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Spectral results on Hamiltonian problem

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ABSTRACT

Let α be a non-negative real number, and let $\Theta(G,\alpha)$ be the largest eigenvalue of $A(G)+\alpha D(G)$. Specially, $\Theta(G,0)$ and $\Theta(G,1)$ are called the spectral radius and signless Laplacian spectral radius of G, respectively. A graph G is said to be Hamiltonian (traceable) if it contains a Hamiltonian cycle (path), and a graph G is called Hamilton-connected if any two vertices are connected by a Hamiltonian path in G. The number of edges of G is denoted by $e(G)$. Recently, the (signless Laplacian) spectral property of Hamiltonian (traceable, Hamilton-connected) graphs received much attention. In this paper, we shall give a general result for all these existed results. To do this, we first generalize the concept of Hamiltonian, traceable, and Hamilton-connected to s-suitable, and we secondly present a lower bound for $e(G)$ to confirm the existence of s-suitable graphs. Thirdly, when $0\le\alpha\le 1$, we obtain a lower bound for $\Theta(G,\alpha)$ to confirm the existence of s-suitable graphs. Consequently, our results generalize and improve all these existed results in this field, including the main results of Chen et al. (2018), Feng et al. (2017), Füredi et al. (2017), Ge et al. (2016), Li et al. (2016), Nikiforov et al. (2016), Wei et al. (2019) and Yu et al. (2013, 2014).

1. Introduction

In this paper, we only consider simple connected undirected graph, and $G=(V,E)$ is a connected graph with n vertices and $e(G)$ edges. Let $N_G(u)$ and $d_G(u)$ be the neighbor set and the degree of vertex u, respectively. Let $\delta(G)$ and $\Delta(G)$ be the minimum degree and maximum degree of G, respectively. If there is no rise of confusion, we always simplify $N_G(u)$ and $d_G(u)$ as $N(u)$ and $d(u)$, respectively. As usual, K_n, C_n, P_n and $K_{1,n-1}$ denote the complete graph, cycle, path and star graph with n vertices, respectively, and $G_1\vee G_2$ denotes the *join graph* of two vertex disjoint graphs G_1 and G_2. In other words, $G_1\vee G_2$ is the graph having vertex set $V\big(G_1\vee G_2\big)=V\big(G_1\big)\cup V\big(G_2\big)$ and edge set $E\big(G_1\vee G_2\big)=E\big(G_1\big)\cup E\big(G_2\big)\cup\Big\{uv: u\in V\big(G_1\big), v\in V\big(G_2\big)\Big\}$. When t is a positive integer, then tK_1 denotes the set of t isolated vertices.

When n, s and k are three integers such that $\max\{0,s\}\le k\le\frac{1}{2}(n+s-2)$, we define the graphs $M_n^{k,s}$ and $N_n^{k,s}$ with n vertices and minimum degree k as follows:

$$N_n^{k,s}\cong K_k\vee\Big(K_{n+s-2k-1}\cup(k+1-s)K_1\Big),\quad\text{and}$$
$$M_n^{k,s}\cong\begin{cases}K_s\vee\big(K_{n-k-1}\cup K_{k+1-s}\big) & \text{for } s>0,\\ K_{n-k-1}\cup K_{k+1} & \text{for } s=0.\end{cases}$$

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If $A(G)$ and $D(G)$, respectively, define the adjacency matrix and the diagonal matrix of G, then the *signless Laplacian matrix* of G is defined as $Q(G) = D(G) + A(G)$. Hereafter, let $\rho(G)$ and $\mu(G)$ be the largest eigenvalues of $A(G)$ and $Q(G)$, respectively, and we call $\rho(G)$ and $\mu(G)$ the spectral radius and signless Laplacian spectral radius of G, respectively. Throughout this paper, α defines a non-negative real number and let $\Theta(G, \alpha)$ be the largest eigenvalue of $A(G) + \alpha D(G)$. From the definition, it is easy to see that $\Theta(G, 0) = \rho(G)$, and $\Theta(G, 1) = \mu(G)$.

A cycle (path) of a graph G that contains every vertex of G is called a *Hamiltonian cycle* (*path*) of G. A graph G is said to be *Hamiltonian* (*traceable*) if it contains a Hamiltonian cycle (path), and a graph G is called *Hamilton-connected* if any two vertices are connected by a Hamiltonian path in G.

Recently, the (signless Laplacian) spectral properties of traceable graph, Hamiltonian graph and Hamilton-connected received more and more attention. For the spectral properties of traceable graph, Fiedler et al. [7] firstly proved that: For a graph G with n vertices, if $\rho(G) \geq n-2$, then G is traceable unless $G \cong M^{0,0}$. In 2016, Li et al. [11] generalized Fiedler's result to: For a graph G with $n \geq \max\left\{6k+10,\ \frac{1}{2}(k^2+7k+8)\right\}$ vertices and minimum degree $\delta(G) \geq k \geq 0$, if $\rho(G) \geq \rho\left(N_n^{k,0}\right)$, then G is traceable unless $G \cong N_n^{k,0}$. Soon later, Nikiforov [14] improved Li's result to:

Theorem 1.1 ([*14*]). *Let G be a graph with $n \geq k^3+k^2+2k+5$ vertices and minimum degree $\delta(G) \geq k \geq 1$. If $\rho(G) \geq n-k-2$, then G is traceable unless $G \in \left\{N_n^{k,0},\ M_n^{k,0}\right\}$.*

For the spectral properties of Hamiltonian graphs, Fiedler et al. [7] firstly proved that: For a graph G with n vertices, if $\rho(G) > n-2$, then G is Hamiltonian unless $G \cong M_n^{1,1}$. In 2016, Li et al. [11] generalized Fiedler's result to: For a graph G with $n \geq \max\left\{6k+5,\ \frac{1}{2}\left(k^2+6k+4\right)\right\}$ vertices and minimum degree $\delta(G) \geq k \geq 1$, if $\rho(G) \geq \rho\left(N_n^{k,1}\right)$, then G is Hamiltonian unless $G \cong N_n^{k,1}$. Soon later, Nikiforov [14] improved Li's result to:

Theorem 1.2 ([*14*]). *Let G be a graph with $n \geq k^3+k+4$ vertices and minimum degree $\delta(G) \geq k \geq 1$. If $\rho(G) \geq n-k-1$, then G is Hamiltonian unless $G \in \left\{N_n^{k,1},\ M_n^{k,1}\right\}$.*

Very recently, Ge et al. [9] improved Nikiforov's result to $n \geq \max\left\{\frac{1}{2}\left(k^3+2k+5\right),\ 6k+5\right\}$. Researchers also concerned with the (signless Laplacian) spectral properties of Hamilton-connected graphs. In this line, Yu et al. [17] proved that: For a graph G with n vertices, if either $\rho(G) > \frac{1}{2}\left(\sqrt{4n^2-12n+17}-1\right)$ or $\mu(G) > 2n-4+\frac{2}{n-1}$, then G is Hamilton-connected unless $G \cong M_n^{2,2}$. In 2017, Zhou et al. [20] proved that: For a graph G with minimum degree $\delta(G) \geq 3$ and $n \geq 9$ vertices, if either $\rho(G) \geq \sqrt{n^2-6n+19}$ or $\mu(G) \geq 2n-6+\frac{14}{n-1}$, then G is Hamilton-connected. Later, Yu et al. [18] generalized the spectral radius version of Zhou's results to: For a graph G with $n \geq 2k^2+1$ vertices and minimum degree $\delta(G) \geq k \geq 2$, if $\rho(G) \geq n-k$, then G is Hamilton-connected unless $G \cong M_n^{k,2}$. Recently, Chen et al. [3] and Yu et al. [19], independently, improved Yu's result of [18] by showing that: For a graph G with $n \geq \max\left\{6k^2-8k+5,\ \frac{1}{2}\left(k^3-k^2+4k-1\right)\right\}$ vertices and minimum degree $\delta(G) \geq k \geq 2$, if $\rho(G) \geq n-k$, then G is Hamilton-connected unless $G \in \left\{N_n^{k,2},\ M_n^{k,2}\right\}$. Soon later, Wei et al. [16] improved Chen's result to $n \geq \max\left\{\frac{1}{2}\left(k^3-k^2+2k+8\right),\ 6k\right\}$, that is,

Theorem 1.3 ([*16*]). *Let G be a graph with $n \geq \max\left\{\frac{1}{2}\left(k^3-k^2+2k+8\right),\ 6k\right\}$ vertices and minimum degree $\delta(G) \geq k \geq 3$. If $\rho(G) \geq n-k$, then G is Hamilton-connected unless $G \in \left\{N_n^{k,2},\ M_n^{k,2}\right\}$.*

For the signless Laplacian spectral properties of traceable graphs, Yu et al. [17] firstly proved that: For a graph G with n vertices, if $\mu(G) \geq 2(n-2)$, then G is traceable unless $G \cong N_n^{0,0}$. In 2016, Li et al. [11] generalized Yu's result to:

Theorem 1.4 ([*11*]). *Let G be a graph with $n \geq \max\left\{6k+10,\ \frac{1}{2}\left(3k^2+9k+8\right)\right\}$ vertices and minimum degree $\delta(G) \geq k \geq 0$. If $\mu(G) \geq \mu\left(N_n^{k,0}\right)$, then G is traceable unless $G \cong N_n^{k,0}$.*

For the signless Laplacian spectral properties of Hamiltonian graphs, Yu et al. [17] proved that: For a graph G with n vertices, if $\mu(G) > 2(n-2)$, then G is Hamiltonian unless $G \cong N_n^{1,1}$. In 2016, Li et al. [11] generalized Yu's result to:

Theorem 1.5 ([*11*]). *Let G be a graph with $n \geq \max\left\{6k+5,\ \frac{1}{2}\left(3k^2+5k+4\right)\right\}$ vertices and minimum degree $\delta(G) \geq k \geq 1$. If $\mu(G) \geq \mu\left(N_n^{k,1}\right)$, then G is Hamiltonian unless $G \cong N_n^{k,1}$.*

By comparing these results of Theorems 1.1–1.5, it is rather interesting for us to consider the following problem:

Problem 1.1. How can we generalize these results of Theorems 1.1–1.5?

To this aim, we need to introduce the concepts of *q-traceable* and *q-Hamiltonian*. For any non-negative integer q, a graph G with $n \geq 3$ vertices is called *q-traceable* if any removal of at most q vertices to G results in a traceable graph, while a graph G with $n \geq 3$ vertices is called *q-Hamiltonian* if any removal of at most q vertices to G results in a Hamiltonian graph. From the definitions, a q-Hamiltonian graph must be a $(q+1)$-traceable graph. However, a $(q+1)$-traceable graph is not necessarily a q-Hamiltonian graph. For instance, the Petersen graph is 1-traceable, but it is not 0-Hamiltonian.

Hereafter, we use $G[X]$ to denote the subgraph of G induced by X. It is easy to see that a traceable graph is also a 0-traceable graph, and a Hamiltonian graph is both a 0-Hamiltonian and a 1-traceable graph. If G is Hamilton-connected, then for any two vertices $\{u, v\}$ of G, there is a Hamiltonian path connecting u and v. Thus, $G\big[V(G)\backslash\{u, v\}\big]$ contains a Hamiltonian path and $G\big[V(G)\backslash\{u\}\big]$ also contains a Hamiltonian path, and hence G is 2-traceable.

A graph G is *q-edge-Hamiltonian* if any collection of vertex-disjoint paths with at most q edges altogether belongs to a Hamiltonian cycle in G. A connected graph G is said to be *q-connected* if it has more than q vertices and remains connected whenever fewer than q vertices are deleted. Similarly, G is *q-edge-connected* if it has at least two vertices and remains connected whenever fewer than q edges are deleted. A graph G is *q-path-coverable* if $V(G)$ can be covered by q or fewer vertex-disjoint paths.

Recently, Feng et al. [6] obtained the generalized result for spectral property of q-Hamiltonian (respectively, q-edge-Hamiltonian) graphs, that is

Theorem 1.6 ([6]). *For a graph G with $n \geq q+6 \geq 7$ vertices and minimum degree at least one, if $\rho(G) \geq \sqrt{(n-2)^2+2q+1}$, then G is q-edge-Hamiltonian and q-Hamiltonian.*

Except for this, Feng et al. [6] also obtained the generalized result for spectral property of q-connected (respectively, q-edge-connected, q-path-coverable) graphs, that is

Theorem 1.7 ([6]). (*i*) *For a graph G with $n \geq q+1 \geq 2$ vertices and minimum degree at least one, if $\rho(G) \geq \sqrt{n(n-4)+2q+1}$, then G is q-connected.* (*ii*) *For a graph G with $n \geq q+1 \geq 3$ vertices and minimum degree at least one, if $\rho(G) \geq \sqrt{n(n-q-5)+2q(q+1)+5}$, then G is q-edge-connected.* (*iii*) *For a graph G with $n \geq 5q+6 \geq 16$ vertices and minimum degree at least one, if $\rho(G) \geq \sqrt{(n-q-2)^2+q+1}$, then G is q-path-coverable.*

For a non-negative integer q, the *q-closure*, denoted by $C_q(G)$, of a graph G is the graph obtained from G by successively joining pairs of nonadjacent vertices whose degree sum is at least q until no such pair remains. It is easy to see that $G \subseteq C_q(G)$.

Let $\mathbb{G}_n$ be the class of graphs with n vertices and q be a non-negative integer. If G has the property P if and only if $C_q(G)$ has property P for each $G \in \mathbb{G}_n$, then the property P is said to be *q-stable* (here, the definition of *q-stable* is a little different from that in [1]).

By an observation to Theorem 1.6, one can easily find that Feng et al. cannot solve Problem 1.1 completely, as Theorems 1.1–1.3 are not a special case of Theorem 1.6. Therefore, positive answer to Problem 1.1 is also valuable, and our main goal of this paper is to solve it. To do this, via analyzing all these referred former results, we find that the key tool to prove them relies on the following k-stable property:

Proposition 1.1 ([1]). *The following stability results hold for graphs with n vertices:*
(*i*) *The property that "G is q-connected" is $(n+q-2)$-stable.*
(*ii*) *The property that "G is q-edge-connected" is $(n+q-2)$-stable.*
(*iii*) *The property that "G is q-Hamiltonian" is $(n+q)$-stable.*
(*iv*) *The property that "G is q-edge-Hamiltonian" is $(n+q)$-stable.*
(*v*) *The property that "G is q-path-coverable" is $(n-q)$-stable.*

In what follows, we shall give the spectral property to $(n+s-1)$-stable property for graphs with n vertices, which will improve and generalize all these results in Theorems 1.1–1.7. To do this, we need more notations in the following:

Definition 1.1. Let n, s, p and k be four integers such that $\max\{0, s\} \leq p \leq k \leq \frac{1}{2}(n+s-2)$. If $p \geq \max\big\{s, 0\big\}+1$, then $\mathbb{G}_n(p, s, k) = \Big\{G : G \cong K_p \vee \big(K_{n+s-k-1-p} \cup H_0\big)$, where H_0 is a $(k-p)$-regular graph with $k+1-s$ vertices$\Big\}$. If $p = s \geq 0$, then $\mathbb{G}_n(p, s, k) = \big\{M_n^{k,s}\big\}$. If $p = 0 > s$, then $\mathbb{G}_n(p, s, k) = \Big\{K_{n+s-k-1} \cup H_0,$ where H_0 is a k-regular graph with $k+1-s$ vertices$\Big\}$.

Here, we need to point out the fact that s being negative is also permitted in Definition 1.1 (see Remark 1.2). From Definition 1.1, it is easy to see that $\mathbb{G}_n(k, s, k) = \big\{N_n^{k,s}\big\}$ and $\mathbb{G}_n(p, s, k)$ are graphs with n vertices and minimum degree $k \geq 0$. Hereafter, if $G \in \mathbb{G}_n(p, s, k)$ and $\max\{s, 1\} \leq p \leq k$, we let $V_1(G)$, $V_2(G)$ and $V_3(G)$ be the vertex sets corresponding to K_p, $K_{n+s-k-1-p}$ and H_0 of G, respectively. Especially, when $k \geq 1$ and $n+s-2k \geq 3$, we define $N_{n,0}^{k,s}$ as the graph obtained from $N_n^{k,s}$ by deleting one edge with two end vertices in $V_2\left(N_n^{k,s}\right)$, namely, in $K_{n+s-k-1-p}$. In what follows, let

$$\Theta_0 = \alpha\left(\frac{2\varepsilon_0}{n-1}+n-2\right)+\frac{1}{2}(1-\alpha)\left(k-1+\sqrt{(k+1)^2+8\varepsilon_0-4nk}\right),$$

where

$$\varepsilon_0 = \binom{n-k-2+s}{2}+(k+1)(k+2-s).$$

Now, we are ready to give the main results of this paper:

Theorem 1.8. *Let s and k be two integers and let G be a graph with* $n \geq 6k + 10 - 5s$ *vertices and minimum degree* $\delta(G) \geq k \geq \max\{1, s\}$. *If* $\Theta(G, \alpha) > \Theta_0$ *and* $0 \leq \alpha \leq 1$, *then either* $C_{n+s-1}(G) \cong K_n$ *or* $C_{n+s-1}(G) \in \mathbb{G}_n(p, s, k)$ *holds for some integer p, where* $\max\{0, s\} \leq p \leq k$.

Theorem 1.9. *Let s and k be two integers and let G be a graph with* $n \geq \max\Big\{6k + 10 - 5s,\ \frac{1}{4}\big(4(k-s)(k+3) + 10k + 25\big),\ \frac{1}{3}\big(2(k-s)(3k-s+5) + 8k + 15\big)\Big\}$ *vertices and minimum degree* $\delta(G) \geq k \geq \max\{1, s\}$. *If* $\Theta(G, \alpha) \geq \Theta\left(N_{n,0}^{k,s}, \alpha\right)$ *and* $0 \leq \alpha \leq 1$, *then unless* $G \cong N_{n,0}^{k,s}$, *either* $C_{n+s-1}(G) \cong K_n$ *or* $G \in \mathbb{G}_n(p, s, k)$ *holds for some integer p, where* $\max\{0, s\} \leq p \leq k$.

Now, one can easily see that our Theorems 1.8 and 1.9 give the spectral property to $(n + s - 1)$-stable property for graphs with n vertices. In [1,2], many q-stable properties had been given. Thus, we can apply Theorem 1.8 or Theorem 1.9 to give a new spectral property to these q-stable properties.

By comparing Theorems 1.8 and 1.9, it is an interesting question that: How large is $\Theta\left(N_{n,0}^{k,s}, \alpha\right)$ or Θ_0. Actually, it is easy to see that $\Theta\left(N_{n,0}^{k,s}, \alpha\right) > \Theta_0$ (see (4.8) in the proof of Theorem 1.9). Furthermore, for $\Theta\left(N_{n,0}^{k,s}, \alpha\right)$, we have

Proposition 1.2. *Let k, s and n be three integers such that* $\max\{1, s\} \leq k \leq \frac{1}{6}(n + 5s - 10)$. *If* $n \geq \max\Big\{\frac{1}{2}\big((k^2+4)(k+1-s)+2\big),\ \frac{1}{2}\big(k^2(k+1)+6\big)\Big\}$, *then* $\rho\left(N_{n,0}^{k,s}\right) < n + s - k - 2$ *and*

$$\mu\left(N_{n,0}^{k,s}\right) < \frac{2n^2 - 2(k+3-s)n + (k-s+1)(3k-s+2)}{n-1}. \tag{1.1}$$

Remark 1.1. Note that K_n is both q-connected and q-edge-connected, $\sqrt{n(n-4)+2q+1} \geq n + q - \delta(G) - 3$ and $\sqrt{n(n-q-5)+2q(q+1)+5} \geq n + q - \delta(G) - 3$ hold for large n and $3q \leq 2\delta(G)$. Thus, by Propositions 1.2 and 1.1 (*i*) and (*ii*), it is easy to see that Theorem 1.9 improves Theorem 1.7 (*i*) and (*ii*) for large n and $3q \leq 2\delta(G)$ by setting $s = q - 1$.

Remark 1.2. Note that K_n is q-path-coverable. Thus, by Propositions 1.2 and 1.1 (*v*), it is easy to see that Theorem 1.9 improves Theorem 1.7 (*iii*) for large n and $\delta(G) \geq 2$ by setting $s = 1 - q$ (here, $s = 1 - q$ may be negative).

Remark 1.3. Note that K_n is both q-edge-Hamiltonian and q-Hamiltonian, and $\delta(G) \geq q + 2$ holds for any q-Hamiltonian graph G. Thus, by Proposition 1.2 and Proposition 1.1 (*iii*) and (*iv*), it is easy to see that Theorem 1.9 improves Theorem 1.6 for large n and $\delta(G) \geq q + 2$ by setting $s = q + 1$.

As referred before, a traceable graph is also a 0-traceable graph, a Hamiltonian graph is both a 0-Hamiltonian graph and a 1-traceable graph, and a Hamilton-connected graph is also a 2-traceable graph. For any graph G with n vertices, it is well-known that [15]: G is traceable if and only if $C_{n-1}(G)$ is traceable. For the general case of q-traceable, we have

Proposition 1.3. *If* $q \geq 0$ *and G is a graph with n vertices, then G is q-traceable if and only if* $C_{n+q-1}(G)$ *is q-traceable.*

Note that K_n is traceable, Hamiltonian and Hamilton-connected. By setting $s = 0, 1, 2$ in Theorem 1.9 and Proposition 1.2, we have: If n is large enough and $\rho(G) \geq n + s - k - 2$, then G is traceable (respectively, Hamiltonian Hamilton-connected) unless $G \in \mathbb{G}_n(p, s, k)$ for some integer p, where $s \leq p \leq k$. However, this is somewhat different from that in Theorems 1.1–1.3. Actually, in [3,11,16], the authors had shown that:

Proposition 1.4 (*[3,11,16]*)**.** *For* $s = 0$ *(respectively,* $s = 1, 2$*), if* $G \in \mathbb{G}_n(p, s, k)$ *for some integer p, where* $s \leq p \leq k \leq \frac{n+s-2}{2}$, *then G is traceable (respectively, Hamiltonian, Hamilton-connected) if and only if* $p \in \{s+1, s+2, \ldots, k-1\}$.

Therefore, to generalize the corresponding results of Theorems 1.1–1.5, we need to generalize Proposition 1.4 to

Proposition 1.5. *Suppose that* $q \geq 0$. (*i*) *If* $q \leq p \leq k \leq \frac{1}{2}(n+q-2)$ *and* $G \in \mathbb{G}_n(p, q, k)$, *then G is q-traceable if and only if* $p \in \{q+1, q+2, \ldots, k-1\}$. (*ii*) *If* $q+1 \leq p \leq k \leq \frac{1}{2}(n+q-1)$ *and* $G \in \mathbb{G}_n(p, q+1, k)$, *then G is q-Hamiltonian if and only if* $p \in \{q+2, q+3, \ldots, k-1\}$.

By combining Propositions 1.1 (*iii*) and 1.3, the property that "G is q-Hamiltonian (respectively, q-traceable)" is $(n+q)$-stable (respectively, $(n+q-1)$-stable). However, Proposition 1.5 and Theorem 1.9 show that: for q-Hamiltonian (respectively, q-traceable) graphs, except for $(n+q)$-stable (respectively, $(n+q-1)$-stable) property, they require more conditions. Now, motivated from Propositions 1.3–1.5, we put forward the concept of *s-suitable graph* as follows:

Definition 1.2. A graph G with n vertices and minimum degree $\delta(G) \geq k$ is called an *s-suitable graph* if G satisfies the following conditions: (*i*) K_n is s-suitable, (*ii*) G is s-suitable if and only if $C_{n+s-1}(G)$ is s-suitable, and (*iii*) If $s_0 \leq p \leq k \leq \frac{1}{2}(n+s-2)$ and $G \in \mathbb{G}(p, s, k)$, then G is s-suitable if and only if $p \in \{s_0+1, s_0+2, \ldots, k-1\}$, where $s_0 = \max\{0, s\}$.

For any non-negative integer q, since K_n is both q-traceable and q-Hamiltonian, by Propositions 1.1 (*iii*), 1.3, 1.5 and Definition 1.2, a q-traceable graph is q-suitable, and a q-Hamiltonian graph is $(q+1)$-suitable. Consequently, a traceable, Hamiltonian, and Hamilton-connected graph is a 0-suitable, 1-suitable and 2-suitable graph, respectively.

Since K_n is s-suitable, by Theorem 1.8, it easily follows that

Corollary 1.1. *Let s and k be two non-negative integers and let G be a graph with $n \geq 6k+10-5s$ vertices and minimum degree $\delta(G) \geq k \geq \max\{1, s\}$. If $\Theta(G, \alpha) > \Theta_0$ and $0 \leq \alpha \leq 1$, then G is s-suitable unless $C_{n+s-1}(G) \in \left\{N_n^{k,s}, M_n^{k,s}\right\}$.*

Note that a Hamilton-connected graph is a 2-suitable graph. Thus, by Corollary 1.1, we have the following remark.

Remark 1.4. In [3], it is shown that: For any graph with $n \geq 6k^2 - 8k + 5$ vertices and minimum degree $\delta(G) \geq k \geq 2$, if either $\rho(G) > \frac{1}{2}(k-1) + \frac{1}{2}\sqrt{4n^2 - 4(3k-1)n + k^2 + 10k - 15}$ or $\mu(G) > 2n - 2k - \frac{2}{n-1}$, then G is Hamilton-connected unless $C_{n+1}(G) \in \left\{N_n^{k,2}, M_n^{k,2}\right\}$. It is easy to see that Corollary 1.1 improves this result for large n by setting $s = 2$.

From the definition of s-suitable and Theorem 1.9, we have

Corollary 1.2. *Let s and k be two non-negative integers and let G be a graph with $n \geq \max\Big\{6k+10-5s,\ \frac{1}{4}\big(4(k-s)(k+3)+10k+25\big),\ \frac{1}{3}\big(2(k-s)(3k-s+5)+8k+15\big)\Big\}$ vertices and minimum degree $\delta(G) \geq k \geq \max\{1, s\}$. If $\Theta(G, \alpha) \geq \Theta\left(N_{n,0}^{k,s}, \alpha\right)$ and $0 \leq \alpha \leq 1$, then G is s-suitable unless $G \in \left\{N_n^{k,s},\ M_n^{k,s},\ N_{n,0}^{k,s}\right\}$.*

Remark 1.5. *Recall that a traceable, Hamiltonian, and Hamilton-connected graph is a 0-suitable, 1-suitable and 2-suitable graph, respectively. By Proposition 1.2, it is easy to see that Corollary 1.2 improves and generalizes these results of Theorems 1.1–1.5 for large n by setting $s = 0$, 1 and 2, respectively.*

As the definition of s-suitable is an extension to the concepts of traceable, Hamiltonian and Hamilton-connected, it is easy to see that the graph with more edges has higher chance to be s-suitable. For a graph with n vertices and minimum degree $\delta(G) \geq k \geq 1$, it is natural and interesting for us to consider the following problem:

Problem 1.2. How large for $e(G)$ can confirm that the s-suitability of a graph G?

The corresponding Problem 1.2 of traceable, Hamiltonian, and Hamilton-connected graph had been, respectively, studied in [3,8,11,16]. Now, we will give partial answer to Problem 1.2, which generalizes the above referred results:

Theorem 1.10. *Let s and k be two integers and let G be a graph with $n \geq 6k+10-5s$ vertices, minimum degree $\delta(G) \geq k \geq \max\{1, s\}$ and $e(G) > \varepsilon_0$ edges. If $C_{n+s-1}(G) \ncong K_n$, then $C_{n+s-1}(G) \in \mathbb{G}_n(p, s, k)$ holds for some integer p, where $\max\{s, 0\} \leq p \leq k$.*

2. The Proofs of Propositions 1.3 and 1.5

The following result generalizes the corresponding result of traceable graph due to Ore [15].

Lemma 2.1. *Let G be a graph with n vertices and q be a non-negative integer. If $d_G(w_1) + d_G(w_2) \geq n + q - 1$ whenever $w_1w_2 \notin E(G)$, then G is q-traceable if and only if $G' = G + w_1w_2$ is q-traceable.*

Proof. It suffices to show the necessity. Let S be any set of at most q vertices such that $S \subseteq V(G)$ and $|S| = t$. Since G' is q-traceable, $G'\big[V(G) \setminus S\big]$ contains a Hamiltonian path, say P, where $V(P) = V(G) \setminus S$.

If $w_1w_2 \notin E(P)$, then P is also a path of $G\big[V(G)\setminus S\big]$ and hence we are done. Otherwise, $w_1w_2 \in E(P)$ and $w_1w_2 \notin E(G)$. Let $G_1 = G\big[V(G)\setminus S\big]$ and let $P = u_1u_2\cdots u_{n-t}$ be the corresponding Hamiltonian path of $G'\big[V(G)\setminus S\big]$, where $w_1 = u_i$ and $w_2 = u_{i+1}$. Recall that $d_G(u_i) + d_G(u_{i+1}) \geq n + q - 1$. Thus, $d_{G_1}(u_i) + d_{G_1}(u_{i+1}) \geq n + q - 2t - 1 \geq n - t - 1$ as $t \leq q$. Furthermore, $u_1u_{i+1} \notin E(G_1)$ and $u_iu_{n-t} \notin E(G_1)$ (otherwise, $G\big[V(G) \setminus S\big]$ also contains a Hamiltonian path, and the result already holds).

We firstly prove the following claim:

Claim 1: There is an index j with either $i+2 \leq j \leq n-t-1$ or $1 \leq j \leq i-2$ such that $u_iu_j \in E(G_1)$ and $u_{i+1}u_{j+1} \in E(G_1)$.

Proof of Claim 1: We suppose that $N_{G_1}(u_{i+1}) = \left\{u_{p_1},\ u_{p_2},\ \ldots,\ u_{p_q},\ u_{i+2}\right\}$, where $\{1, i\} \cap \{p_1, p_2, \ldots, p_q\} = \emptyset$. If Claim 1 does not hold, then $N_{G_1}(u_i) \subseteq \left\{u_1,\ u_2,\ \ldots,\ u_{i-1}, u_{i+2}, u_{i+3}, \ldots, u_{n-t-1}\right\} \setminus \left\{u_{p_1-1},\ u_{p_2-1},\ \ldots,\ u_{p_q-1}\right\}$. Thus, $d_{G_1}(u_i) \leq n - t - 3 - q$ and hence $d_{G_1}(u_i) + d_{G_1}(u_{i+1}) \leq (n-t-3-q) + (q+1) = n - t - 2$, a contradiction. This completes the proof of Claim 1.

By Claim 1, either for some j with $i+2 \leq j \leq n-t-1$, both $u_iu_j \in E(G_1)$ and $u_{i+1}u_{j+1} \in E(G_1)$, whence $u_1u_2\ldots u_iu_ju_{j-1}\ldots u_{i+1}u_{j+1}u_{j+2}\cdots u_{n-t}$ is a Hamiltonian path of $G[V(G) \setminus S]$; or for some j with $1 \leq j \leq i-2$, both

$u_iu_j \in E(G_1)$ and $u_{i+1}u_{j+1} \in E(G_1)$, whence $u_1u_2 \dots u_ju_iu_{i-1} \dots u_{j+1}u_{i+1}u_{i+2} \dots u_{n-t}$ is a Hamiltonian path of $G[V(G) \setminus S]$. This proves the lemma. ■

The Proof of Proposition 1.3. By Lemma 2.1, it is easy to see that Proposition 1.3 holds. ■

Lemma 2.2. *Let q be a non-negative integer. Then G is q-traceable if and only if $K_1 \vee G$ is $(q+1)$-traceable, and G is q-Hamiltonian if and only if $K_1 \vee G$ is $(q+1)$-Hamiltonian.*

Proof. Let $V(K_1) = \{u\}$ and also let $G' = K_1 \vee G$. We firstly suppose that $K_1 \vee G$ is $(q+1)$-traceable (respectively, $(q+1)$-Hamiltonian). Let S be any vertex set of G with t vertices, where $0 \le t \le q$. Then, $S_1 = S \cup \{u\}$ is a vertex set of $K_1 \vee G$ with $t+1$ vertices. In this case, since $t \le q$ and since $K_1 \vee G$ is $(q+1)$-traceable (respectively, $(q+1)$-Hamiltonian), $G'\big[V(G') \setminus S_1\big]$ contains a Hamiltonian path (respectively, Hamiltonian cycle). Note that $G\big[V(G) \setminus S\big] \cong G'\big[V(G') \setminus S_1\big]$. Thus, G is q-traceable (respectively, q-Hamiltonian).

Now, we suppose that G is q-traceable (respectively, q-Hamiltonian) and suppose that S_2 is any vertex set of G' with $t+1$ vertices, where $0 \le t \le q$ (Actually, since G is traceable, it is easy to see that G' is Hamiltonian, and hence we may suppose that $|S_2| \ge 1$). If $u \in S_2$, then let $S_3 = S_2 \setminus \{u\}$. In this case, $G\big[V(G) \setminus S_3\big]$ contains a Hamiltonian path (respectively, Hamiltonian cycle). Since $G'\big[V(G') \setminus S_2\big] \cong G\big[V(G) \setminus S_3\big]$, $G'\big[V(G') \setminus S_2\big]$ also contains a Hamiltonian path (respectively, Hamiltonian cycle). Thus, we assume that $u \notin S_2$.

Since $|S_2| \ge 1$, we choose $v \in S_2$ and let $S_4 = S_2 \setminus \{v\}$. It is easy to see that $G'\big[V(G') \setminus S_4\big] \cong K_1 \vee G\big[V(G) \setminus S_4\big]$. Combining this with $G\big[V(G) \setminus S_4\big]$ containing a Hamiltonian path (respectively, Hamiltonian cycle), we can conclude that $G'\big[V(G') \setminus S_2\big]$ contains a Hamiltonian path (respectively, Hamiltonian cycle). ■

The Proof of Proposition 1.5. We prove Proposition 1.5 by induction on q. By Proposition 1.4, the results hold for $q = 0$. Thus, we assume that $q \ge 1$ and Proposition 1.5 holds for smaller values of q.

(i) $G \in \mathbb{G}_n(p, q, k)$ with $1 \le q \le p \le k \le \frac{1}{2}(n+q-2)$.

As $p \ge q \ge 1$, by Definition 1.1, there exists a graph G_1 such that $G = K_1 \vee G_1$. By Lemma 2.2, G is q-traceable if and only if G_1 is $(q-1)$-traceable. In this case, $G \cong K_p \vee \big(K_{n+q-k-1-p} \cup H_0\big)$ and $G_1 \cong K_{p-1} \vee \big(K_{n+q-k-1-p} \cup H_0\big)$, where H_0 is a $(k-p)$-regular graph with $k+1-q$ vertices.

Recall that $1 \le q \le p \le k \le \frac{1}{2}(n+q-2)$. Thus, $0 \le q-1 \le p-1 \le k-1 \le \frac{1}{2}(n+q-4)$ and hence $\delta(G_1) = k-1$. Therefore, $G_1 \in \mathbb{G}_{n-1}(p-1, q-1, k-1)$. By the induction hypothesis, G_1 is $(q-1)$-traceable if and only if $p-1 \in \big\{q, q+1, \dots, k-2\big\}$. Thus, G is q-traceable if and only if $p \in \big\{q+1, q+2, \dots, k-1\big\}$.

(ii) $G \in \mathbb{G}_n(p, q+1, k)$ with $2 \le q+1 \le p \le k \le \frac{1}{2}(n+q-1)$.

As $p \ge q+1 \ge 2$, we may suppose that $G = K_1 \vee G_1$ by Definition 1.1. By Lemma 2.2, G is q-Hamiltonian if and only if G_1 is $(q-1)$-Hamiltonian. In this case, $G \cong K_p \vee \big(K_{n+q-k-p} \cup H_0\big)$ and $G_1 \cong K_{p-1} \vee \big(K_{n+q-k-p} \cup H_0\big)$, where H_0 is a $(k-p)$-regular graph with $k-q$ vertices.

Recall that $2 \le q+1 \le p \le k \le \frac{1}{2}(n+q-1)$. Thus, $1 \le q \le p-1 \le k-1 \le \frac{1}{2}(n+q-3)$ and hence $\delta(G_1) = k-1$. Therefore, $G_1 \in \mathbb{G}_{n-1}(p-1, q, k-1)$. By the induction hypothesis, G_1 is $(q-1)$-Hamiltonian if and only if $p-1 \in \big\{q+1, q+2, \dots, k-2\big\}$. Thus, G is q-Hamiltonian if and only if $p \in \big\{q+2,\ q+3, \dots, k-1\big\}$. ■

3. The Proofs of Theorems 1.8 and 1.10

For $A, B \subseteq V(G)$ and $A \cap B = \emptyset$, let $e(A, B)$ be the number of edges connecting A and B. Especially, $e(v, B)$ is the number of edges that connect v and B.

Lemma 3.1. *Let s and k be two integers and let G be a graph with $n \ge 6k + 10 - 5s$ vertices, minimum degree $\delta(G) \ge k \ge \max\{1, s\}$ and $e(G) > \varepsilon_0$ edges. If $C_{n+s-1}(G) \ncong K_n$, then $\omega(C_{n+s-1}(G)) \ge n+s-k-1$, where $\omega(C_{n+s-1}(G))$ is the clique number of $C_{n+s-1}(G)$.*

Proof. In the proof of this result, we rewrite $C_{n+s-1}(G)$ as G'. From the definition, it follows that $\delta(G') \ge \delta(G) \ge k$, $e(G') \ge e(G)$ and $d_{G'}(u) + d_{G'}(v) \le n+s-2$ holds for any pair of nonadjacent vertices $\{u, v\} \subseteq V(G')$. Let K be the subset of $V(G')$ containing all vertices which have degree at least $\frac{1}{2}(n+s-1)$. By the definition of G', any two vertices in K are adjacent in G'. Let C be a maximum clique of G' containing all vertices in K and suppose that $|C| = t$. Let $H = G' - C$. Since $G' \ncong K_n$, we can conclude that $H \ne \emptyset$ and $k \le d_{G'}(v) \le \frac{1}{2}(n+s-2)$ holds for each $v \in V(H)$. We consider the following two cases:

Case 1. $0 \le t < \frac{1}{2}(n+s)$.

For every $v \in V(H)$, we have $e(v, C) \le t-1$ and $k \le d_{G'}(v) \le \frac{1}{2}(n+s-2)$, and hence

$$e(H) + e\left(V(H), V(C)\right) = \frac{1}{2}\left(\sum_{v \in V(H)} d_{G'}(v) + \sum_{v \in V(H)} e(v, C)\right) \le \frac{1}{4}(n-t)\left(2t+n+s-4\right).$$

Combining this with $e(G') = e(C) + e(H) + e\big(V(H), V(C)\big)$, it follows that

$$\begin{aligned}\varepsilon_0 < e(G') &= e(C) + e(H) + e\big(V(H), V(C)\big)\\ &\le \binom{t}{2} + \frac{1}{4}(2t+n+s-4)(n-t) = \frac{1}{4}\Big((n-s+2)t + n(n+s-4)\Big)\\ &< \frac{1}{8}(3n-s)(n+s-2). \end{aligned} \tag{3.1}$$

Since $n \ge 6k+10-5s > 2k-s+4$, $8\varepsilon_0 - (3n-s)(n+s-2) = (n-6k+5s-10)(n+s-4-2k) \ge 0$, contrary to (3.1).

Case 2. $\frac{1}{2}(n+s) \le t \le n+s-k-2$.

Since C is a maximum clique of G' and hence every vertex of C has degree at least $t-1$, and since each pair of vertices with sum of degrees at least $n+s-1$ must be adjacent in G' (Recall that $C_{n+s-1}(G) = G'$), we conclude that $d_{G'}(v) \le n+s-t-1$ holds for every $v \in V(H)$. Thus,

$$e(H) + e\left(V(H), V(C)\right) = \sum_{v\in V(H)} d_{G'}(v) - e(H) \le (n-t)(n+s-t-1),$$

and hence

$$\begin{aligned} e(G') &= e(C) + e(H) + e\big(V(H), V(C)\big)\\ &\le \binom{t}{2} + (n-t)(n+s-t-1)\\ &= \frac{1}{2}\Big(3t^2 - (4n+2s-1)t + 2n(n+s-1)\Big). \end{aligned} \tag{3.2}$$

Since $\frac{1}{2}(n+s) \le t \le n+s-k-2$, by (3.2) and $n \ge 6k+10-5s$, we have

$$\begin{aligned} e(G') &\le \frac{1}{2}\Big(3t^2 - (4n+2s-1)t + 2n(n+s-1)\Big)\\ &\le \max\left\{\frac{1}{2}\Big(n^2 - (2k+5-2s)n + (k-s+2)(3k-s+5)\Big), \frac{1}{8}(3n-s)(n+s-2)\right\}\\ &\le \varepsilon_0 < e(G'), \end{aligned}$$

which is a contradiction. ■

The Proof of Theorem 1.10. In the proof of this result, we rewrite $C_{n+s-1}(G)$ as G'. From the definition, it follows that $\delta(G') \ge \delta(G) \ge k$, $e(G') \ge e(G)$ and $d_{G'}(u)+d_{G'}(v) \le n+s-2$ holds for any pair of nonadjacent vertices $\{u, v\} \subseteq V\big(G'\big)$. Let C be a maximum clique of G', and let $H = G'-C$. Note that $G' \ncong K_n$. Thus, $H \ne \emptyset$. By Lemma 3.1, we have $|C| \ge n+s-k-1$.

If $|C| \ge n+s-k$ and $v \in V(H)$, then since $d_{G'}(v) \ge k$ and since $d_{G'}(u) + d_{G'}(v) \le n+s-2$ holds for any pair of nonadjacent vertices $\{u, v\} \subseteq V\big(G'\big)$, we can conclude that v will be adjacent with every vertex of C, contrary to the maximality of C. Otherwise, $|C| = n+s-k-1$ and hence $|V(H)| = k+1-s$.

Since $|C| = n+s-k-1$, each vertex of C has degree at least $n+s-k-2$. We call a vertex in C as a frontier vertex if it has degree at least $n+s-k-1$ in G', i.e., it has at least one neighbor in H. By the maximality of C, we can see that every vertex in H has degree exactly k in G'. Let $F = \big\{u_1, u_2, \dots, u_p\big\}$ be the set of frontier vertices.

If $p \ge 1$, then every vertex in H is adjacent to every frontier vertex in G', as $C_{n+s-1}(G) = G'$. Thus, $G\big[V(H)\big] \cong H_0$ is a $(k-p)$-regular graph. Since $0 \le k-p \le k-s = |V(H)|-1$, we have $s \le p \le k$. Now, we can conclude that $G' \in \mathbb{G}_n(p, s, k)$, where $\max\{s,\ 1\} \le p \le k$.

Otherwise, $p = 0$. In this case, since $C_{n+s-1}(G) = G'$ and $\delta(G') \ge k$, $G\big[V(H)\big] \cong H_0$ is a k-regular graph, and hence $G \cong H_0 \cup K_{n+s-k-1} \in \mathbb{G}_n(0, s, k)$ (here, since $|V(H_0)| = k+1-s$ and H_0 is k-regular, we have $s \le 0$). ■

Lemma 3.2 ([5]). *If G is a graph with n vertices and $e(G)$ edges, then*

$$\mu(G) \le \frac{2e(G)}{n-1} + n - 2,$$

where the equality holds if and only if $G \cong K_{1,n-1}$ or $G \cong K_n$.

Lemma 3.3 ([10,13]). *Let G be a graph with n vertices, $e(G)$ edges and minimum degree $\delta(G)$. If $\delta(G) \ge k \ge 1$, then*

$$\rho(G) \le \frac{1}{2}\left(k - 1 + \sqrt{(k+1)^2 + 8e(G) - 4nk}\right).$$

Theorem 3.1. *Let α be a real number such that $0 \le \alpha \le 1$ and let s and k be two integers. For any graph G with n vertices and minimum degree $\delta(G) \ge k \ge \max\{1, s\}$, if $\Theta(G, \alpha) > \Theta_0$, then $e(G) > \varepsilon_0$.*

Proof. By contradiction, we assume that $e(G) \leq \varepsilon_0$. Since $0 \leq \alpha \leq 1$, we have $A(G) + \alpha D(G) = \alpha Q(G) + (1-\alpha)A(G)$. Combining this with Lemmas 3.2–3.3, it follows that

$$\begin{aligned}\Theta_0 &< \Theta(G,\alpha) \leq \alpha\mu(G) + (1-\alpha)\rho(G)\\ &\leq \alpha\left(\frac{2e(G)}{n-1} + n - 2\right) + \frac{1}{2}(1-\alpha)\left(k - 1 + \sqrt{(k+1)^2 + 8e(G) - 4nk}\right) \leq \Theta_0,\end{aligned}$$

a contradiction. ■

The Proof of Theorem 1.8. We may suppose that $C_{n+s-1}(G) \ncong K_n$. In this case, since $\Theta(G,\alpha) > \Theta_0$, Theorems 1.10 and 3.1 imply that $C_{n+s-1}(G) \in \mathbb{G}_n(p,s,k)$ holds for some integer p, where $\max\{s,\ 0\} \leq p \leq k$. ■

4. The Proof of Theorem 1.9

In what follows, we always suppose that $\alpha \geq 0$. By Rayleigh's theorem, we have

Lemma 4.1 (*see [12]*). *Let G be a connected graph with n vertices and let $\psi = \big(\psi(v_1), \psi(v_2), \ldots, \psi(v_n)\big)^T$ be any non-zero vector defined on $V(G)$. If $H \subset G$, then $\Theta(H,\alpha) < \Theta(G,\alpha) \leq \big(1+\alpha\big)\Delta(G)$. Furthermore,*

$$\Theta(G,\alpha)\psi^T\psi \geq \psi^T\left(A(G) + \alpha D(G)\right)\psi = 2\sum_{uv\in E(G)} f(u)f(v) + \alpha\sum_{i=1}^{n} d_G(v_i)f^2(v_i), \tag{4.1}$$

where the equality holds if and only if ψ is an eigenvector of $\Theta(G,\alpha)$.

If G is connected, since $A(G) + \alpha D(G)$ is non-negative irreducible matrix, there is a unique positive unit eigenvector $f = \big(f(v_1), f(v_2), \ldots, f(v_n)\big)^T$ corresponding to $\Theta(G,\alpha)$. In the sequel, we call f a *Perron vector* of G.

Lemma 4.2 (*[12]*). *Let u, v be two vertices of the connected graph G, and $w_1, w_2, \ldots, w_k$ $\big(1 \leq k \leq d_G(v)\big)$ be some vertices of $N(v) \setminus \big(N(u) \cup \{u\}\big)$. Let $G' = G + w_1u + w_2u + \cdots + w_ku - w_1v - w_2v - \cdots - w_kv$. If f is the Perron vector of G with $f(u) \geq f(v)$, then $\Theta(G',\alpha) > \Theta(G,\alpha)$.*

Given two distinct vertices u, v in a graph G such that $N_G(v) \setminus \big(N_G(u) \cup \{u\}\big) \neq \emptyset \neq N_G(u) \setminus \big(N_G(v) \cup \{v\}\big)$, we construct a new graph $G' = G'(u,v)$ via replacing all edges vw by uw for each $w \in N_G(v) \setminus \big(N_G(u) \cup \{u\}\big)$. This operation is called the *Kelmans transformation* from v to u (see [4]).

Corollary 4.1. *Let G be a connected graph. If G' is a graph obtained from G by some Kelmans transformation, then $\Theta(G',\alpha) > \Theta(G,\alpha)$.*

Proof. We use the idea as that in [4] and we suppose that G' is obtained from G by a Kelmans transformation from the vertex v to vertex u. Let f be the Perron vector of G. The key observation is that up to isomorphism G' is independent of u or v being the beneficiary if we apply the transformation from v to u. Indeed, in G' one of u or v will be adjacent to $N_G(u) \cup N_G(v)$, the other will be adjacent to $N_G(u) \cap N_G(v)$ (and if the two vertices are adjacent in G then they will remain adjacent also). Now, we may assume that $f(u) \geq f(v)$. Then, the result follows from Lemma 4.2. ■

Corollary 4.2. *Let $G \in \mathbb{G}_n(p,s,k)$ and let G_0 be the graph obtained from G by deleting one edge with two end vertices in $V_2(G)$ such that $\delta(G_0) \geq k \geq p \geq \max\{s, 1\}$ and $n \geq p + k + 3 - s$. If G' is a graph obtained from G by deleting one edge and $\delta(G') \geq k$, then $\Theta(G',\alpha) \leq \Theta(G_0,\alpha)$, where the equality holds if and only if $G' \cong G_0$.*

Proof. Suppose that G' is a graph obtained from G by deleting one edge (say $e = w_0z_0$). Since $\delta(G) \geq k$ and $d_G(w) = k$ holds for each vertex $w \in V_3(G)$, we only need to consider the following three cases by symmetry:

(1) $\{w_0, z_0\} \subseteq V_1(G)$, (2) $z_0 \in V_1(G)$ and $w_0 \in V_2(G)$, (3) $\{w_0, z_0\} \subseteq V_2(G)$.

Let $G_1 = G - w_0z_0$ for $\{w_0, z_0\} \subseteq V_1(G)$, let $G_2 = G - w_0z_0$ for $z_0 \in V_1(G)$ and $w_0 \in V_2(G)$, and let $G_0 = G - w_0z_0$ for $\{w_0, z_0\} \subseteq V_2(G)$. To complete the proof of this result, it suffices to show that $\Theta(G_1,\alpha) < \Theta(G_2,\alpha) < \Theta(G_0,\alpha)$. Since $k \geq \max\{s, 1\}$, we get $|V_3(G)| = k + 1 - s \geq 1$. We choose $w \in V_3(G)$.

For G_1, we choose $v \in V_2(G_1)$ and $w \in V_3(H)$, and we rewrite z_0 as u. In this case, $\{w_0\} = N_{G_1}(v) \setminus (N_{G_1}(u) \cup \{u\})$ and $w \in N_{G_1}(u) \setminus (N_{G_1}(v) \cup \{v\})$. It is easy to see that G_2 is isomorphic to the graph obtained from G_1 by a Kelmans transformation from v to u. By Corollary 4.1, $\Theta(G_1,\alpha) < \Theta(G_2,\alpha)$.

For G_2, we choose $v \in V_2(G_2) \setminus \{w_0\}$, and we rewrite z_0 as u. In this case, $\{w_0\} = N_{G_2}(v) \setminus (N_{G_2}(u) \cup \{u\})$ and $w \in N_{G_1}(u) \setminus (N_{G_1}(v) \cup \{v\})$. It is easy to see that G_0 is isomorphic to the graph obtained from G_2 by a Kelmans transformation from v to u. By Corollary 4.1, $\Theta(G_2,\alpha) < \Theta(G_0,\alpha)$. ■

Lemma 4.3. *Let s and k be two integers and let G be a proper subgraph of $\mathbb{G}_n(p, s, k)$ such that G contains $n \geq 6k + 10 - 5s$ vertices and minimum degree $\delta(G) \geq k \geq \max\{1, s\}$. If $\max\{0, s\} \leq p \leq k$, then $\Theta\big(G, \alpha\big) \leq \Theta\left(N_{n,0}^{k,s}, \alpha\right)$, with equality holding if and only if $G \cong N_{n,0}^{k,s}$.*

Proof. By Corollary 4.2 and Lemma 4.1, the result already holds for $p = k$. Thus, we may suppose that $0 \leq p \leq k - 1$ in what follows. We consider the following two cases:

Case 1. $p = 0$.

In this case, $s \leq 0$ and hence $G \subset K_{n+s-k-1} \cup H_0$, where H_0 is a k-regular graph with $k + 1 - s$ vertices. Recall that $\delta(G) \geq k$. Thus, G is obtained from $K_{n+s-k-1} \cup H_0$ by deleting some edges from $K_{n+s-k-1}$. Since $n \geq 6k + 10 - 5s$, and $K_{n+s-k-2} \subset K_{n+s-k-1} - e \subset N_{n,0}^{k,s}$, Lemma 4.1 implies that

$$\Theta\big(H_0, \alpha\big) \leq (1+\alpha)k < (1+\alpha)(n+s-k-3) = \Theta\Big(K_{n+s-k-2}, \alpha\Big) < \Theta\big(K_{n+s-k-1} - e, \alpha\big) < \Theta\left(N_{n,0}^{k,s}, \alpha\right).$$

Thus,

$$\Theta\big(G, \alpha\big) \leq \Theta\big(K_{n+s-k-1} - e, \alpha\big) < \Theta\left(N_{n,0}^{k,s}, \alpha\right),$$

and hence this result holds.

Case 2. $1 \leq p \leq k - 1$.

In this case, since $p \geq \max\{0, s\} \geq s$ and since $n \geq 6k + 10 - 5s$, we have $\max\{1, s\} \leq p \leq k - 1$ and $n > p + k + 9 - s$. Suppose that $G_1 \in \mathbb{G}_n(p, s, k)$ and $G_2 \in \mathbb{G}_n(p + 1, s, k)$. Let G and G' be the graphs obtained from G_1 and G_2 by deleting one edge from $V_2(G_1)$ and $V_2(G_2)$, respectively. By Corollary 4.2 and Lemma 4.1, it suffices to show that

$$\Theta\big(G, \alpha\big) < \Theta\big(G', \alpha\big). \tag{4.2}$$

For convenience, we may suppose that G is obtained from G_1 by deleting the edge w_0z_0 with $\{w_0, z_0\} \subseteq V_2(G_1)$, and we rewrite $\Theta\big(G, \alpha\big)$ as Θ. Let f be the Perron vector of G. We firstly prove the following claim:

Claim 1. For any pairs of vertices $\{u, v\} \subseteq V_3(G)$, we have $f(u) = f(v)$.

Proof of Claim 1. By contradiction, we assume that Claim 1 does not hold. Let $\{w_1, w_2\} \subseteq V_3(G)$ such that $f(w_1) = \max\big\{f(w) : w \in V_3(G)\big\}$ and $f(w_2) = \min\big\{f(w) : w \in V_3(G)\big\}$. Then, $f(w_1) > f(w_2)$. Note that $H_0 \cong G\big[V_3(G)\big]$. Thus,

$$\begin{aligned}\Theta\big(f(w_1) - f(w_2)\big) &= \alpha k\big(f(w_1) - f(w_2)\big) + \sum_{w \in N_{H_0}(w_1)} f(w) - \sum_{z \in N_{H_0}(w_2)} f(z)\\ &\leq \alpha k\big(f(w_1) - f(w_2)\big) + (k-p)\big(f(w_1) - f(w_2)\big).\end{aligned}$$

Recall that $f(w_1) - f(w_2) > 0$. Thus, $\Theta - \alpha k - (k - p) \leq 0$.

On the other hand, Lemma 4.1 implies that $\Theta > \Theta\big(K_{n+s-k-2}, \alpha\big) = (1 + \alpha)(n + s - k - 3)$. Combining this with $n \geq 6k - 5s + 10$, we have $\Theta - \alpha k - (k - p) \geq (1 + \alpha)(n + s - k - 3) - \alpha k - (k - p) = \alpha(n + s - 3 - 2k) + n + s + p - 3 - 2k \geq \alpha(4k - 4s + 7) + 4k - 4s + 7 + p > 0$, a contradiction.

Now, we can conclude that $f(w_1) = f(w_2)$. This completes the proof of Claim 1.

With the similar reason with Claim 1, we can set $x_1 = f(w)$ for $w \in V_1(G)$, set $x_2 = f(w)$ for $w \in V_2(G)\backslash\{w_0, z_0\}$, set $x_3 = f(w_0) = f(z_0)$, and set $x_4 = f(w)$ for $w \in V_3(G)$. Now, from $\big(A(G) + \alpha D(G)\big)f = \Theta f$, it follows that

$$\begin{cases} \Theta x_1 = \alpha(n-1)x_1 + (k+1-s)x_4 + (p-1)x_1 + (n+s-k-3-p)x_2 + 2x_3,\\ \Theta x_2 = \alpha(n+s-k-2)x_2 + px_1 + (n+s-k-p-4)x_2 + 2x_3,\\ \Theta x_3 = \alpha(n+s-k-3)x_3 + px_1 + (n+s-k-p-3)x_2,\\ \Theta x_4 = \alpha k x_4 + (k-p)x_4 + px_1. \end{cases} \tag{4.3}$$

By the second to fourth equations of (4.3), we have

$$\begin{cases} x_2 = \frac{p\big(\Theta+2-\alpha(n+s-k-3)\big)}{\big(\Theta+2+p-(\alpha+1)(n+s-k-2)\big)\big(\Theta+2-\alpha(n+s-k-3)\big)-2\big(\Theta+1-\alpha(n+s-k-2)\big)}x_1,\\ x_3 = \frac{p\big(\Theta+1-\alpha(n+s-k-2)\big)}{\big(\Theta+2+p-(\alpha+1)(n+s-k-2)\big)\big(\Theta+2-\alpha(n+s-k-3)\big)-2\big(\Theta+1-\alpha(n+s-k-2)\big)}x_1,\\ x_4 = \frac{px_1}{\Theta-(\alpha+1)k+p}. \end{cases} \tag{4.4}$$

Now, from Lemma 4.1 we can deduce that

$$\begin{aligned}\Theta\big(G', \alpha\big) - \Theta\big(G, \alpha\big) &\geq f^T\big(A(G') + \alpha D(G')\big)f - f^T\big(A(G) + \alpha D(G)\big)f\\ &= (k+1-s)(2x_2 - x_4)x_4 + \alpha(k+1-s)x_2^2.\end{aligned} \tag{4.5}$$

By (4.5), to prove (4.2), it suffices to show that $2x_2 > x_4$, that is equivalent to

$$\Phi(\Theta) > 0, \tag{4.6}$$

where $\Phi(\Theta) = \Theta^2 + \Big(n+p+s+\alpha(1-2k)-3k\Big)\Theta - \alpha(\alpha+1)n^2 + \Big(\alpha\Big((\alpha+1)(4k-2s+5)+2-p\Big)+2\Big)n - \Big((k-s)(3k-s+5)+6k+6\Big)\alpha^2 - \Big((k-s)(3k-s+7-p)-3p+10k+12\Big)\alpha - 2(3k-p-s+3)$.

Combining this with $n \geq 6k-5s+10$ and $\Theta > (1+\alpha)(n+s-k-3)$, we have $\Phi'(\Theta) = 2\Theta+n+p+s+\alpha(1-2k)-3k > 2(1+\alpha)(n+s-k-3)+n+p+s+\alpha(1-2k)-3k = (2n+2s-4k-5)\alpha+3n+3s+p-5k-6 \geq 3n+3s+p-5k-6 \geq 13k+p-12s+24 > 0$.

Let $\Phi_1(n) = 2n^2-(6k+7-4s)n+2(2k-s)(k-s)+9k-7s+3$. Since $\Phi_1'(n) = 4n-(6k+7-4s) \geq 18k-16s+33 > 0$ by $n \geq 6k+10-5s$, $\Phi_1(n) \geq \Phi_1(6k+10-5s) = 8(k-s)(5k-4s)+(147k-132s)+133 > 0$. Combining with $\Phi'(\Theta) > 0$, $\Phi_1(n) > 0$ and $n \geq 6k-5s+10$, it follows that

$$\begin{aligned}
\Phi(\Theta) &> \Phi((1+\alpha)(n+s-k-3)) \\
&= \Big(2n^2-(6k+7-4s)n+2(2k-s)(k-s)+9k-7s+3\Big)\alpha \\
&\quad + (2n-4k+p+2s-3)(n+s-k-3)+2(n+p+s-3k-3) \\
&\geq (2n-4k+p+2s-3)(n+s-k-3)+2(n+p+s-3k-3) \\
&= 4k^2-(6n+6s+p-9)k+(n+s)(2n+2s+p-7)+3-p = \Phi_2(k).
\end{aligned}$$

Since $n \geq 6k+10-5s$, we have

$$\Phi_2'(k) = 8k-(6n+6s+p-9) \leq \frac{4}{3}(n+5s-10)-(6n+6s+p-9) = -\frac{1}{3}(14n+3p-2s+13) < 0,$$

which implies that

$$\begin{aligned}
\Phi(\Theta) &> \Phi_2(k) \geq \Phi_2\left(\frac{1}{6}(n+5s-10)\right) \\
&= \frac{1}{18}\Big(20n^2+(15p-16s+41)n+4(3p-4)-s(11+4s-3p)\Big) = \Phi_3(n).
\end{aligned}$$

If $s \geq 1$, since $n \geq 6k+10-5s \geq 10+s$, it follows that $18\Phi_3'(n) = 40n+15p-16s+41 \geq 24s+15p+441 > 0$, and hence

$$\Phi(\Theta) > \Phi_2(k) \geq \Phi_3(n) \geq \Phi_3(s+10) = 9p+15s+ps+133 > 0.$$

If $s \leq 0$, since $n \geq 6k+10-5s \geq 10-5s$, it follows that $18\Phi_3'(n) = 40n+15p-16s+41 > 0$, and hence

$$\Phi(\Theta) > \Phi_2(k) \geq \Phi_3(n) \geq \Phi_3(10-5s) = 32s^2-(4p+132)s+9p+133 > 0.$$

Now, (4.6) holds and this completes the proof of this result. ■

Lemma 4.4. *Let α and q be two real numbers such that $0 \leq \alpha \leq 1$ and $0 \leq q < 1$. If $n \geq \max\Big\{\frac{1}{2}\big(3k+3+2q-2s\big), \frac{1}{2(1-q)}\big((k-q)(k+q-s)+(q-4)(s-3)-k(3q-7)-4\big), \frac{1}{2-q}\big((k-s)(3k-s+5)+4(k+2)-q\big)\Big\}$, then $(1+\alpha)(n+s-k-2)-q \geq \Theta_0$.*

Proof. Let $\Phi(\alpha) = (1+\alpha)(n+s-k-2)-q-\Theta_0$. Note that

$$\Phi'(\alpha) = s-k-\frac{2\varepsilon_0}{n-1}+\frac{1}{2}\left(k-1+\sqrt{(k+1)^2+8\varepsilon_0-4nk}\right).$$

Thus, $\Phi(\alpha) \geq \min\big\{\Phi(0), \Phi(1)\big\}$. To complete the proof of this result, it suffices to show that

$$\min\big\{\Phi(0),\ \Phi(1)\big\} \geq 0. \tag{4.7}$$

Since $2(1-q)n \geq (k-q)(k+q-s)+(q-4)(s-3)-k(3q-7)-4$, we have

$$\Phi(0) = n+s-\frac{3}{2}k-\frac{3}{2}-q-\sqrt{n^2-(3k-2s+5)n+\frac{(13k-3s)(k-s)+s(s-28)+46k+41}{4}} \geq 0.$$

Furthermore, since $(2-q)n \geq (k-s)(3k-s+5)+4(k+2)-q$, we have

$$\begin{aligned}
\Phi(1) &= n+2s-2k-2-q-\frac{1}{n-1}\Big((n-k-2+s)(n-k-3+s)+2(k+1)(k+2-s)\Big) \\
&= \frac{1}{n-1}\Big((2-q)n-(k-s)(3k-s+5)-4(k+2)+q\Big) \geq 0.
\end{aligned}$$

Now, we can conclude that (4.7) holds. ■

Lemma 4.5. *Let $\alpha \geq 0$ and $q > 0$ be two real numbers. If $n \geq \frac{1}{q}\Big((1-q)(\alpha-q)+\alpha+2\Big)$, then $\Theta\Big(K_n-e,\alpha\Big) \geq (1+\alpha)(n-1)-q$.*

Proof. Throughout the proof of this result, we rewrite $K_n - e$ as G and simplify $\Theta(G,\alpha)$ as Θ, where $e = w_0z_0$. Let f be the Perron vector of Θ. For convenience, let $x_1 = f(w)$ for $w \in V(G) \setminus \{w_0, z_0\}$ and let $x_2 = f(w)$ for $w \in \{w_0, z_0\}$. Now, from $\big(A(G) + \alpha D(G)\big)f = \Theta(G,\alpha)f$, it follows that

$$\begin{cases} \Theta x_1 = \alpha(n-1)x_1 + (n-3)x_1 + 2x_2, \\ \Theta x_2 = \alpha(n-2)x_2 + (n-2)x_1. \end{cases}$$

Thus, Θ satisfies

$$\Phi(\Theta) = \Theta^2 - \Big(\alpha(2n-3) + n - 3\Big)\Theta + (\alpha+1)(n-2)\Big(\alpha(n-1) - 2\Big),$$

which implies

$$\Theta = \frac{1}{2}\left(\alpha(2n-3) + n - 3 + \sqrt{n^2 + 2(\alpha+1)n + (\alpha+1)(\alpha-7)}\right).$$

Note that $qn \geq (1-q)(\alpha-q) + \alpha + 2$. Thus, $\Theta \geq (1+\alpha)(n-1) - q$. ■

The Proof of Theorem 1.9. When $q = \frac{1}{2}$, since $n \geq 6k - 5s + 10 > \frac{1}{2}\big(3k + 4 - 2s\big) = \frac{1}{2}\big(3k + 3 + 2q - 2s\big)$ and $0 \leq \alpha \leq 1$, we have

$$n + s - k - 1 \geq 6k - 5s + 10 + s - k - 1 > \frac{13}{2} \geq 3\alpha + \frac{7}{2} = \frac{1}{q}\Big((1-q)(\alpha-q) + \alpha + 2\Big).$$

Combining this with $K_{n+s-k-1} - e \subset N_{n,0}^{k,s}$ and $\alpha \geq 0$, by setting $q = \frac{1}{2}$ in Lemmas 4.4 and 4.5 it follows that

$$\Theta\left(N_{n,0}^{k,s}, \alpha\right) > \Theta\big(K_{n+s-k-1} - e, \alpha\big) \geq \Theta_0. \tag{4.8}$$

If $C_{n+s-1}(G) \ncong K_n$, then $k \leq \frac{1}{2}\big(n + s - 2\big)$ and Theorem 1.8 implies that $C_{n+s-1}(G) \in \mathbb{G}_n(p, s, k)$ for some integer p, where $\max\{0, s\} \leq p \leq k$. Hence, Theorem 1.9 follows from Lemma 4.3. ■

Remark 4.1. By an observation to the proof of Theorem 1.9, we can improve the lower bound for n of Theorem 1.9 by setting suitable q in Lemmas 4.4 and 4.5, where $0 < q < 1$.

5. The proof of Proposition 1.2

This section dedicates to the proof of Proposition 1.2.

The Proof of Proposition 1.2. In the proof of this result, we rewrite $N_{n,0}^{k,s}$ as G and we suppose that $N_{n,0}^{k,s} = N_n^{k,s} - w_0z_0$. Note that $e(G) = \frac{1}{2}(n+s-k-1)(n+s-k-2) + k(k+1-s) - 1$. Thus, (1.1) follows from Lemma 3.2. To complete the proof of this result, it suffices to show that $\rho(G) < n + s - k - 2$.

Let f be the Perron vector of G, and let $\rho = \rho(G)$. For convenience, let $x_1 = f(w)$ for $w \in V_1(G)$, let $x_2 = f(w)$ for $w \in V_2(G)\backslash\{w_0,\ z_0\}$, let $x_3 = f(w_0) = f(z_0)$, and let $x_4 = f(w)$ for $w \in V_3(G)$. Now, from $(A(G))f = \rho f$, it follows that

$$\begin{cases} \rho x_1 = (k-1)x_1 + (n-2k-3+s)x_2 + 2x_3 + (k+1-s)x_4, \\ \rho x_2 = kx_1 + (n+s-2k-4)x_2 + 2x_3, \\ \rho x_3 = kx_1 + (n+s-2k-3)x_2, \\ \rho x_4 = kx_1. \end{cases} \tag{5.1}$$

From the second and third equations of (5.1), we have $(\rho+1)x_2 = (\rho+2)x_3$. Now, from the second equation of (5.1), we have

$$x_2 = \frac{k(\rho+2)}{(\rho+4+2k-n-s)(\rho+2) - 2(\rho+1)}x_1. \tag{5.2}$$

Furthermore, combining with $(\rho+1)x_2 = (\rho+2)x_3$, by the first and fourth equations of (5.1), we have

$$x_2 = \frac{(\rho+2)\big((\rho-k)(\rho+1) - k(k-s)\big)}{\rho\big((n-2k-3+s)(\rho+2) + 2(\rho+1)\big)}x_1. \tag{5.3}$$

By (5.2) and (5.3), ρ satisfies $0 = \Phi(\rho) = \rho^4 - (n-k+s-5)\rho^3 - \Big((k-2)(k-s) + 3n + s - 10\Big)\rho^2 - \Big(k(k-s)(2k-n-s+5) - (n-k-2)k + 2(n+s-3)\Big)\rho + 2k(k-s+1)(n+s-2k-3)$.

Let $\Phi\big(n+s-k-2\big)=2n^2-\Big((k^2+4)(k-s)+k^2+6\Big)n+(k-s+1)(k^3-sk^2-2s+4)=\Phi_1(n)$.

Claim 1. $\Phi\big(n+s-k-2\big)=\Phi_1(n)>0$.

Proof of Claim 1. We consider the following two cases:

Case 1. $k\geq s+1$.

In this case, since $n\geq \frac{1}{2}\big((k^2+4)(k+1-s)+2\big)$, we have

$$\Phi_1'(n)\geq \Phi_1'\left(\frac{1}{2}\big((k^2+4)(k+1-s)+2\big)\right)=4(k-s)+k^2(k-s+1)+6>0.$$

Combining this with $k\geq s+1$, we can conclude that

$$\begin{aligned}\Phi_1(n) &\geq \Phi_1\left(\frac{1}{2}\big((k^2+4)(k+1-s)+2\big)\right)\\ &=(k-s+1)\big(k^2(k-s)-2s+4\big)\geq 2\big(k^2-2s+4\big)>0,\end{aligned}$$

and hence Claim 1 holds.

Case 2. $k=s$.

In this case, $k=s\geq 1$. Since $n\geq \frac{1}{2}\big(k^2(k+1)+6\big)$ and $k=s$, we have

$$\Phi_1'(n)\geq \Phi_1'\left(\frac{1}{2}\big(k^2(k+1)+6\big)\right)=2s^3+s^2+6>0.$$

Now, we can conclude that

$$\Phi_1(n)\geq \Phi_1\left(\frac{1}{2}\big(k^2(k+1)+6\big)\right)=\frac{1}{2}(s+2)\left(s^4(s-1)+2s^3+2(s-1)^2+2\right)>0,$$

and hence Claim 1 holds. This completes the proof of Claim 1.

In what follows, let $\rho_1\geq\rho_2\geq\rho_3\geq\rho_4$ be the four roots of $\Phi(\rho)=0$. Then, $\rho_1+\rho_2+\rho_3+\rho_4=n-k+s-5$. Now, $\Phi(\rho)\to+\infty$ as $\rho\to+\infty$ and $\Phi\big(n+s-k-2\big)>0$ by Claim 1. Since the sum of four roots of $\Phi(\rho)=0$ is equal to $n+s-k-5$, either no roots are in $\big(n+s-k-2,+\infty\big)$ or exactly two roots are in $\big(n+s-k-2,+\infty\big)$.

If no roots are in $\big(n+s-k-2,+\infty\big)$, then $\rho(G)=\rho_1<n+s-k-2$ and we are done. Otherwise, we must have exactly two roots are in $\big(n+s-k-2,+\infty\big)$, that is, $\rho_1\geq\rho_2>n+s-k-2$ in what follows.

When $k(k+1-s)\geq 2$, then $\Phi(-2)=4-2k(k+1-s)\leq 0$. Combining this with $\Phi(n+s-k-2)>0$ by Claim 1, there is at least one root in $[-2,n+s-k-2)$, and hence $\rho_3\geq -2$. Since the absolute value of any eigenvalue of $A(G)$ is not greater than the spectral radius of G, namely, ρ_1, we have $\rho_1+\rho_4\geq 0$. Thus,

$$n+s-k-5=\sum_{i=1}^{4}\rho_i\geq\rho_2+\rho_3>n+s-k-2-2>n+s-k-5,\quad \text{a contradiction.}$$

When $k(k+1-s)\leq 1$, then $k=s=1$. In this case, we have

$$\Phi(\rho)=\rho^4-(n-5)\rho^3-(3n-9)\rho^2-(n-1)\rho+2n-8.$$

Recall that $n\geq 6k-5s+10=11$. Thus, it is easy to see that $\Phi(0)=2(n-4)>0$ and $\Phi(n-3)=-n^3+8n^2-21n+16=-n^2(n-8)-21n+16<0$. Combining this with $\Phi(n-2)>0$ by Claim 1, we have $0<\rho_4<n-3<\rho_3$. Since $\rho_1+\rho_2+\rho_3+\rho_4=n+s-k-5$ and $\rho_1>n+s-k-2$, we must have $\rho_4<0$, a contradiction. ■

6. Further discussion

Recently, Zhou et al. [21] showed that: If $\delta(G)\geq k\geq 2$ and $\mu(G)\geq 2(n-k)$, where $n\geq k^4+5k^3+2k^2+8k+12$, then G is Hamilton-connected unless G is either obtained from $N_n^{k,2}$ deleting at most $\frac{1}{4}k(k-1)$ edges or G is obtained from $M_n^{k,2}$ by deleting at most $\frac{1}{2}(k-1)$ edges. This result strengthens that of Theorem 1.9 for $s=2$ and $\alpha=1$.

In Lemma 4.4, we have showed that $\Theta_0\leq(1+\alpha)(n+s-k-2)-q$ for $0\leq\alpha\leq 1$, $0\leq q<1$, and $n\geq\max\Big\{\frac{1}{2}\big(3k+3+2q-2s\big),\ \frac{1}{2(1-q)}\big((k-q)(k+q-s)+(q-4)(s-3)-k(3q-7)-4\big),\ \frac{1}{2-q}\big((k-s)(3k-s+5)+4(k+2)-q\big)\Big\}$. Note that $\Theta_0\leq 2(n-k)-q\leq 2n-2k$ for $s=2$, $0\leq q<1$ and $\alpha=1$. By Theorem 1.8 and Proposition 1.4, to improve Zhou's result, it suffices to give the characterization of those proper subgraphs G of $N_n^{k,2}$ and $M_n^{k,2}$ such that $\Theta(G,\alpha)>\Theta_0$.

As in Remark 4.1, we can improve the lower bounds for n in Theorem 1.9 such that

$$\Theta_0\leq(1+\alpha)(n+s-k-2)-q\leq\Theta\left(K_{n+s-k-1}-e,\ \alpha\right)$$

by setting suitable positive real number q.

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