σ) and (G', σ) is also lower-optimal. Then it follows from Lemma 5.1 that every connected component of (G', σ) is lower-optimal. Applying the induction hypothesis to each connected component of (G', σ) yields:

(a) each cycle C_q of (G', σ) is either $q \equiv 0 \pmod 4$ and $\sigma(C_q) = +$ or $q \equiv 2 \pmod 4$ and $\sigma(C_q) = -$;

(b) $m(T_{G'}) = m([T_{G'}])$.

Assertion (a) implies that each cycle (if any) C_q of (G, σ) either is $q \equiv 0 \pmod 4$ and $\sigma(C_q) = +$ or $q \equiv 2 \pmod 4$ and $\sigma(C_q) = -$ since all cycles of (G, σ) belong to (G', σ) in this case. Hence, (ii) holds in this case. Note that u is also a pendant vertex of T_G (resp., $[T_G]$) which is adjacent to v and $T_{G'} = T_G - \{u, v\}$ and (resp., $[T_{G'}] = [T_G] - \{u, v\}$). By Lemma 2.7 and (b), one has that

$$m(T_G) = m(T_{G'}) + 1 = m([T_{G'}]) + 1 = m([T_{G'}]).$$

Thus (iii) holds in this case.

Case 2. u lies on some pendant signed cycle of (G, σ).

Let $C_1', C_2', \cdots, C_k'$ be all cycles of (G, σ), where C_1' is a pendant signed cycle. Then (ii) immediately follows from Lemma 5.5.

So it suffices to show that $m(T_G) = m([T_G])$. Let u be the unique vertex of C_1' of degree 3 and $(G_1, \sigma) = (G, \sigma) - C_1'$ and $(G_2, \sigma) = (G_1, \sigma) + x$. By Lemma 5.6, (G, σ) has a maximum matching M such that $M \cap \mathscr{F}(G) = \emptyset$. Consequently,

$$m(G) = m([T_G]) + \frac{\sum_{j=1}^{k} |V(C_j')|}{2}.$$

By Lemma 5.5, (G_2, σ) is lower-optimal. Hence, (G_2, σ) has a maximum matching M_2 such that $M_2 \cap \mathscr{F}(G_2) = \emptyset$, from which it follows that

$$m(G_2) = m([T_{G_2}]) + \frac{\sum_{j=2}^{k} |V(C_j')|}{2}.$$

By induction hypothesis for (G_2, σ), one has $m(T_{G_2}) = m([T_{G_2}])$. Moreover, by Lemma 5.5, we have $m(G_1) = m(G_2)$. Note that $T_G \cong T_{G_2}$. By Lemma 2.6, one has that

$$m(G) = m(C_1') + m(G_1).$$

Thus,

$$\begin{aligned}
m(T_G) &= m(T_{G_2}) \\
&= m([T_{G_2}]) \\
&= m(G_2) - \frac{\sum_{j=2}^{k} |V(C_j')|}{2} \\
&= m(G_1) - \frac{\sum_{j=2}^{k} |V(C_j')|}{2} \\
&= m(G_1) + \frac{|V(C_1')|}{2} - \frac{\sum_{j=1}^{k} |V(C_j')|}{2} \\
&= m(G) - m(C_1') + \frac{|V(C_1')|}{2} - \frac{\sum_{j=1}^{k} |V(C_j')|}{2} \\
&= m(G) - \frac{\sum_{j=1}^{k} |V(C_j')|}{2} \\
&= m([T_G]).
\end{aligned}$$

This completes the proof.

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