Graphs and Combinatorics (2019) 35:67–89
https://doi.org/10.1007/s00373-018-1987-4

ORIGINAL PAPER

Pancyclicity of 4-Connected $\{K_{1,3}, Z_8\}$-Free Graphs

Hong-Jian Lai[1,2] · Mingquan Zhan[3] · Taoye Zhang[4] · Ju Zhou[5]

Received: 24 March 2015 / Revised: 6 June 2017 / Published online: 29 November 2018

Abstract

A graph G is said to be *pancyclic* if G contains cycles of lengths from 3 to $|V(G)|$. For a positive integer i, we use Z_i to denote the graph obtained by identifying an endpoint of the path P_{i+1} with a vertex of a triangle. In this paper, we show that every 4-connected claw-free Z_8-free graph is either pancyclic or is the line graph of the Petersen graph. This implies that every 4-connected claw-free Z_6-free graph is pancyclic, and every 5-connected claw-free Z_8-free graph is pancyclic.

Keywords Claw-free · Pancyclic · Forbidden subgraphs

1 Introduction

We use [1] for terminology and notation not defined here, and consider finite simple graphs only. Let G be a graph. If $v \in V(G)$ and $S \subseteq V(G)$, $G[S]$ is the *subgraph* induced by S in G, $N_G(v)$ is the *neighborhood* of v in G, and $N_G(S) = \bigcup_{v \in S} N_G(v)$. Throughout this paper, we will assume that all cycles C have an inherent clockwise orientation. For a vertex $v \in V(C)$ we will denote the first, second, and i-th *successor* of v as v^+, v^{++}, and v^{+i}, respectively. Similarly, we denote the first, second, and i-th *predecessor* of v as v^-, v^{--}, and v^{-i} respectively. If $u, v \in V(C)$, then $C[u, v]$ denotes the consecutive vertices on C from u to v in the chosen direction of C, and $C(u, v] = C[u, v] - \{u\}$, $C[u, v) = C[u, v] - \{v\}$, $C(u, v) = C[u, v] - \{u, v\}$. The

✉ Mingquan Zhan
Mingquan.Zhan@millersville.edu

1 Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

2 College of Mathematics and System Sciences, Xinjiang University, Ürümqi 830046, Xinjiang, People's Republic of China

3 Department of Mathematics, Millersville University of Pennsylvania, Millersville, PA 17551, USA

4 Department of Mathematics, Penn State Worthington Scranton, Dunmore, PA 18512, USA

5 Department of Mathematics, Kutztown University of Pennsylvania, Kutztown, PA 19530, USA

same vertices, in the reverse order, are denoted by $\overleftarrow{C}[v, u]$, $\overleftarrow{C}[v, u)$, $\overleftarrow{C}(v, u]$ and $\overleftarrow{C}(v, u)$, respectively. A *hop* in a cycle is a chord that joins some v to v^{++}.

Given a family $\mathcal{F}$ of graphs, G is said to be *$\mathcal{F}$-free* if G contains no member of $\mathcal{F}$ as an induced subgraph. If $\mathcal{F} = \{K_{1,3}\}$, then G is said to be *claw-free*. A graph G is *hamiltonian* if it contains a spanning cycle and *pancyclic* if it contains cycles of lengths from 3 to $|V(G)|$. In 1984, Matthews and Sumner [6] conjectured that every 4-connected claw-free graph is hamiltonian. This conjecture is still open, and has also fostered a large body of research into other structural properties of cycles for claw-free graphs. In this paper we are specifically interested in the pancyclicity of highly connected claw-free graphs.

Let Ł denote the graph obtained by connecting two disjoint triangles with a single edge, and let $N(i, j, k)$ denote the net obtained by identifying an endpoint of each the paths P_{i+1}, P_{j+1}, P_{k+1} with distinct vertices of a triangle. $N(i, 0, 0)$ is also denoted by Z_i.

Theorem 1.1 (Gould, Łuczak, Pfender [4]) *Let X and Y be connected graphs on at least three vertices. If neither X nor Y is P_3 and Y is not $K_{1,3}$, then every 3-connected $\{X, Y\}$-free graph G is pancyclic if and only if $X = K_{1,3}$ and Y is a subgraph of one of the graphs in the family*

$$\mathcal{F} = \{P_7, Ł, N(4, 0, 0), N(3, 1, 0), N(2, 2, 0), N(2, 1, 1)\}.$$

Motivated by the Matthews–Sumner Conjecture and Theorem 1.1, Ron Gould came up with the following problem at the 2010 SIAM Discrete Math Meeting in Austin, TX.

Problem 1.2 *Characterize the pairs of forbidden subgraphs that imply a 4-connected graph is pancyclic.*

Theorem 1.3 (Ferrara, Gould, Gehrke, Magnant, and Powell [2]) *Every 4-connected $\{K_{1,3}, N(i, j, k)\}$-free graph with $i + j + k = 5$ is pancyclic.*

Theorem 1.4 (Ferrara, Morris, Wenger [3]) *Every 4-connected $\{K_{1,3}, P_{10}\}$-free graph is either pancyclic or is the line graph of the Petersen graph.*

The result of this paper is as follows.

Theorem 1.5 *Every 4-connected $\{K_{1,3}, Z_8\}$-free graph is either pancyclic or is the line graph of the Petersen graph.*

Notice that if a graph is P_{10}-free, it must be Z_8-free. Theorem 1.5 generalizes Theorem 1.4. The line graph of the Petersen graph is 4-connected and $\{K_{1,3}, Z_7\}$-free, but not Z_6-free, and it contains no cycle of length 4 (Fig. 1). This immediately implies the following corollary.

Corollary 1.6 *Every 4-connected $\{K_{1,3}, Z_6\}$-free graph is pancyclic.*

Corollary 1.7 *Every 5-connected $\{K_{1,3}, Z_8\}$-free graph is pancyclic.*

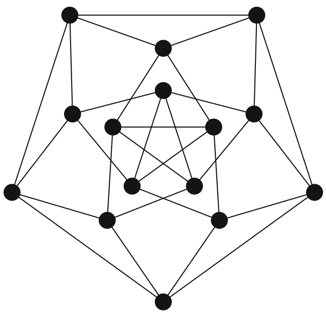

Fig. 1 The line graph of the Petersen graph is the unique 4-connected claw-free, Z_8-free graph that is not pancyclic

We would like to point out that the idea underlying our proofs comes from [3]. In Sect. 2, we will show that every 4-connected $\{K_{1,3}, Z_8\}$-free graph G contains cycles of all lengths from 10 to n by showing that if G contains a t-cycle ($t \geq 11$), then G also contains a $(t-1)$-cycle. The existence of a 9-cycle follows from the existence of 10-cycles, which will be given in Sect. 3. The existence of a 3-cycle follows immediately from the fact that G is claw-free. For 4-cycles, we use similar arguments based on the longest induced graphs Z_k. The proof of the existence of 4-cycles will be given in Sect. 4. The proof of the existence of t-cycles ($t = 5, 6, 7, 8$) will be given in Sect. 5.

2 Long Cycles

Let C be a cycle in G and $v \in V(C)$ and $u \notin V(C)$ such that $uv \in E(G)$. If C is hop-free, then we have either $uv^+ \in E(G)$ or $uv^- \in E(G)$ as G is claw-free. Let $x_1, x_2, \ldots, x_k \in V(C)$ lie on C along the orientation of C and let $w_1, w_2, \ldots, w_k$ be distinct vertices not in $V(C)$ so that $w_i x_i \in E(G)$. The *claw-extension* at $x_1, x_2, \ldots, x_k$ of C is the extension of C by inserting $w_1, w_2, \ldots, w_k$ into C one by one as follows.

For $i = 1, 2, \ldots, k$, do:

Cases	Methods
$x_{i+1} \neq x_i^+$ or $x_i w_{i+1} \notin E(G)$	Insert w_i into C by replacing $x_i^- x_i x_i^+$ by $x_i^- w_i x_i x_i^+$ or $x_i^- x_i w_i x_i^+$. Set $i := i+1$
$x_{i+1} = x_i^+$ and $x_i w_{i+1} \in E(G)$, and $x_i^- w_i \in E(G)$.	Insert w_i and w_{i+1} into C by replacing $x_i^- x_i x_{i+1}$ by $x_i^- w_i x_i w_{i+1} x_{i+1}$. Set $i := i+2$.
$x_{i+1} = x_i^+$ and $x_i w_{i+1} \in E(G)$, and $x_i^- w_i \notin E(G)$	Then $w_i x_{i+1} \in E(G)$. Consider $G[\{x_i, x_i^-, w_i, w_{i+1}\}]$, we have either $w_{i+1} x_i^- \in E(G)$, or $w_i w_{i+1} \in E(G)$. • If $w_{i+1} x_i^- \in E(G)$, insert w_i and w_{i+1} into C by replacing $x_i^- x_i x_{i+1}$ by $x_i^- w_{i+1} x_i w_i x_{i+1}$. Set $i := i+2$. • If $w_i w_{i+1} \in E(G)$, insert w_i and w_{i+1} into C by replacing $x_i x_{i+1}$ by $x_i w_i w_{i+1} x_{i+1}$. Set $i := i+2$.

Lemma 2.1 *Let G be a 4-connected $\{K_{1,3}, Z_8\}$-free graph of order n and let C be a cycle of length $t \geq 11$ in G. If G contains no $(t-1)$-cycles, then C contains a chord.*

Proof Suppose that C is chordless. Since G is 4-connected, C is not a hamiltonian cycle. Thus, for any $v \in V(C)$, there is a vertex $x \notin V(C)$ such that $vx \in E(G)$. As $v^+v^- \notin E(G)$, we have either $v^+x \in E(G)$ or $v^-x \in E(G)$. Without loss of generality, we assume that $xv^- \in E(G)$. Denote $u = v^-$. Then $uv \in E(C)$ and $G[\{x, u, v\}]$ is a clique in G.

Claim 1 $xv^+, xu^-, xv^{++}, xu^{--}, xv^{+3}, xu^{-3} \notin E(G)$.

Assume that $xv^+ \in E(G)$. Since G contains no $(t-1)$-cycles, $xv^{++}, xu^- \notin E(G)$. As G is claw-free and C is chordless, for any $z \in C(v^{++}, u^-)$, $xz \notin E(G)$. Thus the subgraph induced by $\{x, v, v^+\} \cup \{v^{++}, \ldots, v^{+9}\}$ is Z_8, a contradiction. This contradiction implies that $xv^+ \notin E(G)$. Similarly, $xu^- \notin E(G)$. As G contains no $(t-1)$-cycles, $xv^{++}, xu^{--}, xv^{+3}, xu^{-3} \notin E(G)$. Claim 1 holds.

Since G is Z_8-free and since C is chordless and $t \geq 11$, $N_G(x) \cap (V(C) - \{u, v\}) \neq \emptyset$. Let j be a positive integer so that $xv^+, xv^{++}, \ldots, xv^{+(j-1)} \notin E(G)$, and $xv^{+j} \in E(G)$. By Claim 1, $j \geq 4$. Choose $uv \in E(C)$ $(u = v^-)$ and $x \notin V(C)$ so that j is as small as possible.

Consider the neighborhoods of u, u^{--}, and u^{-3}. Since G is 4-connected, there exits a vertex $w_1 \notin V(C) \cup \{x\}$ such that $uw_1 \in E(G)$. Since G is claw-free, we have either $w_1v \in E(G)$ or $w_1u^- \in E(G)$. By Claim 1, $w_1u^{--}, w_1u^{-3} \notin E(G)$. As $xu^{--}, xu^{-3} \notin E(G)$, there are distinct vertices $w_2, w_3 \notin V(C) \cup \{x, w_1\}$ such that $w_2u^{--}, w_3u^{-3} \in E(G)$. If $xv^{+4} \in E(G)$, then the $(t-2)$-cycle $C[v^{+4}, v]xv^{+4}$ can be extended to a $(t-1)$-cycle via claw-extension at u^{--}; if $xv^{+5} \in E(G)$, then the $(t-3)$-cycle $C[v^{+5}, v]xv^{+5}$ can be extended to a $(t-1)$-cycle via claw-extensions at u^{--} and u^{-3}; if $xv^{+6} \in E(G)$, then the $(t-4)$-cycle $C[v^{+6}, v]xv^{+6}$ can be extended to a $(t-1)$-cycle via claw-extensions at u, u^{--} and u^{-3}. This implies that $j \geq 7$.

Consider the neighborhoods of u^{-5} and u^{-6}. By the choice of uv and x, $(N_G(u^{-5}) \cup N_G(u^{-6})) \cap \{w_1, w_2, w_3, x\} = \emptyset$. As G is 4-connected, there are distinct vertices $w_4, w_5 \notin V(C) \cup \{x, w_1, w_2, w_3\}$ such that $w_4u^{-5}, w_5u^{-6} \in E(G)$. If $xv^{+7} \in E(G)$, then the $(t-5)$-cycle $C[v^{+7}, v]xv^{+7}$ can be extended to a $(t-1)$-cycle via claw-extensions at u, u^{--}, u^{-3}, and u^{-5}; if $xv^{+8} \in E(G)$, then the $(t-6)$-cycle $C[v^{+8}, v]xv^{+8}$ can be extended to a $(t-1)$-cycle via claw-extensions at u, u^{--}, u^{-3}, u^{-5}, and u^{-6}. Therefore, $j \geq 9$. Thus the subgraph induced by $\{x, u, v\} \cup \{v^+, \ldots, v^{+8}\}$ is Z_8, a contradiction. □

Lemma 2.2 *Let G be a claw-free graph with minimum degree at least 4, let C be a cycle of length $t \geq 6$, and let X be the set of vertices in C that are not on any chord of C. If $x_1, x_2, \ldots, x_5 \in V(C) \cap X$, then $N_G(x_1) \cap N_G(x_2) \cap N_G(x_3) \cap N_G(x_4) \cap N_G(x_5) = \emptyset$.*

Proof Assume that $x_1, x_2, \ldots, x_5$ lie on C in order along the orientation of C. Since $|V(C)| \geq 6$, without loss of generality, we assume that $x_1x_5 \notin E(C)$. If $w \in N_G(x_1) \cap N_G(x_2) \cap N_G(x_3) \cap N_G(x_4) \cap N_G(x_5)$, then $G[\{w, x_1, x_3, x_5\}] = K_{1,3}$, a contradiction. □

Theorem 2.3 (Gould, Łuczak, Pfender, Lemma 3.1 in [4]) *Let G be a claw-free graph with minimum degree at least 3, let C be a cycle of length $t \geq 5$ without hops, and let*

X be the set of vertices in C that are not on any chord of C. If some chord xy of C satisfies $|X \cap C(x, y)| \leq 2$, then G contains cycles of lengths $t-1$ and $t-2$.

Let C be a cycle without hops in G, and let X be the set of vertices in C that are not on any chord of C. Let xy be a chord of C so that (i). $|C(x, y) \cap X|$ is minimum, and (ii). subject to (i), $|C[x, y]|$ is minimum.

In order to prove the following lemmas, we need the following technique to insert some vertices of $C(x, y)$ into the cycle $xC[y, x]$ along the orientation of C. Let $p \in C(x, y) - X$. Then, by the choice of xy, we conclude that p has a neighbor q in $C(y, x)$. Since G is claw-free and C is hop-free, we have either $pq^+ \in E(G)$ or $pq^- \in E(G)$. Without loss of generality, we may assume that $pq^+ \in E(G)$. Then we can insert p into $C(y, x)$ by replacing qq^+ with qpq^+. Such a vertex p is called an *insertable vertex*, and the edge qq^+ is called the *insertion edge* for p. If there are two vertices $p_1, p_2 \in C(x, y) - X$ such that ww^+ is the insertion edge for both p_1 and p_2, then vertices in the path $C[p_1, p_2]$ can be inserted into $C(y, x)$ by replacing ww^+ with $wC[p_1, p_2]w^+$. Such path $C[p_1, p_2]$ is called the *insertable path* with respect to the insertion edge ww^+. If there is no $p' \in C(x, p_1) \cup C(p_2, y) - X$ such that ww^+ is also the insertion edge for p', the path $C[p_1, p_2]$ is called the *maximal insertable path* in $C(x, y)$ with respect to the insertion edge ww^+. The path $C[p_1, p_2]$ is trivial if $p_1 = p_2$ (Fig. 2).

Let x_1 be the first vertex in $C(x, y) - X$ along the orientation of C. Then x_1 is an insertable vertex in $C(x, y)$ with respect to an insertion edge $w_1w_1^+$. Let $P_1 = C[x_1, y_1]$ be the maximal insertable path in $C(x, y)$ with respect to insertion edge $w_1w_1^+$. Let x_2 be the first vertex in $C(y_1, y) - X$ along the orientation of C. Then x_2 is an insertable vertex in $C(y_1, y)$ with respect to an insertion edge $w_2w_2^+$. By the choice of P_1, $w_2 \neq w_1$. Let $P_2 = C[x_2, y_2]$ be the maximal insertable path in $C(y_1, y)$ with respect to insertion edge $w_2w_2^+$. Repeat this process until $C(y_s, y) - X = \emptyset$. Now $P_1, P_2, \ldots, P_s$ are maximal insertable paths in $C(x, y), C(y_1, y), \ldots, C(y_{s-1}, y)$, with respect to insertion edges $w_1w_1^+, w_2w_2^+, \ldots, w_sw_s^+$, respectively. The set $\{P_1, P_2, \ldots, P_s\}$ is called a *maximal insertable path set* in $C(x, y)$. Denote by W the set of all vertices in these paths, then $C(x, y) - W \subseteq X$.

Lemma 2.4 *Let G be a claw-free graph with minimum degree at least 4, let C be a cycle of length $t \geq 6$ without hops, and let X be the set of vertices in C that are not on*

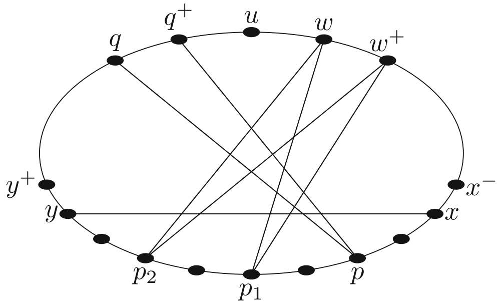

Fig. 2 Insertable vertices and insertable paths in $C(x, y)$

any chord of C. If some chord xy of C satisfies $|X \cap C(x, y)| \leq 4$, then G contains cycles of lengths $t-1$ and $t-2$.

Proof Choose the chord xy of C such that

(a) $|C(x, y) \cap X|$ is minimized.
(b) subject to Condition (a), $|C[x, y]|$ is minimized.

By Theorem 2.3, we assume that $|X \cap C(x, y)| \geq 3$. Thus $|C(x, y) \cap X| \in \{3, 4\}$. By Conditions (a) and (b), $yx^+, xy^- \notin E(G)$. As G is claw-free and C is hop-free, $xy^+, yx^- \in E(G)$. If $x^-y^+ \notin E(G)$, as $G[\{y, y^+, y^-, x^-\}] \neq K_{1,3}$, we have $x^-y^- \in E(G)$. Similarly, $x^+y^+ \in E(G)$. Thus the cycles $C[y^+, x^-]\overleftarrow{C}[y^-, x^+]y^+$ and $C[y^+, x^-]\overleftarrow{C}[y^-, x]y^+$ are cycles of lengths $t-2$ and $t-1$, respectively. Therefore, we assume $x^-y^+ \in E(G)$.

If $C(x, y) - X \neq \emptyset$, then let $\{P_1, \ldots, P_s\}$ be a maximal insertable path set in $C(x, y)$. Denote by W the set of all vertices in these paths. Assume that C' is the cycle obtained by inserting vertices of W into the cycle $xC[y, x]$. Then $C(x, y) - W \neq \emptyset$ (otherwise, the cycles $C'[y^+, x]y^+$ and $C'[y^+, x^-]y^+$ are cycles of lengths $t-1$ and $t-2$). Let $X' = C(x, y) - W$. Then $X' \subseteq X$ and $|C(y, x) \cap X| \geq |X'|$. Let $k = |X'|$. Then the length of the cycle C' is $|V(C)| - k = t - k$, and $|C(y, x) \cap X| \geq |C(x, y) \cap X| \geq k$.

If $k = 1$, then the cycles C' and $C'[y, x^-]y$ are cycles of lengths $t-1$ and $t-2$. If $k = 2$, then C' is a $(t-2)$-cycle. Let $x_0 \in C(y, x) \cap X$, then the $(t-2)$-cycle C' can be extended to a $(t-1)$-cycle via claw-extension at x_0. If $k = 3$, note that $|C(y, x) \cap X| \geq k = 3$. Let $y_1, y_2, y_3 \in C(y, x) \cap X$. Since $\delta(G) \geq 4$, there are vertices $w_1, w_3 \notin V(C)$ such that $y_1w_1, y_3w_3 \in E(G)$. Then the $(t-3)$-cycle C' can be extended to a $(t-1)$-cycle via claw-extensions at y_1, y_3, and to a $(t-2)$-cycle via claw-extension at y_1, Thus $k = 4$. Assume $C(x, y) - W = \{x_1, x_2, x_3, x_4\}$ and x_1, x_2, x_3, x_4 are labeled with respect to the orientation of C.

Let $C(y, x) \cap X = \{y_1, y_2, \ldots, y_m\}$ be the set of vertices labeled with respect to the orientation of C (as well as the orientation of C'). As each of $y_i (i = 1, 2, \ldots, m)$ has at least two neighbors not on C, let $w_1y_1, w_2y_2 \in E(G)$, where $w_1, w_2 \notin V(C)$. Then the $(t-4)$-cycle C' can be extended to a $(t-2)$-cycle via claw-extensions at y_1 and y_2. Next we will find a $(t-1)$-cycle in G.

If $N_G(\{y_3, \ldots, y_m\}) - \{w_1, w_2\} \neq \emptyset$, say $w_3y_3 \in E(G)$, then the $(t-4)$-cycle C' can be extended to a $(t-1)$-cycle via claw-extensions at y_1, y_2 and y_3. Therefore, we assume $N_G(\{y_3, \ldots, y_m\}) = \{w_1, w_2\}$. Then $w_1y_i, w_2y_i \in E(G)$ for $i = 3, \ldots, m$. By Lemma 2.2, $m = 4$. By the minimality of xy, $|C(x, y) \cap X| = 4$, and so $|V(C) \cap X| = 8$.

If $(N_G(y_1) - V(C)) - \{w_1, w_2\} \neq \emptyset$, then there exists $w_4 \in N_G(y_1) - (V(C) \cup \{w_1, w_2\})$ such that $y_1w_4 \in E(G)$. As $w_1y_3, w_2y_4 \in E(G)$, the $(t-4)$-cycle C' can be extended to a $(t-1)$-cycle via claw-extensions at y_1, y_3 and y_4. So we may assume that $N_G(y_1) - V(C) = \{w_1, w_2\}$. Similarly, $N_G(y_i) - V(C) = \{w_1, w_2\} (i = 2, 3, 4)$. As G is claw-free, $y_2 = y_1^+$ and $y_4 = y_3^+$, but $|C(y_2, y_3)| \geq 1$ (otherwise, the cycle $C[y_4, y_1]w_1y_4$ is a $(t-1)$-cycle).

Consider y_2^+. Then y_2^+ is an endpoint of a chord on C. Let y_2' be the other endpoint of this chord. By the minimality of xy and $|V(C) \cap X| = 8$, we have $y_2' \in C(x_2, x_3)$.

Without loss of generality, we assume that y_2' is the last vertex in $C(x_2, x_3)$ adjacent to y_2^+. Then $y_2'y_2^+$ is the only chord that joins a pair of vertices in $C[y_2', y_2^+]$ and $|C(y_2', y_2^+) \cap X| = 4$. Thus the chord $y_2'y_2^+$ also satisfies Conditions (a) and (b). Applying the same discussion mentioned above on the chord $y_2'y_2^+$ instead of xy, we have $N_G(x_1) - V(C) = \{w_1, w_2\}$ and $N_G(x_2) - V(C) = \{w_1, w_2\}$, contradicting Lemma 2.2. □

Lemma 2.5 *Let G be a 4-connected $\{K_{1,3}, Z_8\}$-free graph. If G contains a cycle of length $t \geq 11$, then G contains a cycle of length $t-1$.*

Proof Let C be a cycle of length t in G and suppose that G contains no $(t-1)$-cycles. Then C does not contain hops. By Lemma 2.1, C contains at least one chord. Let X be the set of vertices of C that are not endpoints of chords of C. Let xy be a chord of C. Then, by Lemma 2.4, $|X \cap C(x, y)| \geq 5$. Choose xy such that

(a) $|C(x, y) \cap X|$ is minimized.
(b) subject to Condition (a), $|C(x, y)|$ is minimized. Therefore, xy is the only chord that joins a pair of vertices in $C[x, y]$.

Claim 1 $xy^+, yx^-, x^-y^+ \in E(G)$, and $zx^-, zy^+ \notin E(G)$ for any $z \in C(x, y)$.

By Conditions (a) and (b), $yx^+, xy^- \notin E(G)$. As G is claw-free and C is hop-free, $xy^+, yx^- \in E(G)$. If $x^+y^+ \in E(G)$, then the cycle $C[x^+, y]\overleftarrow{C}[x^-, y^+]x^+$ is a $(t-1)$-cycle, a contradiction. Thus $x^+y^+ \notin E(G)$. Similarly, $x^-y^- \notin E(G)$. Since $G[\{x, y^+, x^-, x^+\}]$ is not a claw, $x^-y^+ \in E(G)$. By Conditions (a) and (b), $y^+z \notin E(G)$ for $z \in C(x^+, y)$, and $x^-z \notin E(G)$ for $z \in C(x, y^-)$. Claim 1 holds.

Claim 2 Let $x_1, x_2, x_3, x_4 \in C(y, x) \cap X$. Then $N_G(x_1) \cap N_G(x_2) \cap N_G(x_3) \cap N_G(x_4) = \emptyset$.

We assume that $w \in N_G(x_1) \cap N_G(x_2) \cap N_G(x_3) \cap N_G(x_4)$. We also assume that x_1, x_2, x_3, x_4 lie on C in order along the orientation of C. By Claim 1, $|C(x_4, x_1)| \geq |C(x, y)| + |\{x, x^-, y, y^+\}| \geq 9$. As G is claw-free and C is hop-free, $x_2 = x_1^+$ and $x_4 = x_3^+$, and $|C(x_2, x_3)| \geq 3$. Consider the subgraph induced by $\{x_3, x_4, w\} \cup \{x_1, x_1^-, x_1^{--}, \ldots, x_1^{-7}\}$. Then $wz \notin E(G)$ for $z \in \{x_1^-, \ldots, x_1^{-7}\}$ (Otherwise, $G[\{w, z, x_2, x_3\}] = K_{1,3}$, a contradiction). Since $G[\{x_3, x_4, w\} \cup \{x_1, x_1^-, \ldots, x_1^{-7}\}]$ is not Z_8, $G[\{x_1^-, \ldots, x_1^{-7}\}]$ contains an edge. Since $|C(x, y)| \geq 5$, by minimality of xy, $x_1^-x_1^{-7} \in E(G)$ but $x_1^{--}x_1^{-7} \notin E(G)$. Thus $G[\{x_1^-, x_1, x_1^{--}, x_1^{-7}\}] = K_{1,3}$, a contradiction. Claim 2 holds.

Claim 3 $|C(x, y)| \geq 6$.

By way of contradiction, assume that $|C(x, y)| \leq 5$. By Lemma 2.4, $|C(x, y)| = |C(x, y) \cap X| = 5$. As $|C(y, x) \cap X| \geq 5$, let $x_1, x_2, \ldots, x_5 \in C(y, x) \cap X$. Consider the bipartite graph H with partitions $\{x_1, x_2, x_3, x_4, x_5\}$ and $\bigcup_{i=1}^{5} \mathbf{N}_G(x_i) - C$. As each x_i has at least two neighbors not in C, by Claim 2, $|N_H(S)| \geq |S| - 1$ for any $S \subseteq \{x_1, x_2, \ldots, x_5\}$. Thus H has a matching M with 4 edges. Without loss of generality, we assume that $\{x_1, x_2, x_3, x_4\} \subseteq V(M)$. Then the $(t-5)$-cycle $xC[y, x]$ can be extended to a $(t-1)$-cycle via claw-extensions at x_1, x_2, x_3, and x_4. Claim 3 holds.

Claim 4 Let $x_1, x_2, x_3 \in C(y, x) \cap X$. Then $N_G(x_1) \cap N_G(x_2) \cap N_G(x_3) = \emptyset$.

Assume that $w \in N_G(x_1) \cap N_G(x_2) \cap N_G(x_3)$. Also we assume that x_1, x_2, x_3 lie on the cycle C in the order along the orientation of C. As G is claw-free and $x_1, x_2, x_3 \in X$, we have either $x_2 = x_1^+$ or $x_3 = x_2^+$. Without loss of generality, we assume that $x_2 = x_1^+$. By Claim 3, $|\overleftarrow{C}(x_1, x_3)| \geq |C(x, y)| + |\{x, x^-, y, y^+\}| \geq 10$. Since $x_1, x_2, x_3 \in X$ and G is claw-free, we have $x_2 x_1^- \notin E(G)$ and $zx_1, zx_2, zw \notin E(G)$ for $z \in \{x_1^{--}, x_1^{-3}, \ldots, x_1^{-8}\}$.

If $G[\{x_1^-, x_1^{--}, \ldots, x_1^{-8}\}]$ contains a chord, by Claim 3 and the minimality of xy, $x_1^- x_1^{-8} \in E(G)$ but $x_1^{--} x_1^{-8} \notin E(G)$. Thus $G[\{x_1^-, x_1, x_1^{--}, x_1^{-8}\}] = K_{1,3}$, a contradiction. Hence, $G[\{x_1^-, x_1^{--}, \ldots, x_1^{-8}\}] = P_8$. As $G[\{w, x_1, x_2\} \cup \{x_1^-, x_1^{--}, \ldots, x_1^{-8}\}]$ is not Z_8, $wx_1^- \in E(G)$. It implies that $x_3 \neq x_2^+$ (otherwise, the cycle $C[x_3, x_1^-]wx_3$ is a $(t-1)$-cycle, a contradiction). Therefore, $G[\{w, x_1^-, x_2, x_3\}] = K_{1,3}$, a contradiction. Claim 4 holds.

Let $\{P_1, \ldots, P_s\}$ be a maximal insertable path set in $C(x, y)$. Denote by W the set of all vertices in these paths. Assume that C' is the cycle obtained by inserting vertices of W into the cycle $xC[y, x]$. Then $C(x, y) - W \neq \emptyset$ (otherwise, the cycle $C'[y^+, x]y^+$ is a $(t-1)$-cycle). Let $X' = C(x, y) - W$. Then $X' \subseteq X$ and $|C(y, x) \cap X| \geq |X'|$. Let $k = |X'|$. Then the length of the cycle C' is $|V(C)| - k = t - k$, and so $k \geq 2$.

As $|X \cap C(x, y)| \geq |C(x, y) - W| \geq k$, by Condition (a), $|C(y, x) \cap X| \geq k$. Let $x_1, x_2, \ldots, x_k \in C(y, x) \cap X$ and they occur on C in order along the orientation of C. Obviously, $x_1, x_2, \ldots, x_k$ are not endpoints of insertion edges. Since G is 4-connected, we assume that $u_i, v_i \notin C$ are adjacent to x_i. Consider the bipartite graph H with partitions $\{x_1, x_2, \ldots, x_k\}$ and $\bigcup_{i=1}^{k}\{u_i, v_i\}$. By Claim 4, for any $S \subseteq \{x_1, x_2, \ldots, x_k\}$, $|N_H(S)| \geq |S|$. Thus H has a matching M covering $C(y, x) \cap X$. Assume that $M = \{x_1 w_1, x_2 w_2, \ldots, x_k w_k\}$. Then the $(t-k)$-cycle C' can be extended a $(t-1)$-cycle via claw-extensions at $x_1, x_2, \ldots, x_{k-1}$, a contradiction. □

Theorem 2.6 (Lai et al. [5]) *Every 3-connected $\{K_{1,3}, Z_8\}$-free graph is hamiltonian.*

By Lemmas 2.5 and Theorem 2.6, G contains cycles of lengths 10 through $|V(G)|$.

3 Existence of 9-Cycles

Lemma 3.1 *If G is a 4-connected $\{K_{1,3}, Z_8\}$-free graph, then G contains a 9-cycle.*

Proof Suppose that G does not contain a 9-cycle. By Lemma 2.5 and Theorem 2.6, G contains a 10-cycle C, and we let $\{v_1, v_2, \ldots, v_{10}\}$ be the vertex set of C labeled in order. By Lemma 2.4, C is chordless.

Claim 1 Let $a \notin V(C)$ have a neighbor in $V(C)$. Then $|N_G(a) \cap V(C)| \leq 3$. Moreover, if $|N_G(a) \cap V(C)| = 3$, then these three vertices are consecutive on C.

Since $a \notin V(C)$ has a neighbor in $V(C)$, we assume $av_1 \in E(G)$. As G is claw-free and has no chords of C, either $av_2 \in E(G)$ or $av_{10} \in E(G)$. Without loss of generality, we assume that $av_{10} \in E(G)$. As G has no 9-cycles, $N_G(a) \cap \{v_3, v_4, v_7, v_8\} = \emptyset$. Thus $N_G(a) \cap V(C) \subseteq \{v_1, v_{10}, v_2, v_9, v_5, v_6\}$.

If $av_5 \in E(G)$, then $av_6 \in E(G)$ since G is claw-free and C is chordless. Since $av_3 \notin E(G)$, let $b \in N_G(v_3)$ such that $b \notin V(C) \cup \{a\}$. Then the 8-cycle $v_{10}v_1v_2v_3v_4v_5v_6av_{10}$ can be extended to a 9-cycle via claw-extension at v_3. This tells us that $av_5 \notin E(G)$ and so $av_6 \notin E(G)$. Therefore, $N_G(a) \cap V(C) \subseteq \{v_1, v_{10}, v_2, v_9\}$.

If both $av_2 \in E(G)$ and $av_9 \in E(G)$, then the cycle $v_2v_3v_4v_5v_6v_7v_8v_9av_2$ is a 9-cycle. Thus we have $N_G(a) \cap V(C) \in \{\{v_1, v_{10}, v_2\}, \{v_1, v_{10}, v_9\}, \{v_1, v_{10}\}\}$. Claim 1 holds.

Claim 2 There is a vertex $a \notin V(C)$ such that $|N_G(a) \cap V(C)| = 2$.

By way of contradiction, we assume that for any $a \notin V(C)$, $|N_G(a) \cap V(C)| \neq 2$. By Claim 1, every vertex with a neighbor on C has exactly three neighbors on C which are consecutive. For $1 \leq i \leq 10$, let $V_i = N_G(v_{i-1}) \cap N_G(v_i) \cap N_G(v_{i+1})$, where indices are taken modulo 10. If there is a vertex $w \notin V(C) \cup \bigcup_{i=1}^{10} V_i$ that has a neighbor w_i in some V_i, then $\{w_i, v_{i-1}, v_{i+1}, w\}$ induces a claw. Thus we may assume that the sets $V_1, V_2, \ldots, V_{10}$ partition $V(G) \setminus V(C)$. If there is an edge joining V_i and V_j when $|i - j| \geq 2 \pmod{10}$, then G contains a 9-cycle. If there are two nonconsecutive values $i < j$ such that V_i and V_j are empty, then $\{v_i, v_j\}$ is a cut set, a contradiction. Thus for some $1 \leq i \leq 10$, the sets V_i, V_{i+1}, V_{i+2}, and V_{i+3} are all non-empty. Let w_j be any vertex in V_j for $i \leq j \leq i+3$. It follows that $v_iw_iv_{i+1}w_{i+2}v_{i+3}v_{i+4}w_{i+3}v_{i+2}w_{i+1}v_i$ is a 9-cycle. Claim 2 holds.

By Claim 2, let $N_G(x_1) \cap V(C) = \{v_1, v_2\}$. Since G is 4-connected, let $\{y_1, y_2, v_1, v_2\} \subseteq N_G(x_1)$. As G has no 9-cycles, $N_G(w) \cap \{v_5, v_6, v_7, v_8\} = \emptyset$ for $w \in N_G(x_1) - \{v_1, v_2\}$.

Claim 3 For any $w \in N_G(x_1) - \{v_1, v_2\}$, $N_G(w) \cap \{v_3, v_4, v_9, v_{10}\} \neq \emptyset$.

By way of contradiction, assume that $N_G(y_1) \cap \{v_3, v_4, v_9, v_{10}\} = \emptyset$. If $y_1v_2 \in E(G)$, then the subgraph induced by $\{x_1, y_1, v_2\} \cup \{v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ is Z_8. Thus $y_1v_2 \notin E(G)$. Similarly, $y_1v_1 \notin E(G)$, and therefore $N_G(y_1) \cap V(C) = \emptyset$. As G has no 9-cycles, $N_G(w) \cap \{v_6, v_7\} = \emptyset$ for any $w \in N_G(y_1) - \{x_1\}$.

Claim 3.1 For any $w \in N_G(x_1) - \{v_1, v_2, y_1\}$, $N_G(w) \cap \{v_3, v_4, v_9, v_{10}\} \neq \emptyset$.

Otherwise, by the discussion above, $wv_1, wv_2 \notin E(G)$. As G is claw-free, $y_1w \in E(G)$. Thus the subgraph induced by $\{x_1, y_1, w\} \cup \{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ is Z_8, a contradiction. Claim 3.1 holds.

Claim 3.2 Let $z \in N_G(y_1) - \{x_1\}$. Then $N_G(z) \cap \{v_5, v_8\} = \emptyset$.

By way of contradiction, we assume that $zv_8 \in E(G)$. As $N_G(y_1) \cap V(C) = \emptyset$ and $N_G(x_1) \cap V(C) = \{v_1, v_2\}$, and as G is 4-connected, there is a vertex $y_9' \notin V(C) \cup \{x_1, y_1, z\}$ such that $v_9v_9' \in E(G)$. Then the 8-cycle $v_2x_1y_1zv_8v_9v_{10}v_1v_2$ can be extended to a 9-cycle via claw-extension at v_9, a contradiction. Therefore, Claim 3.2 holds.

Claim 3.3 Let $z \in N_G(y_1) - \{x_1\}$. Then $N_G(z) \cap \{v_4, v_9\} = \emptyset$.

By way of contradiction, we assume that $zv_9 \in E(G)$. As $zv_8 \notin E(G)$, $zv_{10} \in E(G)$. By Claim 1, $N_G(z) \subseteq \{v_9, v_{10}, v_1\}$. Considering the subgraph induced by $\{z, v_9, v_{10}\} \cup \{v_8, v_7, v_6, v_5, v_4, v_3, v_2, x_1\}$, we have $zx_1 \in E(G)$.

Consider the neighborhood of v_3. As $N_G(v_3) \cap \{x_1, y_1, z\} = \emptyset$, there is a vertex $v_3' \in N_G(v_3)$ such that $v_3' \notin V(C) \cup \{x_1, y_1, z\}$. As G has no 9-cycles, $v_3'x_1, v_3'y_1, v_3'v_{10} \notin E(G)$. As $x_1v_{10} \notin E(G)$ and as G is claw-free, $v_3'z \notin E(G)$. Since the subgraph induced by $\{x_1, y_1, z\} \cup \{v_9, v_8, v_7, v_6, v_5, v_4, v_3, v_3'\}$ is not Z_8, $v_3'v_4 \in E(G)$. If $v_3'v_5 \notin E(G)$, then the subgraph induced by $\{v_3', v_3, v_4\} \cup \{v_5, v_6, v_7, v_8, v_9, v_{10}, v_1, x_1\}$ is Z_8; if $v_3'v_5 \in E(G)$, then the subgraph induced by $\{v_3', v_4, v_5\} \cup \{v_6, v_7, v_8, v_9, v_{10}, v_1, x_1, y_1\}$ is Z_8, a contradiction. Claim 3.3 holds.

Claim 3.4 There exist at least two vertices $z \in N_G(y_1) - \{x_1\}$ such that $N_G(z) \cap \{v_3, v_{10}\} \neq \emptyset$.

By way of contradiction, assume that there is at most one vertex $z \in N_G(y_1) - \{x_1\}$ such that $N_G(z) \cap V(C) \cap \{v_3, v_{10}\} \neq \emptyset$. Since G is 4-connected, there are at least two vertices $z_1, z_2 \in N_G(y_1) - \{x_1\}$ such that $N_G(z_1) \cap \{v_3, v_{10}\} = \emptyset$ and $N_G(z_2) \cap \{v_3, v_{10}\} = \emptyset$. By Claim 3.3, $N_G(z_1) \cap \{v_3, v_4, v_9, v_{10}\} = \emptyset$ and $N_G(z_2) \cap \{v_3, v_4, v_9, v_{10}\} = \emptyset$. By Claim 3.1, $z_1x_1, z_2x_1 \notin E(G)$. Thus $z_1z_2 \in E(G)$. As $G[\{v_2, v_3, x_1, z_1\}] \neq K_{1,3}$, we have $z_1v_2 \notin E(G)$. Similarly, $z_2v_2 \notin E(G)$. Therefore, the subgraph induced by $\{y_1, z_1, z_2\} \cup \{x_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ is Z_8, a contradiction. Claim 3.4 holds.

By Claim 3.4, we assume that $z_1, z_2 \in N_G(y_1) - \{x_1\}$ with $N_G(z_1) \cap \{v_3, v_{10}\} \neq \emptyset$ and $N_G(z_2) \cap \{v_3, v_{10}\} \neq \emptyset$. Without loss of generality, we assume that $z_1v_{10} \in E(G)$. Then $z_1v_1 \in E(G)$. By Claim 1, $z_1v_3 \notin E(G)$. If $z_1x_1 \in E(G)$, then the subgraph induced by $\{x_1, y_1, z_1\} \cup \{v_{10}, v_9, v_8, v_7, v_6, v_5, v_4, v_3\}$ would be Z_8. This contradiction implies that $z_1x_1 \notin E(G)$. Similarly, $z_2x_1 \notin E(G)$ and so $z_1z_2 \in E(G)$. Since $G[\{v_2, x_1, v_3, z_1\}]$ is not a claw, $z_1v_2 \notin E(G)$. Then $N_G(z_1) \cap V(C) = \{v_1, v_{10}\}$.

Consider the neighborhood of z_2. If $z_2v_3 \notin E(G)$, then $z_2v_{10} \in E(G)$, and so $N_G(z_2) \cap V(C) = \{v_1, v_{10}\}$. It implies that the subgraph induced by $\{y_1, z_1, z_2\} \cup \{x_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ is Z_8. This contradiction tells us that $z_2v_3 \in E(G)$. Thus $N_G(z_2) \cap V(C) = \{v_2, v_3\}$.

We will finish the proof of Claim 3 by considering the neighborhood of x_1. As $N_G(x_1) \cap V(C) = \{v_1, v_2\}$ and $z_1x_1, z_2x_1 \notin E(G)$, there is a vertex $y_2 \in N_G(x_1)$ such that $y_2 \notin V(C) \cup \{y_1, z_1, z_2\}$. By Claim 3.1, $N_G(y_2) \cap \{v_3, v_4, v_9, v_{10}\} \neq \emptyset$. By symmetry, we assume that either $y_2v_4 \in E(G)$ or $y_2v_3 \in E(G)$. If $y_2v_4 \in E(G)$, then the cycle $v_4y_2x_1y_1z_2z_1v_1v_2v_3v_4$ is a 9-cycle; if $y_2v_3 \in E(G)$, then the cycle $v_3y_2x_1y_1z_2z_1v_{10}v_1v_2v_3$ is a 9-cycle. This contradiction finishes the proof of Claim 3.

Claim 4 For any $w \in N_G(x_1) - \{v_1, v_2\}$, $N_G(w) \cap \{v_4, v_9\} = \emptyset$. Therefore, $N_G(w) \cap \{v_3, v_{10}\} \neq \emptyset$.

By way of contradiction, we assume $y_1, y_2 \in N_G(x_1) - \{v_1, v_2\}$ and $y_1v_9 \in E(G)$. Then $y_1v_{10} \in E(G)$ since $y_1v_8 \notin E(G)$. By Claim 1, $y_1v_2 \notin E(G)$. As G has no 9-cycles, $y_2v_4 \notin E(G)$. If $y_2v_3 \in E(G)$, then we consider the 8-cycle $C' = v_9v_{10}v_1v_2v_3y_2x_1y_1v_9$. As G is 4-connected, there is a vertex $a \notin V(C')$ so that a is adjacent to one of $V(C') - \{v_3, v_9, x_1\}$. If $ay_2 \in E(G)$, then either $av_3 \in E(G)$ or $ax_1 \in E(G)$. Thus C' can be extended to a 9-cycle by replacing $v_3y_2x_1$ to be $v_3ay_2x_1$ or $v_3y_2ax_1$. If a is adjacent to any other vertex in $V(C') - \{v_3, v_9, x_1\}$, we can still use this method to insert a into C' to get a 9-cycle. This contradiction implies that $y_2v_3 \notin E(G)$.

Next we will prove that $wv_2 \notin E(G)$ for any $w \in N_G(x_1) - \{v_1, v_2\}$. By way of contradiction, we may assume that $y_2v_2 \in E(G)$. Then $y_2v_1 \in E(G)$. By Claims 1 and 3, $y_2v_{10} \in E(G)$ and $y_2v_9 \notin E(G)$. Since the subgraph induced by $\{v_1, x_1, y_2\} \cup \{y_1, v_9, v_8, v_7, v_6, v_5, v_4, v_3\}$ is not Z_8, we have either $y_1y_2 \in E(G)$ or $y_1v_1 \in E(G)$. Since $d_G(v_9) \geq 4$, let $y_9' \in N_G(v_9) - (V(C) \cup \{y_1, y_2, x_1\})$. If $v_9'v_8 \in E(G)$, as G has no 9-cycles, $N_G(v_9') \cap \{y_1, y_2, v_{10}, v_1\} = \emptyset$. Since the subgraph induced by

$$\begin{cases} \{y_1, v_1, v_{10}\} \cup \{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9'\}, & \text{if } y_1v_1 \in E(G) \\ \{y_1, y_2, v_{10}\} \cup \{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9'\}, & \text{if } y_1y_2 \in E(G) \end{cases},$$

is not Z_8, we have $v_9'v_7 \in E(G)$. Thus the subgraph induced by $\{v_7, v_9', v_8\} \cup \{v_6, v_5, v_4, v_3, v_2, x_1, y_1, v_{10}\}$ is Z_8. This contradiction implies that $v_9'v_8 \notin E(G)$. Thus $v_9'v_{10} \in E(G)$. As $G[\{v_9, v_9', y_1, v_8\}] \neq K_{1,3}$, $y_1v_9' \in E(G)$. Let H be a subgraph induced by $\{v_1, v_2, v_{10}, v_9, v_9', y_1, y_2, x_1\}$. Since G is 4-connected, there is a vertex b adjacent to a vertex in $V(H) - \{v_2, v_9, v_9'\}$. If $by_2 \in E(G)$, by $G[\{y_2, b, v_2, v_{10}\}]$, we have either $bv_2 \in E(G)$ or $bv_{10} \in E(G)$. Thus

$$C' = \begin{cases} v_2by_2x_1y_1v_9'v_9v_{10}v_1v_2, & \text{if } bv_2 \in E(G) \\ v_2y_2bv_{10}v_9v_9'y_1x_1v_1v_2, & \text{if } bv_{10} \in E(G) \end{cases}$$

is a 9-cycle in G. If b is adjacent to any other vertex in $V(H) - \{v_2, v_9, v_9'\}$, we can still use this method to insert b into H to get a 9-cycle. This contradiction implies that $wv_2 \notin E(G)$ for any $w \in N_G(x_1) - \{v_1, v_2\}$.

As $y_1v_2 \notin E(G)$, we have $y_1y_2 \in E(G)$. By Claim 3, we have $y_2v_{10} \in E(G)$. Let $v_8' \in N_G(v_8)$ such that $v_8' \notin V(C) \cup \{y_1, y_2, x_1\}$. Obviously, $x_1, y_1, y_2 \notin N_G(v_8')$. Considering the subgraph induced by $\{x_1, y_1, y_2\} \cup \{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_8'\}$, we have $v_8'v_7 \in E(G)$. By the subgraph induced by $\{v_7, v_8, v_8'\} \cup \{v_6, v_5, v_4, v_3, v_2, x_1, y_2, v_{10}\}$, we have $v_8'v_6 \in E(G)$. Since the subgraph induced by $\{v_6, v_7, v_8'\} \cup \{v_5, v_4, v_3, v_2, x_1, y_2, v_{10}, v_9\}$ is not Z_8, $y_2v_9 \in E(G)$. Again, since the subgraph induced by $\{v_9, y_1, y_2\} \cup \{v_8, v_7, v_6, v_5, v_4, v_3, v_2, v_1\}$ is not Z_8, we have either $y_2v_1 \in E(G)$ or $y_1v_1 \in E(G)$. By symmetry, we assume that $y_2v_1 \in E(G)$.

Consider the neighborhood of v_2. As $y_1v_2, y_2v_2 \notin E(G)$, let $v_2' \in N_G(v_2)$ such that $v_2' \notin V(C) \cup \{y_1, y_2, x_1\}$. As $wv_2 \notin E(G)$ for any $w \in N_G(x_1) - \{v_1, v_2\}$, $v_2'x_1 \notin E(G)$. Since $G[\{v_2, v_2', v_3, x_1\}]$ is not a claw, $v_2'v_3 \in E(G)$. Thus $v_2'v_1, v_2'y_1, v_2'y_2 \notin E(G)$ since G has no 9-cycles. By the subgraph induced by $\{x_1, v_1, y_2\} \cup \{v_9, v_8, v_7, v_6, v_5, v_4, v_3, v_2'\}$, we have $v_2'v_4 \in E(G)$. By Claim 1, $v_2'v_5 \notin E(G)$. Thus the subgraph induced by $\{v_3, v_4, v_2'\} \cup \{v_5, v_6, v_7, v_8, v_9, v_{10}, v_1, x_1\}$ is Z_8, a contradiction. Claim 4 holds.

By Claim 4, for any $w \in N_G(x_1)$, either $wv_{10} \in E(G)$ or $wv_3 \in E(G)$. If there are two vertices, say $y_1, y_2 \in N_G(x_1) - \{v_1, v_2\}$, such that $y_1v_{10}, y_2v_3 \in E(G)$. Then $y_1v_1, y_2v_2 \in E(G)$. Let H be the subgraph induced by $\{v_{10}, v_1, v_2, v_3, x_1, y_1, y_2\}$. Since G is 4-connected, there are two vertices q_1, q_2 such that $q_1, q_2 \notin V(H)$ adjacent to different vertices in $V(H) - \{v_3, v_{10}\}$. Since G is claw-free, by Claim 4, $N_G(q_i) \cap \{v_3, v_{10}\} \neq \emptyset (i = 1, 2)$. By symmetry, we assume that $q_1v_{10} \in E(G)$. Then $q_1v_9 \notin E(G)$ (otherwise, the subgraph induced by $V(H) \cup \{q_1, v_9\}$ contains a 9-cycle). Thus $q_1v_1 \in E(G)$. Using this discussion on q_2, we have either $q_2v_3, q_2v_2 \in E(G)$

or $q_2v_{10}, q_2v_1 \in E(G)$. If $q_2v_3, q_2v_2 \in E(G)$, then $v_{10}y_1x_1y_2v_3q_2v_2v_1q_1v_{10}$ is a 9-cycle; if $q_2v_{10}, q_2v_1 \in E(G)$, then $q_1q_2 \in E(G)$ (otherwise, $G[\{v_{10}, q_1, q_2, v_9\}]$ is a claw), and so $v_{10}q_2q_1v_1v_2v_3y_2x_1y_1v_{10}$ is a 9-cycle. This contradiction implies that either $N_G(v_3) \cap (N_G(x_1) - \{v_1, v_2\}) = \emptyset$ or $N_G(v_{10}) \cap (N_G(x_1) - \{v_1, v_2\}) = \emptyset$. Without loss of generality, we assume that $N_G(v_3) \cap (N_G(x_1) - \{v_1, v_2\}) = \emptyset$. Thus for any $w \in N_G(x_1) - \{v_1, v_2\}$, $N_G(w) \cap (V(C) - \{v_1, v_2\}) = \{v_{10}\}$.

Consider the neighborhood of x_1, and let $N_G(x_1) = \{v_1, v_2, y_1, y_2, \ldots, y_k\}(k \geq 2)$. Then $y_iv_{10} \in E(G)(i = 1, 2, \ldots, k)$. By Claim 4, the subgraph induced by $\{y_1, y_2, \ldots, y_k\}$ is a clique, and $y_iv_1 \in E(G)(i = 1, 2, \ldots, k)$. Let H' be the subgraph induced by $N_G(x_1) \cup \{x_1, v_{10}\}$. Since G is 4-connected, there are at least two vertices $q_3, q_4 \notin V(H')$ adjacent to different vertices in $V(H') - \{v_2, v_{10}\}$. Since G is claw-free, by Claim 4, $q_3v_{10}, q_4v_{10} \in E(G)$. If $k \geq 3$, then $q_3v_9 \notin E(G)$ (otherwise, the subgraph induced by $V(H') \cup \{q_3, v_9\}$ contains a 9-cycle). Similarly, $q_4v_9 \notin E(G)$. Thus $q_3q_4, q_3v_1, q_4v_1 \in E(G)$, and so $v_{10}q_3q_4v_1v_2x_1y_1y_2y_3v_{10}$ is a 9-cycle. This contradiction implies that $k = 2$ and $N_G(x_1) = \{v_1, v_2, y_1, y_2\}$. Notice that q_3, q_4 are adjacent to different vertices in $\{y_1, y_2, v_1\}$. By symmetry, we have either $q_3y_1, q_4v_1 \in E(G)$, or $q_3y_1, q_4y_2 \in E(G)$. For each of these two cases, $q_3v_9, q_4v_9 \notin E(G)$ since G has no 9-cycles. Therefore, $q_3q_4 \in E(G)$ and $\{y_1, y_2, v_1\} \subseteq N_G(q_i)$ for $i = 3, 4$.

Since the subgraph induced by $\{q_3, q_4, v_1\} \cup \{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ is not Z_8, we have either $q_3v_2 \in E(G)$ or $q_4v_2 \in E(G)$. By symmetry, we assume that $q_3v_2 \in E(G)$. Since G has no 9-cycles, for any $x \in \{y_1, y_2, x_1, v_1, q_3\}$, $N_G(x) \subseteq H' \cup \{q_3, q_4\}$. This implies that $\{v_1, v_{10}, q_4\}$ is a 3-cut, a contradiction. □

4 Existence of 4-Cycles

In this section we will prove that if G is a 4-connected, claw-free and Z_8-free graph, then G is the line graph of the Petersen graph if G has no 4-cycles. Suppose that G is a 4-connected, claw-free and Z_8-free graph and that G does not have 4-cycles. Since G is claw-free, the neighborhood of every vertex is either connected or two cliques. Since G is 4-connected, the minimum degree of G is at least 4. If the neighborhood of a vertex is connected, then the neighborhood of this vertex contains a path of order 3, yielding a 4-cycle. Thus the neighborhood of every vertex is two cliques. If a vertex has degree at least 5, then one of the cliques has at least three vertices, yielding a 4-cycle. Thus we have the following properties for the graph G.

(P0) G is 4-regular and, for any $v \in V(G)$, $G[N_G(v) \cup \{v\}]$ are two triangles identified at v.

(P1) Any two distinct vertices in G can have at most one common neighbor.

By Theorem 1.3, G has an induced subgraph Z_5. Let $H = Z_t$ be an induced subgraph of G such that t is maximized. Since G is Z_8-free, $t \in \{5, 6, 7\}$. Let $V(H) = \{v, v_1, v_2, \ldots, v_{t+2}\}$ and $E(H) = \{vv_1, vv_2, v_1v_2, v_2v_3, \ldots, v_tv_{t+1}, v_{t+1}v_{t+2}\}$. By the choice of H, v_{t+2} has no neighbors in $V(H) \setminus \{v_{t+1}\}$. By (P0), let y_1, y_2, y_3 be the three neighbors of v_{t+2} which are not in $V(H) \setminus \{v_{t+1}\}$ and we may assume, without loss of generality, that y_3 is adjacent to v_{t+1} and that y_1 and y_2 are adjacent.

Since G is claw-free and G does not have 4-cycles, y_1, y_2, and y_3 satisfy the following properties.

(P2) By the choice of H (the maximum of t), both y_1 and y_2 have neighbors in $V(H) \setminus \{v_{t+2}\}$.
(P3) y_1 (also y_2) is not adjacent to v_{t+1} or v_t, and y_3 is not adjacent to v_{t-1}, v_t (since G has no 4-cycles).
(P4) Any vertex not in H that is adjacent to v_i for $i \in \{2, 3, \ldots, t+1\}$ is also adjacent to v_{i+1} or v_{i-1} (since G is claw-free).

Lemma 4.1 *Let G be a 4-connected $\{K_{1,3}, Z_8\}$-free graph, and let $H = Z_t$ be an induced subgraph of G such that t is maximized. If G has no 4-cycles, then $t \neq 5$.*

Proof Assume that $t = 5$. First of all, we claim that $N_G(v_3) \cap \{y_1, y_2\} \neq \emptyset$. By way of contradiction, we assume that $N_G(v_3) \cap \{y_1, y_2\} = \emptyset$. By (P3), $N_G(v_5) \cap \{y_1, y_2\} = \emptyset$. By (P4), $N_G(v_4) \cap \{y_1, y_2\} = \emptyset$. By (P2), $N_G(y_1) \cap \{v, v_1, v_2\} \neq \emptyset$ and $N_G(y_2) \cap \{v, v_1, v_2\} \neq \emptyset$. Note that $v_7 \in N_G(y_1) \cap N_G(y_2)$. By (P1), y_1 and y_2 are adjacent to two distinct vertices in $\{v, v_1, v_2\}$, implying a 4-cycle in G. This contradiction implies that $N_G(v_3) \cap \{y_1, y_2\} \neq \emptyset$. Without loss of generality, we assume that $v_3 y_2 \in E(G)$.

Next we claim that $v_4 y_2 \in E(G)$. Otherwise, by (P4), $v_2 y_2 \in E(G)$. As G has no 4-cycles, $N_G(y_1) \cap \{v_1, v_2, v_3, v_4, v\} = \emptyset$. By (P3), $N_G(y_1) \cap (V(H) - \{v_7\}) = \emptyset$, contradicting (P2). Therefore, $v_4 y_2 \in E(G)$. By (P1), $N_G(y_1) \cap \{v_2, v_3, v_4, v_5, v_6, y_3\} = \emptyset$. By (P2), $N_G(y_1) \cap \{v_1, v\} \neq \emptyset$. By symmetry, we assume that $y_1 v_1 \in E(G)$. Then $v_1 y_2, v_1 y_3 \notin E(G)$.

Consider $N_G(v_1)$. As $d_G(v_1) = 4$, we assume that $N_G(v_1) = \{v, v_2, y_1, a\}$, where $a \notin V(H) \cup \{y_1, y_2, y_3\}$. By (P0), $a y_1 \in E(G)$. As G has no 4-cycles, $N_G(a) \cap \{v_2, v_3, v_4, v_6, y_3\} = \emptyset$. By (P4), $v_5 a \notin E(G)$. As G has no 4-cycles again, $N_G(y_3) \cap \{v_3, v_4, v_5\} = \emptyset$. As $d_G(v_1) = 4$, $y_3 v_1 \notin E(G)$. By (P3), $y_3 v_2 \notin E(G)$. Thus the subgraph induced by $\{a, y_1, v_1\} \cup \{v_2, \ldots, v_6, y_3\}$ is Z_6. It contradicts the maximality of t. □

Lemma 4.2 *Let G be a 4-connected $\{K_{1,3}, Z_8\}$-free graph, and let $H = Z_t$ be an induced subgraph of G such that t is maximized. If G has no 4-cycles, then $t \neq 7$.*

Proof Assume that $t = 7$.

Claim 1 Either $v_4 \in N_G(y_1) \cup N_G(y_2)$ or $v_5 \in N_G(y_1) \cup N_G(y_2)$.

Assume that $v_4, v_5 \notin N_G(y_1) \cup N_G(y_2)$. By (P3), $v_7, v_8 \notin N_G(y_1) \cup N_G(y_2)$. By (P4), $v_6 \notin N_G(y_1) \cup N_G(y_2)$. Therefore, $N_G(y_1) \cap \{v, v_1, v_2, v_3\} \neq \emptyset$ and $N_G(y_2) \cap \{v, v_1, v_2, v_3\} \neq \emptyset$, contradicting (P1). Claim 1 holds.

Claim 2 $v_4 \in N_G(y_1) \cup N_G(y_2)$.

Assume $v_4 \notin N_G(y_1) \cup N_G(y_2)$. By Claim 1, $v_5 \in N_G(y_1) \cup N_G(y_2)$. Without loss of generality, we assume that $y_2 v_5 \in E(G)$. By (P4), $y_2 v_6 \in E(G)$. By (P1) and (P3), $N_G(y_1) \cap \{v_4, v_5, v_6, v_7, v_8\} = \emptyset$. By (P2), $N_G(y_1) \cap \{v, v_1, v_2, v_3\} \neq \emptyset$.

We claim that $y_1 v_2 \notin E(G)$. By way of contradiction, we assume that $y_1 v_2 \in E(G)$. By (P1), $N_G(y_1) \cap \{v, v_1\} = \emptyset$. By (P4), $y_1 v_3 \in E(G)$. As G has no 4-cycles, $N_G(y_3) \cap \{v_2, v_3, v_5, v_6, v_7\} = \emptyset$. By (P4), $y_3 v_4 \notin E(G)$. As $d_G(v_3) = 4$, let $N_G(v_3)$

$= \{v_2, v_4, y_1, v_3'\}$, where $v_3' \notin V(H) \cup \{y_1, y_2, y_3\}$. By (P0), $v_3'v_4 \in E(G)$. As G has no 4-cycles, $N_G(v_3') \cap \{v, v_1, v_2, v_5, v_6\} = \emptyset$. As G is 4-regular, $v_3'v_9, v_3'y_1 \notin E(G)$. Since the subgraph induced by $\{v, v_1, v_2\} \cup \{y_1, v_9, v_8, v_7, v_6, v_5, v_4, v_3'\}$ is not Z_8, by (P4), we have $v_3'v_7, v_3'v_8 \in E(G)$. By (P1), we have either $v_1y_3 \notin E(G)$ or $vy_3 \notin E(G)$. Without loss of generality, we assume that $v_1y_3 \notin E(G)$. Then $vy_3 \notin E(G)$ and the subgraph induced by $\{y_3, v_9, v_8\} \cup \{v_7, v_6, \ldots, v_1\}$ is Z_7. By (P0), we assume that $N_G(v_1) = \{v, v_2, z_1, z_2\}$, where $z_1, z_2 \notin V(H) \cup \{y_1, y_2, y_3, v_3'\}$. Then $z_1z_2 \in E(G)$. By symmetry and Claim 1, $\{v_5, v_6\} \cap (N_G(z_1) \cup N_G(z_2)) \neq \emptyset$. Since G is 4-regular, we assume that $N_G(z_1) \cap \{v_5, v_6\} \neq \emptyset$. Then we have either $z_1v_6, z_1v_7 \in E(G)$ or $z_1v_4, z_1v_5 \in E(G)$. For each of these two cases, $N_G(z_2) \cap \{v_2, v_3, \ldots, v_9\} = \emptyset$. By the maximality of t, $z_2y_3 \in E(G)$. Let $z_3 \in N_G(y_3) - \{v_8, v_9, z_2\}$. Then $z_3z_2 \in E(G)$. Let $z_4 \in N_G(v_5) - \{v_4, v_6, y_2\}$ if $z_1v_6, z_1v_7 \in E(G)$, or $z_4 \in N_G(v_6) - \{v_5, v_7, y_2\}$ if $z_1v_4, z_1v_5 \in E(G)$. Since G is 4-regular, $\{v, z_3, z_4\}$ is a 3-cut in G, a contradiction. So $y_1v_2 \notin E(G)$.

By (P4), $v_3y_1 \notin E(G)$, and so $N_G(y_1) \cap \{v, v_1\} \neq \emptyset$. We assume that $v_1y_1 \in E(G)$. Then $v_1y_3 \notin E(G)$. Consider $N_G(v_1)$. Assume that $N_G(v_1) = \{v, v_2, y_1, a\}$, where $a \notin V(H) \cup \{y_1, y_2, y_3\}$. By (P0), $ay_1 \in E(G)$. As G has no 4-cycles, $N_G(a) \cap \{v, v_2, v_3, v_5, v_6, v_8, v_9, y_3\} = \emptyset$. By (P4), $av_4, av_7 \notin E(G)$. Notice that the subgraph induced by $\{a, v_1, y_1\} \cup \{v_2, v_3, \ldots, v_8, y_3\}$ is not Z_8. We have $N_G(y_3) \cap \{v_2, v_3, v_4\} \neq \emptyset$. Then $y_3v_3 \in E(G)$.

Consider the neighborhood of v_7, and let $N_G(v_7) = \{b, c, v_6, v_8\}$, where $b, c \notin V(H) \cup \{a, y_1, y_2, y_3\}$. By (P0), we assume $bv_6, cv_8 \in E(G)$. Then $N_G(b) \cap \{v_1, v_4, v_5, v_8, v_9, y_1, y_2, y_3, c\} = \emptyset$ and $N_G(c) \cap \{v_1, v_5, v_6, v_9, y_1, y_2, y_3\} = \emptyset$. We consider the following two cases.

Case 1 $bv \notin E(G)$.

Considering the subgraph induced by $\{v, v_1, v_2\} \cup \{v_3, v_4, v_5, y_2, v_9, v_8, v_7, b\}$, we have $bv_2, bv_3 \in E(G)$. As G is 4-regular, $y_3v_2 \notin E(G)$. Thus $y_3v_4 \in E(G)$. Consider the neighborhood of v_5 and let $N_G(v_5) = \{r, v_4, v_6, y_2\}$. Then $rv_4 \in E(G)$. Since G has no 4-cycles, $r \notin \{v, a, c\}$. As G is 4-regular, $N_G(r) \cap \{v_1, v_2, v_3, v_6, v_7, v_8, v_9, y_1, y_2, y_3, b\} = \emptyset$. As $G[\{r, v_4, v_5\} \cup \{v_3, b, v_7, v_8, v_9, y_1, v_1, v\}] \neq Z_8$, we have $rv \in E(G)$. Let $r' \in N_G(r) - \{v_4, v_5, v\}$. Then $r'v \in E(G)$, and so $\{r', a, c\}$ is a 3-cut in G, a contradiction.

Case 2 $bv \in E(G)$.

As G has no 4-cycles, $ab, vc \notin E(G)$. As $by_3 \notin E(G)$, $vy_3 \notin E(G)$. Since the subgraph induced by $\{a, v_1, y_1\} \cup \{y_2, v_5, v_4, v_3, y_3, v_8, v_7, b\}$ is not Z_8, we have $y_3v_4 \in E(G)$. Also, since the subgraph induced by $\{y_1, y_2, v_9\} \cup \{v_5, v_4, v_3, v_2, v, b, v_7, c\}$ is not Z_8, we have $cv_2, cv_3 \in E(G)$. Consider the neighborhood of v_5. Assume $N_G(v_5) = \{r, v_4, v_6, y_2\}$. Then $rv_4 \in E(G)$. Since G has no 4-cycles, $r \notin \{v, a, b, c\}$ and $rb \notin E(G)$. Let $b' \in N_G(b) - \{v_6, v_7, v\}$. Then $b'v \in E(G)$. So $\{b', a, r\}$ is a 3-cut in G, a contradiction.

By Claim 2, we assume that $v_4y_2 \in E(G)$.

Claim 3 $v_5y_2 \notin E(G)$. Therefore, $v_3y_2 \in E(G)$.

By way of contradiction, we assume that $v_5y_2 \in E(G)$. Then $N_G(y_1) \cap \{v_3, v_4, \ldots, v_8\} = \emptyset$. By (P2), $N_G(y_1) \cap \{v, v_1, v_2\} \neq \emptyset$. Then $y_1v_2 \notin E(G)$ (otherwise, by (P4), $y_1v_1 \in E(G)$. Then $vv_1y_1v_2v$ is a 4-cycle). Thus $N_G(y_1) \cap \{v, v_1\} \neq \emptyset$. Without loss of generality, we assume $v_1y_1 \in E(G)$. Let $N_G(v_1) = \{y_1, v_2, v, a\}$, where $a \notin V(H) \cup \{y_1, y_2, y_3\}$. By (P0), $ay_1 \in E(G)$. As G has no 4-cycles, $N_G(a) \cap \{v_2, v_3, v_4, v_5, v_8, y_3\} = \emptyset$.

We claim that $av_6, av_7 \in E(G)$. Otherwise, considering the subgraph induced by $\{a, v_1, y_1\} \cup \{v_2, v_3, \ldots, v_8, y_3\}$, we have $y_3v_2, y_3v_3 \in E(G)$. Consider the neighborhood of v, and let $N_G(v) = \{v_1, v_2, b, c\}$, where $b, c \notin V(H) \cup \{a, y_1, y_2, y_3\}$. Then $\{v_1, v_2, v_3, v_9, a, y_1, y_2, y_3\} \cap (N_G(b) \cup N_G(c)) = \emptyset$. As $y_2v_4, y_2v_5 \in E(G)$, by (P4), $v_4 \notin N_G(b) \cup N_G(c)$. As $G[\{v, b, c\} \cup \{v_1, y_1, v_9, y_3, v_3, v_4, v_5, v_6\}] \neq Z_8$, we have $\{v_5, v_6\} \cap (N_G(b) \cup N_G(c)) \neq \emptyset$. Without loss of generality, we assume that $\{v_5, v_6\} \cap N_G(c) \neq \emptyset$. By (P4), we have either $cv_5, cv_6 \in E(G)$ or $cv_6, cv_7 \in E(G)$. If $cv_5, cv_6 \in E(G)$, then $v_7, v_8 \notin N_G(b) \cup N_G(c)$ and so the subgraph induced by $\{v, b, c\} \cup \{v_6, v_7, v_8, y_3, v_3, v_4, y_2, y_1\}$ is Z_8. If $cv_6, cv_7 \in E(G)$, the subgraph induced by $\{v, b, c\} \cup \{v_6, v_5, v_4, v_3, y_3, v_9, y_1, a\}$ is Z_8, a contradiction. Therefore, $av_6, av_7 \in E(G)$.

Since G has no 4-cycles, let $b \in N_G(v_4) - \{v_3, v_5, y_2\}$ and $c \in N_G(v_5) - \{v_4, v_6, y_2\}$ and $b \neq c$. Since G has no 4-cycles, we have $bv_3, cv_6 \in E(G)$, and $N_G(b) \cap \{v_5, v_6, v_9, y_1, v_1, v\} = \emptyset$ and $N_G(c) \cap \{v_7, v_8, v_9, y_1, v_1, v_3, v_4\} = \emptyset$. By (P4), $cv_2 \notin E(G)$. Since the subgraph induced by $\{b, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8, v_9, y_1, v_1, v\}$ is not Z_8, we have $bv_7, bv_8 \in E(G)$. Since the subgraphs induced by $\{c, v_5, v_6\} \cup \{y_2, y_1, v_1, v_2, v_3, b, v_8, y_3\}$ and $\{c, v_5, v_6\} \cup \{a, y_1, v_9, v_8, b, v_3, v_2, v\}$ are not Z_8, we have $cy_3, cv \in E(G)$. Thus $vy_3 \in E(G)$. As G is 4-regular, $\{v_2, v_3\}$ is a 2-cut in G, a contradiction. Therefore, Claim 3 holds.

By Claim 3, $y_2v_3, y_2v_4 \in E(G)$. By (P3) and (P1), $N_G(y_1) \cap \{v_2, v_3, v_4, v_5, v_7, v_8\} = \emptyset$. By (P4), $v_6y_1 \notin E(G)$. By (P2), we assume that $y_1v_1 \in E(G)$. Thus $y_3v_1 \notin E(G)$. Let $N_G(v_1) = \{v, y_1, v_2, a\}$, where $a \notin V(H) \cup \{y_1, y_2, y_3\}$. By (P0), $ay_1 \in E(G)$. As G has no 4-cycles, $N_G(y_3) \cap \{v_1, v_3, v_4, v_6, v_7\} = \emptyset$. By (P4), $v_2y_3, v_5y_3 \notin E(G)$. Then the subgraph induced by $\{y_3, v_8, v_9\} \cup \{v_7, v_6, \ldots, v_1\}$ is Z_7. By symmetry (discussion used in Claims 2 and 3), we assume that $av_6, av_7 \in E(G)$.

Consider the neighborhoods of v_3 and v_4. Let $N_G(v_3) = \{v_2, v_4, y_2, b\}$ and $N_G(v_4) = \{v_3, v_5, y_2, c\}$. Then $b \neq c$ and $b, c \notin V(H) \cup \{a, y_1, y_2, y_3\}$. Also, we have $bv_2, cv_5 \in E(G)$. Considering the subgraph induced by $\{c, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9, y_1, v_1, v_2, b\}$, we conclude that $bv_7, bv_8 \in E(G)$. Considering the subgraph induced by $\{y_3, v_8, v_9\} \cup \{b, v_3, v_4, v_5, v_6, a, v_1, v\}$, we have $vy_3 \in E(G)$. Considering the subgraph induced by $\{c, v_4, v_5\} \cup \{v_6, v_7, b, v_2, v, y_3, v_9, y_1\}$, we have $N_G(c) \cap \{v, y_3\} \neq \emptyset$. By (P0), $cv, cy_3 \in E(G)$. As G is 4-regular, $\{v_5, v_6\}$ is a 2-cut, a contradiction. ◻

Lemma 4.3 *If G is a 4-connected $\{K_{1,3}, Z_8\}$-free graph, then G has a 4-cycles unless G is the line graph of the Petersen graph.*

Proof Suppose that G does not have 4-cycles. By Theorem 1.3, G has an induced subgraph Z_5. Let $H = Z_t$ be an induced subgraph of G such that t is maximized. Since G is Z_8-free, $t = 5, 6, 7$. By Lemmas 4.1 and 4.2, $t = 6$. Let H be the graph obtained from $P_8 = v_1v_2 \ldots v_8$ by adding a vertex v and joining v to v_1 and v_2. By the choice of H, v_8 has no neighbors in $V(H) \setminus \{v_7\}$. By (P0), let y_1, y_2, y_3 be the three neighbors of v_8 which are not in $V(H) \setminus \{v_7\}$ and we may assume, without loss of generality, that y_3 is adjacent to v_7 and that y_1 and y_2 are adjacent. By (P3), $v_6, v_7 \notin N_G(y_1) \cup N_G(y_2)$ and $y_3v_5, y_3v_6 \notin E(G)$.

Claim 1 $v_4 \in N_G(y_1) \cup N_G(y_2)$.

Assume that $v_4 \notin N_G(y_1) \cup N_G(y_2)$. By (P4), $v_5 \notin N_G(y_1) \cup N_G(y_2)$. If $v_2y_1 \in E(G)$, by (P0) and (P1), $N_G(y_2) \cap (V(H) - \{v_8\}) = \emptyset$, contradicting (P2). Thus $v_2y_1 \notin E(G)$. Similarly, $v_2y_2 \notin E(G)$. By (P4), $v_3 \notin N_G(y_1) \cup N_G(y_2)$. By (P2) and (P1), we may assume that $v_1y_1, vy_2 \in E(G)$. This results in a 4-cycle $vv_1y_1y_2v$, a contradiction. Claim 1 holds.

By Claim 1, we assume that $v_4y_2 \in E(G)$. By (P1) and (P4), $\{v_3, v_4, v_5\} \cap N_G(y_1) = \emptyset$. If $v_2y_1 \in E(G)$, then $v_1y_1 \in E(G)$ by (P4). This would result in a 4-cycle $vv_1y_1v_2v$. Therefore, $v_2y_1 \notin E(G)$. By (P2) and by symmetry, we assume that $v_1y_1 \in E(G)$. Thus $v_1y_3, v_1y_2 \notin E(G)$. As $d_G(v_1) = 4$, we assume that $N_G(v_1) = \{v, v_2, y_1, y_1'\}$, where $y_1' \notin V(H) \cup \{y_1, y_2, y_3\}$. By (P0), $y_1y_1' \in E(G)$. Then $N_G(y_1) \cap \{v, v_2, v_3, \ldots, v_7, y_3\} = \emptyset$.

Claim 2 $y_2v_5 \notin E(G)$.

Assume that $y_2v_5 \in E(G)$. Since G has no 4-cycles, $\{v_2, v_3, v_4, v_5, v_7, y_3\} \cap N_G(y_1') = \emptyset$. By (P4), $v_6y_1' \notin E(G)$. Considering the subgraph induced by $\{y_1', y_1, v_1\} \cup \{v_2, \ldots, v_7, y_3\}$, we have that $y_3v_2, y_3v_3 \in E(G)$. Let $N_G(v) = \{b, c, v_1, v_2\}$, where $b, c \notin V(H) \cup \{y_1, y_2, y_3, y_1'\}$. Thus $(N_G(b) \cup N_G(c)) \cap \{v_1, v_2, v_3, v_8, y_1, y_3\} = \emptyset$. As $y_2v_4, y_2v_5 \in E(G)$, by (P4), $v_4 \notin N_G(b) \cup N_G(c)$. Since the subgraph induced by $\{v, b, c\} \cup \{v_1, y_1, v_8, y_3, v_3, v_4, v_5\}$ is not Z_7, $v_5 \in N_G(b) \cup N_G(c)$. Without loss of generality, we assume that $cv_5 \in E(G)$. By (P0), $cv_6 \in E(G)$. Since G has no 4-cycles, $(N_G(b) \cup N_G(c)) \cap \{v_7, y_1\} = \emptyset$. As G is 4-regular, $y_2 \notin N_G(b) \cup N_G(c)$. This implies that the subgraph induced by $\{v, b, c\} \cup \{v_1, y_1, y_2, v_4, v_3, y_3, v_7\}$ is Z_7, contradicting the maximality of $t = 6$. Claim 2 holds.

By Claim 2 and (P4), $y_2v_3 \in E(G)$. As G has no 4-cycles, $\{v_2, v_3, v_4, v_7, y_3\} \cap N_G(y_1') = \emptyset$. Since G is Z_7-free, considering the subgraph induced by $\{y_1', y_1, v_1\} \cup \{v_2, \ldots, v_7, y_3\}$, $N_G(y_1') \cap \{v_5, v_6\} \neq \emptyset$. By (P4), $y_1'v_5, y_1'v_6 \in E(G)$. Again, as G has no 4-cycles, $N_G(y_3) \cap \{v_1, v_3, v_4, v_5, v_6\} = \emptyset$. By (P4), $y_3v_2 \notin E(G)$.

Claim 3 $vy_3 \in E(G)$.

Assume that $vy_3 \notin E(G)$. Let $N_G(y_3) = \{v_7, v_8, a, b\}$, where $a, b \notin V(H) \cup \{y_1', y_1, y_2\}$. By (P0), $ab \in E(G)$. Notice that the subgraph induced by $(V(H) - \{v_8\}) \cup \{y_3\}$ is still Z_6. Using the discussion in Claims 1 and 2, we have either $av_3, av_4 \in E(G)$ or $bv_3, bv_4 \in E(G)$, implying a 4-cycle $av_3y_2v_4a$ or $bv_3y_2v_4b$, a contradiction. Claim 3 holds.

Let $N_G(y_3) = \{v_7, v_8, v, x_2\}$. By (P0), $vx_2 \in E(G)$. As G has no 4-cycles, $N_G(x_2) \cap \{v_2, v_3, v_6, v_7\} = \emptyset$. By (P0), $N_G(x_2) \cap \{v_1, v_8, y_1, y_2, y_1'\} = \emptyset$.

Claim 4 $x_2v_4 \in E(G)$. Therefore, $x_2v_5 \in E(G)$.

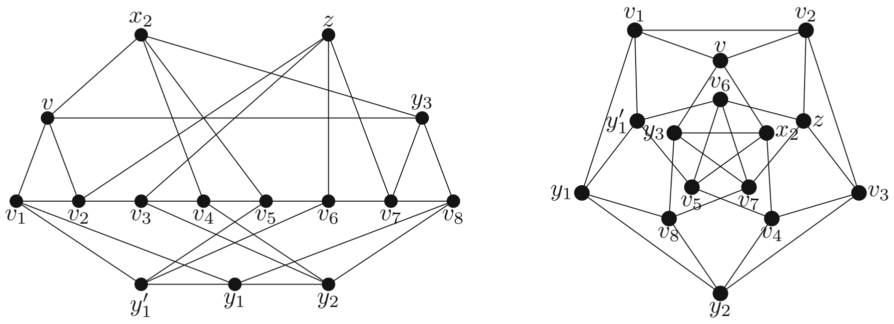

Fig. 3 Two drawings of the line graph of the Petersen graph

By way of contradiction, we assume that $x_2v_4 \notin E(G)$. By (P4), $x_2v_5 \notin E(G)$. Thus we assume that $N_G(x_2) = \{v, y_3, s, t\}$, where $s, t \notin V(H) \cup \{y_1, y_2, y_3, y_1'\}$. By (P0), $st \in E(G)$. As G has no 4-cycles, $v_2 \notin N_G(s) \cup N_G(t)$. As $y_2v_3, y_2v_4 \in E(G)$, by (P4), $v_3 \notin N_G(s) \cup N_G(t)$.

If $v_6 \notin N_G(s) \cup N_G(t)$, then $G[\{s, t, x_2\} \cup \{v, v_2, v_3, y_2, y_1, y_1', v_6\}] = Z_7$, contradicting the maximality of $t = 6$. Without loss of generality, we assume that $v_6t \in E(G)$. As $y_1'v_5, y_1'v_6 \in E(G)$, $v_7t \in E(G)$. Thus $G[\{x_2, s, t\} \cup \{v_7, v_8, y_2, v_3, v_2, v_1, y_1'\}] = Z_7$, contradicting the maximality of $t = 6$ again. Claim 4 holds.

We will get the line graph of Peterson graph by considering the neighborhood of v_2. As G is 4-regular, we assume that $N_G(v_2) = \{v, v_1, v_3, z\}$, where $z \notin V(H) \cup \{y_1, y_2, y_3, y_1', x_2\}$. By (P4), $zv_3 \in E(G)$. As $G[\{z, v_2, v_3\} \cup \{y_2, y_1, y_1', v_6, v_7, y_3, x_2\}] \neq Z_7$, by (P4), $zv_6, zv_7 \in E(G)$. Since G is 4-regular, G is the left graph in Fig. 3. It is easy to check that G is the line graph of Peterson graph. □

5 Existence of *t*-Cycles ($t = 5, 6, 7, 8$)

Lemma 5.1 *If G is a 4-connected $\{K_{1,3}, Z_8\}$-free graph, then G has a 5-cycle.*

Proof Suppose that G does not have 5-cycles. Since the line graph of the Petersen graph has 5-cycles, G is not the line graph of the Petersen graph. By Theorem 1.4, G has an induced path P_{10}. Let $P_k = v_1v_2 \cdots v_k$ be a longest induced path of G, and let $Y = N_G(v_1) - \{v_2\} = \{y_1, y_2, \ldots, y_s\}$, $Y_1 = N_G(v_1) \cap N_G(v_2) = \{y_1, \ldots, y_r\}$, and $Y_2 = Y - Y_1$. Then $k \geq 10$, $s \geq 3$, $r \geq 0$. Since G is claw-free, $G[Y_2]$ is a complete graph.

(Q1) For $w \notin V(P_k)$, if $wv_i \in E(G)(1 < i < k)$, then either $wv_{i-1} \in E(G)$ or $wv_{i+1} \in E(G)$.

(Q2) For $w \notin V(P_k)$, if $wv_i \in E(G)$ $(1 \leq i \leq k-2)$, then $wv_{i+2} \notin E(G)$. (Otherwise, let $a \in N_G(v_{i+1}) - \{v_i, v_{i+2}\}$. Then either $av_i \in E(G)$ or $av_{i+2} \in E(G)$. Thus either $v_iav_{i+1}v_{i+2}wv_1$ or $v_iv_{i+1}av_{i+2}wv_1$ is a 5-cycle.) In addition, $wv_{i+3} \notin E(G)$ if $i \leq k-3$. Thus, $N_G(y_i) \cap \{v_3, v_4, v_5\} = \emptyset$ for $y_i \in Y_1$, and

$N_G(y_i) \cap \{v_2, v_3, v_4\} = \emptyset$ for $y_i \in Y_2$. As G is claw-free, $G[Y_1]$ is a complete graph.

Claim 1 $|Y_2| \leq 2$. Therefore, $|Y_1| \geq 1$.

Assume that $Y_2 = \{y_{r+1}, \ldots, y_s\} = \{u_1, u_2, \ldots, u_{s-r}\}$ $(s - r \geq 3)$. By (Q2), $N_G(u_i) \cap \{v_2, v_3, v_4\} = \emptyset$.

We claim that $N_G(v_5) \cap \{u_1, u_2, u_3\} = \emptyset$. Otherwise, we assume $u_3v_5 \in E(G)$. By (Q1), $u_3v_6 \in E(G)$. Since G is claw-free, $N_G(u_3) \cap V(P_k) = \{v_1, v_5, v_6\}$. As G has no 5-cycles, $N_G(u_i) \cap \{v_5, \ldots, v_8\} = \emptyset$ for $i = 1, 2$. As G is Z_8-free, there is a vertex in $\{u_1, u_2\}$, say u_2, such that $u_2v_9 \in E(G)$. Then $N_G(u_2) \cap V(P_k) = \{v_1, v_9, v_{10}\}$ and $N_G(u_1) \cap \{v_2, \ldots, v_{10}\} = \emptyset$. By the choice of P_k, $k \geq 11$. As $u_1v_{11} \notin E(G)$, $k \geq 12$. As $u_1v_{12} \notin E(G)$, $k \geq 13$. Consider $N_G(v_2)$ and let $w \in N_G(v_2) - \{v_1, v_3\}$. Since G has no 5-cycles, $N_G(w) \cap \{u_1, u_2, u_3, v_4, v_5, v_6, v_9, v_{10}\} = \emptyset$. If $wv_1 \in E(G)$, then $N_G(w) \cap \{v_3, v_7, v_8\} = \emptyset$. This implies that $G[\{w, v_1, v_2, \ldots, v_{10}\}] = Z_8$, a contradiction. So $wv_1 \notin E(G)$. By (Q1), $wv_3 \in E(G)$. Since $G[\{w, v_2, v_3, \ldots, v_9, u_2, u_1\}] \neq Z_8$, $wv_7, wv_8 \in E(G)$. So $N_G(w) \cap V(P_k) = \{v_2, v_3, v_7, v_8\}$. Hence $G[\{w, v_7, v_8, v_3, v_4, v_5, u_3, u_2, v_{10}, v_{11}, v_{12}\}] = Z_8$, a contradiction. So $N_G(v_5) \cap \{u_1, u_2, u_3\} = \emptyset$.

If $N_G(u_3) \cap \{v_6, v_7, v_8, v_9\} \neq \emptyset$, as G has no 5-cycles, by (Q1), $N_G(u_i) \cap \{v_6, \ldots, v_9\} = \emptyset$ for $i = 1, 2$. This implies that $G[\{u_1, u_2, v_1, \ldots, v_9\}] = Z_8$, a contradiction. So $N_G(u_3) \cap \{v_6, v_7, v_8, v_9\} = \emptyset$. Similarly, we have $N_G(u_2) \cap \{v_6, v_7, v_8, v_9\} = \emptyset$. So $G[\{u_2, u_3, v_1, \ldots, v_9\}] = Z_8$, a contradiction. Claim 1 holds.

Claim 2 $|Y_1| \leq 1$.

Assume that $v_2y_1, v_2y_2 \in E(G)$. By (Q2), $N_G(y_i) \cap \{v_3, v_4, v_5\} = \emptyset$ for $i = 1, 2$, $y_1y_2 \in E(G)$ and $N_G(y_3) \cap \{v_2, v_3, v_4\} = \emptyset$. As G has no 5-cycles, $G_G(y_3) \cap \{y_1, y_2\} = \emptyset$. Since G is Z_8-free, $N_G(y_i) \cap \{v_6, v_7, \ldots, v_{10}\} \neq \emptyset$ for $i = 1, 2$. Furthermore, if $y_1v_i, y_2v_j \in E(G)$, where $i, j \in \{6, \ldots, 10\}$, then $|j - i| \geq 3$. Thus, by (Q1), we may assume that $y_1v_6, y_1v_7, y_2v_{10} \in E(G)$. As G has no 5-cycles, $N_G(y_3) \cap \{v_2, v_3, \ldots, v_{10}\} = \emptyset$, and so $k \geq 11$ and $d_G(v_1) = 4$. Hence $y_2v_{11} \in E(G)$ and $y_3v_{11} \notin E(G)$. Therefore, $k \geq 12$. Let $z_1, z_2, z_3 \in N_G(y_3) - \{v_1\}$. Then $z_1z_2, z_1z_3, z_2z_3 \in E(G)$. For $i = 1, 2, 3$, $N_G(z_i) \cap \{y_1, y_2, v_1, v_2, v_3, v_6, v_7, v_{10}, v_{11}\} = \emptyset$. If $z_iv_4 \in E(G)$, then $z_iv_5 \in E(G)$ and $z_iv_8, z_iv_9 \notin E(G)$. Thus $G[\{z_i, v_4, v_5, v_3, v_2, y_1, v_7, \ldots, v_{11}\}] = Z_8$. If $z_iv_8, z_iv_9 \in E(G)$, then $z_iv_4, z_iv_5 \notin E(G)$ and $G[\{z_i, v_9, v_8, \ldots, v_2, y_2, v_{11}\}] = Z_8$. So $N_G(z_i) \cap \{v_4, v_5, v_8, v_9\} = \emptyset$. This implies that $G[\{z_1, z_2, y_3, v_1, \ldots, v_8\}] = Z_8$, a contradiction. Claim 2 holds.

By Claims 1 and 2, $Y_1 = \{y_1\}$ and $Y_2 = \{y_2, y_3\}$. Thus $y_2y_3 \in E(G)$. As G has no 5-cycles, $N_G(y_1) \cap \{y_2, y_3, v_3, v_4, v_5\} = \emptyset$ and $N_G(y_i) \cap \{v_2, v_3, v_4\} = \emptyset (i = 2, 3)$. As G is Z_8-free, $N_G(y_1) \cap \{v_6, \ldots, v_{10}\} \neq \emptyset$ and $\cup_{i=2}^{3} N_G(y_i) \cap \{v_5, v_6, \ldots, v_9\} \neq \emptyset$. We assume that $T = N_G(y_3) \cap \{v_5, v_6, \ldots, v_9\} \neq \emptyset$. Let $w \in N_G(v_2) - \{v_1, v_3, y_1\}$. Then $wv_1 \notin E(G)$ and so $wv_3 \in E(G)$. By (Q2), $N_G(w) \cap \{v_4, v_5, v_6\} = \emptyset$. As G has no 5-cycles, $N_G(w) \cap \{y_1, y_2, y_3\} = \emptyset$.

We claim that $N_G(y_1) \cap \{v_6, \ldots, v_9\} = \emptyset$. Otherwise, by (Q1), $N_G(y_1) \cap V(P_k) = \{v_1, v_2, v_{i_0}, v_{i_0+1}\}$, where $i_0 = 6, 7, 8, 9$. As G has no 5-cycles, Thus $\cup_{i=2}^{3} N_G(y_i) \cap \{v_{i_0-1}, v_{i_0}, v_{i_0+1}\} = \emptyset$ and $y_2v_{i_0+2}, y_3v_{i_0+2} \notin E(G)$ if $i \neq 9$. Thus $i_0 \neq 7$.

If $i_0 = 6$, then $T = \{v_9, v_{10}\}$; if $i_0 = 8$, then $T = \{v_5, v_6\}$; if $i_0 = 9$, then T is either $\{v_5, v_6\}$ or $\{v_6, v_7\}$. For these three cases, $N_G(y_2) \cap \{v_6, v_7, \dots, v_{10}\} = \emptyset$. By the choice of P_k, $k \geq 11$. For $i_0 = 6$, as G has no 5-cycles, $N_G(w) \cap \{v_7, \dots, v_{10}\} = \emptyset$. So $G[\{w, v_2, v_3, \dots, v_9, y_3, y_2\}] = Z_8$, a contradiction. For $i_0 = 9$, $N_G(w) \cap \{v_8, \dots, v_{11}\} = \emptyset$. By (Q1), $wv_7 \notin E(G)$. So $G[\{w, v_2, v_3, \dots, v_{11}\}] = Z_8$, a contradiction. For $i_0 = 8$, let $z_1, z_2 \in N_G(y_2) - \{v_1, y_3\}$. As $d_G(v_1) = 4$, $z_1z_2 \in E(G)$. As G has no 5-cycles, $N_G(z_i) \cap \{v_1, v_2, \dots, v_9\} = \emptyset$ for $i = 1, 2$. Thus $G[\{z_1, z_2, y_2, v_1, v_2, \dots, v_8\}] = Z_8$, a contradiction. So $N_G(y_1) \cap \{v_6, \dots, v_9\} = \emptyset$.

Notice that $N_G(y_1) \cap \{v_6, \dots, v_{10}\} \neq \emptyset$. We have $y_1v_{10} \in E(G)$. As G has no 5-cycles, $N_G(y_i) \cap \{v_9, v_{10}\} = \emptyset$ for $i = 2, 3$. . Thus $T \subseteq \{v_5, \dots, v_8\}$, and so $N_G(y_2) \cap \{v_2, \dots, v_{10}\} = \emptyset$. By the choice of P_k, $k \geq 11$, and so $y_1v_{11} \in E(G)$ and $y_2v_{11}, y_3v_{11} \notin E(G)$. This implies that $k \geq 12$ and $y_2v_{12}, y_3v_{12} \notin E(G)$. As G has no 5-cycles, $N_G(w) \cap \{v_9, \dots, v_{12}\} = \emptyset$. As $G[\{w, v_2, v_3, \dots, v_{11}\}] \neq Z_8$, $wv_7, wv_8 \in E(G)$. Thus $y_3v_7, y_3v_8 \notin E(G)$ and $y_3v_5, y_3v_6 \in E(G)$. So $G[\{y_3, v_5, v_6, v_1, v_2, w, v_8, \dots, v_{12}\}] = Z_8$, a contradiction. □

The next lemma states that G has 6-, 7-, and 8-cycles if G is a 4-connected $\{K_{1,3}, Z_8\}$-free graph. In the proof Lemma 5.2, we follow the setup originated by Ferrara, Morris, and Wenger in [3], utilizing an argument based on the neighborhoods of vertices in smaller cycles. The Figs. 4 and 5 below are also originally from [3].

Lemma 5.2 *If G is a 4-connected $\{K_{1,3}, Z_8\}$-free graph, then G has cycles of length 6, 7, and 8.*

Proof By Lemma 5.1, G has a 5-cycle. Let t be the largest integer less than 8 such that G has a t-cycle but no $(t+1)$-cycle. Let C be a t-cycle in G and X be the set of vertices in C that have neighbors not in C. Since G is 4-connected, $|X| = l \geq 4$. Assume $X = \{v_1, v_2, \dots, v_l\}$. If $w_i \in N_G(v_i) - V(C)$, then $w_iv_i^+, w_iv_i^- \notin E(G)$ since G does not have a $(t+1)$-cycle. Since G is claw-free, we have $v_i^+v_i^- \in E(G)$. Using similar arguments, we have $x_iy_i \in E(G)$ if $x_i, y_i \in N_G(v_i) \cap V(C)$. Continue this process, we have that $G[V(C)]$ contains one of the graphs in Fig. 4 as a subgraph, where v_1, v_2, v_3, and v_4 are the vertices incident to the dashed edges.

For any two vertices v_i and v_j in X, if $t = 5$, $G[V(C)]$ contains paths of length 1 through $t - 1 = 4$ joining v_i and v_j; if $t \in \{6, 7\}$, then $G[V(C)]$ contains paths of length 2 through $t-1$ joining any two vertices v_i and v_j. Let $P(i, j)$ be a shortest path in $G[V(C)]$ connecting v_i and v_j that does not contain v_k for any k distinct from i and j. For $1 \leq i \leq l$, let S_i be the set of vertices in $V(G) - V(C)$ that are adjacent to v_i, S_i' be the set of vertices in $V(G) - V(C)$ that have distance 2 to v_i, and S_i'' be the set of vertices in $V(G) - V(C)$ that have distance 3 to v_i. We conclude that the following claims hold for $1 \leq i < j \leq l$.

(W1) For any $x \in S_i$, $N_G(x) \cap V(C) = \{v_i\}$. Therefore, for any $i \neq j$, $S_i \cap S_j = \emptyset$.
(W2) For $i \neq j$, and for any $x \in S_i' \cup S_i''$, $N_G(x) \cap (V(C) \cup S_j) = \emptyset$. Therefore, for any $i \neq j$, $S_i' \cap (S_j \cup S_j') = \emptyset$, and there are no edges joining $S_i \cup S_i'$ and $S_j \cup S_j'$.
(W3) $S_i' \neq \emptyset$ and $S_i'' \neq \emptyset$ for $1 \leq i \leq l$.
(W4) No vertex can have a neighbor in S_i' for three distinct values of i.

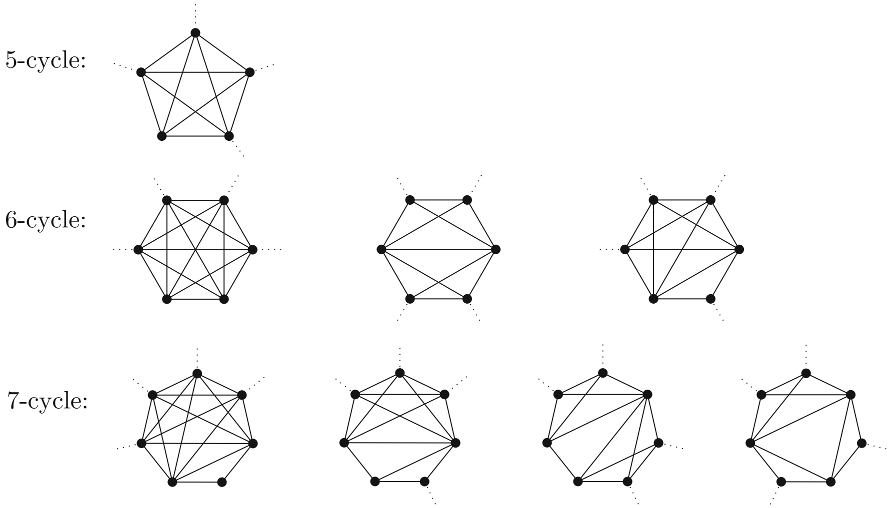

Fig. 4 Necessary subgraphs in $G[V(C)]$ with v_1, v_2, v_3, and v_4 incident to the dashed edges

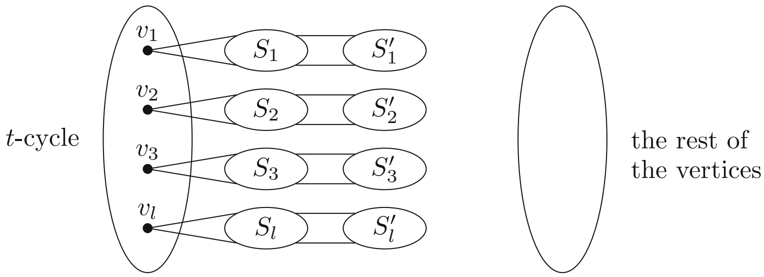

Fig. 5 The structure of G

The structure of $G[V(C)]$ and the assumption that G does not contain a $(t+1)$-cycle imply (W1) and (W2). Since G is 4-connected, (W3) must hold, as otherwise v_i is a cut vertex. (W4) comes from (W2) since G is claw-free. Thus G has the structure shown in Fig. 5.

For $1 \le i \le l$, we use s_i to denote a general vertex in S_i, use s_i' to denote a general vertex in S_i', and use s_i'' to denote a general vertex in S_i'' such that $v_i s_i, s_i s_i', s_i' s_i'' \in E(G)$.

Claim 1 There are distinct values $i, j \in \{1, 2, \ldots, l\}$ such that $S_i'' \cap S_j'' \neq \emptyset$.

By way of contradiction, we assume that for any $i \neq j$, $S_i'' \cap S_j'' = \emptyset$. Consider the graph H from $G - E(C) - (V(G) - X)$ by contracting $v_i \cup S_i \cup S_i' \cup S_i''$ for $1 \le i \le l$, and denote the contracted vertices be x_i. If H is disconnected, one component of H contains at most 3 vertices of $\{x_1, x_2, \ldots, x_l\}$. Without loss of generality, we may assume that $x_1, \ldots, x_k$ are in the same component of H with $k \le 3$, then $\{v_1, \ldots, v_k\}$ is a vertex-cut of G, contradiction to the fact that G is 4-connected. Therefore H is connected, and there is a path from each S_i'' to each S_j'' in G, where $i \neq j$, that contains no vertices in C. Let P' be a shortest such path connecting S_i'' and S_j'' over all choices of i and j. Without loss of generality, we assume that $i = 1$ and $j = 2$. Since P' is minimal, $V(P') \cap S_k = \emptyset$ and $V(P') \cap S_k' = \emptyset$ for all $k \in \{1, 2, \ldots, l\}$, and $V(P') \cap S_1'' = \{s_1''\}$ and $V(P') \cap S_2'' = \{s_2''\}$. Thus $Q = s_1 s_1' s_1'' P' s_2'' s_2' s_2 v_2 P(2, 3) v_3 s_3 s_3'$ is an induced path on at least 10 vertices. Consider the neighborhood of s_1. Since G is 4-connected, let $\{z_1, z_2, v_1, s_1'\} \subseteq N_G(s_1)$.

If both $z_1 v_1 \in E(G)$ and $z_2 v_1 \in E(G)$, then $z_1, z_2 \in S_1$. If both $z_1 s_1' \notin E(G)$ and $z_2 s_1' \notin E(G)$, then $z_1 z_2 \in E(G)$ and the subgraph induced by $\{z_1, z_2\} \cup V(Q)$ contains $Z_t (t \geq 9)$. Otherwise, assume that $z_1 s_1' \in E(G)$. As $z_1 \in S_1$, $z_1 s_1'' \notin E(G)$. Thus the subgraph induced by $\{z_1\} \cup V(Q)$ contains $Z_t (t \geq 8)$. This contradiction implies that either $z_1 v_1 \notin E(G)$ or $z_2 v_1 \notin E(G)$. Without loss of generality, we assume that $z_1 v_1 \notin E(G)$. Then $z_1 s_1' \in E(G)$ and $z_1 \in S_1'$. If $z_1 s_1'' \notin E(G)$, then the subgraph induced by $V(Q) \cup \{z_1\}$ would be $Z_t (t \geq 8)$. This contradiction implies that $z_1 s_1'' \in E(G)$. Notice that $S_i'' \cap S_j'' = \emptyset$ for $i \neq j$. If $|V(P')| \geq 3$, then the subgraph induced by $\{z_1\} \cup (V(Q) - \{s_1\})$ would be $Z_t (t \geq 8)$. This implies that $P' = s_1'' s_2''$. Consider the subgraph induced by $\{z_1, s_3''\} \cup (V(Q) - \{s_1\})$. We have either $s_1'' s_3'' \in E(G)$ or $s_2'' s_3'' \in E(G)$. If $s_2'' s_3'' \in E(G)$, then the subgraph induced by $\{z_1, s_1, s_1'\} \cup (V(P(1, 2)) \cup \{s_2, s_2', s_2'', s_3'', s_3', s_3\}$ is $Z_t (t \geq 8)$. Thus $s_2'' s_3'' \notin E(G)$ and $s_1'' s_3'' \in E(G)$.

Next we consider the neighborhood of s_3. Applying the method used on z_1 and z_2 to the neighborhood of s_3, there is a vertex $a \in N_G(s_3)$ such that $a v_3 \notin E(G)$ and $a s_3', a s_3'' \in E(G)$. Thus the subgraph induced by $\{a, s_3, s_3'\} \cup (V(P(2, 3)) \cup \{s_2, s_2', s_2'', s_1'', s_1', s_1\}$ is $Z_t (t \geq 8)$, a contradiction. So Claim 1 holds.

By Claim 1, we may assume that $x_{12} \in S_1'' \cap S_2''$. Consider S_3''. By (W3), $S_3'' \neq \emptyset$. Since there is a path $K(v_1, v_2)$ of length 2 joining v_1 and v_2 in C, then $K(v_1, v_2) s_2 s_2' x_{12} s_1' s_1 v_1$ forms an 8-cycle. So $t = 5, 6$.

Claim 2 $S_3'' \cap S_4'' = \emptyset$.

By way of contradiction, we assume that $x_{34} \in S_3'' \cap S_4''$. Since G is claw-free, by (W4), $x_{12} x_{34} \notin E(G)$, implying that $Q = s_1 s_1' x_{12} s_2' s_2 v_2 P(2, 3) v_3 s_3 s_3' x_{34} s_4' s_4$ is an induced path on at least 12 vertices. Since G is 4-connected, we assume that $\{z_3, z_4, s_1, x_{12}\} \subseteq N_G(s_1')$.

Let us consider z_3 first. Since G is claw-free, we have either $z_3 s_1 \in E(G)$ or $z_3 x_{12} \in E(G)$. If $z_3 \in S_1 \cup S_1'$, then $N_G(z_3) \cap V(Q) \subseteq \{s_1, s_1', x_{12}\}$. Thus the subgraph induced by $V(Q) \cup \{z_3\}$ contains Z_9, a contradiction. This contradiction implies that $z_3 \notin S_1 \cup S_1'$. As $z_3 s_1' \in E(G)$, $z_3 \in S_1''$, and so $z_3 s_1 \notin E(G)$ and $z_3 x_{12} \in E(G)$. Applying this argument on z_4, we have $z_4 \in S_1''$ and $z_4 x_{12} \in E(G)$. As G is claw-free, $z_3 z_4 \in E(G)$.

If $z_3 s_2' \in E(G)$, by (W4), $z_3 s_3', z_3 s_4' \notin E(G)$. Thus $z_3 x_{34} \notin E(G)$, implying that the subgraph induced by $(V(Q) - \{s_1, s_1'\}) \cup \{z_3\}$ is $Z_t (t \geq 8)$, a contradiction. So $z_3 s_2' \notin E(G)$. Similarly, $z_4 s_2' \notin E(G)$

If $z_3 x_{34} \notin E(G)$, then $z_3 s_3' \notin E(G)$ (otherwise, $G[\{s_3', s_3, z_3, x_{34}\}]$ is a claw, a contradiction). Similarly, $z_3 s_4' \notin E(G)$. Thus the subgraph induced by $(V(Q) - \{s_1\}) \cup \{z_3\}$ is $Z_t (t \geq 9)$, a contradiction. So $z_3 x_{34} \in E(G)$. Similarly, $z_4 x_{34} \in E(G)$.

Next let us consider the neighborhood of s_2'. Since G is 4-connected and since $z_3, z_4, x_{34} \notin N_G(s_2')$, we assume that $\{z_5, z_6, s_2, x_{12}\} \subseteq N_G(s_2')$. Using the method we used for z_3 and z_4 on the vertices z_5 and z_6, we have $G[\{z_5, z_6, s'_2, x_{12}\}]$ is a clique and $z_5x_{34}, z_6x_{34} \in E(G)$. Thus the subgraph induced by $\{z_3, z_4, z_5, z_6, s'_1, s'_2, x_{12}, x_{34}\}$ contains cycles of lengths 6,7,8, a contradiction. Claim 2 holds.

Claim 3 $S''_3 \cap (S''_1 \cup S''_2) \neq \emptyset$ and $S''_4 \cap (S''_1 \cup S''_2) \neq \emptyset$.

We prove $S''_3 \cap (S''_1 \cup S''_2) \neq \emptyset$ by contradiction. The proof for $S''_4 \cap (S''_1 \cup S''_2) \neq \emptyset$ is similar. Suppose that $S''_3 \cap (S''_1 \cup S''_2) = \emptyset$. Then there is a vertex s'''_3 such that $s''_3s'''_3 \in E(G)$ and the distance between s'''_3 and v_3 is 4. As G has no $(t+1)$-cycles, $N_G(s'''_3) \cap V(C) = \emptyset$. By Claim 2 and the assumption of $S''_3 \cap (S''_1 \cup S''_2) = \emptyset$, $N_G(s'''_3) \cap (S_1 \cup S_2 \cup S_3 \cup S_4) = \emptyset$ and $N_G(s''_3) \cap \{s'_1, s'_2, x_{12}\} = \emptyset$.

Consider the neighborhood of s_1 and let $\{z_7, z_8, v_1, s'_1\} \subseteq N_G(s_1)$. If both $z_7v_1 \in E(G)$ and $z_8v_1 \in E(G)$, then $z_7, z_8 \in S_1$ and $z_7z_8 \in E(G)$. If $z_7s'_1 \notin E(G)$ and $z_8s'_1 \notin E(G)$, the subgraph induced by $\{z_7, z_8, s_1\} \cup \{s'_1, x_{12}, s'_2, s_2\} \cup V(P(2,3)) \cup \{s_3, s'_3\}$ is $Z_t(t \geq 8)$. Otherwise, assume that $z_7s'_1 \in E(G)$. Then the subgraph induced by $\{z_7, s_1, s'_1\} \cup \{x_{12}, s'_2, s_2\} \cup V(P(2,3)) \cup \{s_3, s'_3, s''_3\}$ is $Z_t(t \geq 8)$. This contradiction implies that either $z_7v_1 \notin E(G)$ or $z_8v_1 \notin E(G)$. Without loss of generality, we assume that $z_7v_1 \notin E(G)$. Then $z_7s'_1 \in E(G)$ and $z_7 \in S'_1$. If $z_7x_{12} \notin E(G)$, the subgraph induced by $\{z_7, s_1, s'_1\} \cup \{x_{12}, s'_2, s_2\} \cup V(P(2,3)) \cup \{s_3, s'_3, s''_3\}$ would $Z_t(t \geq 8)$. This contradiction implies that $z_7x_{12} \in E(G)$.

Considering the subgraph induced by $\{z_7, s'_1, x_{12}\} \cup \{s'_2, s_2\} \cup V(P(2,3)) \cup \{s_3, s'_3, s''_3, s'''_3\}$, we have $N_G(s'''_3) \cap \{z_7, s'_1, s'_2, x_{12}\} \neq \emptyset$. Notice that if $x_{12}s'''_3 \notin E(G)$, then $N_G(s'''_3) \cap \{z_7, s'_1, s'_2\} \neq \emptyset$. Thus either $G[\{s'_1, s_1, x_{12}, s'''_3\}] = K_{1,3}$, or $G[\{s'_2, s_2, x_{12}, s'''_3\}] = K_{1,3}$, or $G[\{z_7, s_1, x_{12}, s'''_3\}] = K_{1,3}$. This contradiction implies that $x_{12}s'''_3 \in E(G)$. By $G[\{x_{12}, s'''_3, s'_1, s'_2\}]$, we have either $s'''_3s'_1 \in E(G)$ or $s'''_3s'_2 \in E(G)$. Without loss of generality, we assume that $s'_2s'''_3 \in E(G)$ (otherwise, we can consider the neighborhood of s_2 instead). As $S''_3 \cap (S''_1 \cup S''_2 \cup S''_4) = \emptyset$, $N_G(s'''_3) \cap \{s'_1, z_7, s'_4\} = \emptyset$ (otherwise, $G[\{s'''_3, s'_2, s''_3, w\}] = K_{1,3}$, where $w \in \{s'_1, z_7, s'_4\}$, a contradiction). Then the subgraph induced by $\{z_7, s'_1, x_{12}\} \cup \{s'''_3, s''_3, s'_3, s_3\} \cup V(P(3,4)) \cup \{s_4, s'_4\}$ is $Z_t(t \geq 8)$, a contradiction. Therefore, Claim 3 holds.

By Claim 3, without loss of generality, we assume that $S''_3 \cap S''_1 \neq \emptyset$. Let $x_{13} \in S''_1 \cap S''_3$. Applying the argument used in Claim 2 on S''_2 and S''_4, we have $S''_2 \cap S''_4 = \emptyset$. By Claim 3, $S''_1 \cap S''_4 \neq \emptyset$. Let $x_{14} \in S''_1 \cap S''_4$. Since G is claw-free, $x_{12}x_{13}, x_{12}x_{14}, x_{13}x_{14} \in E(G)$. By (W4), $S'_4 \cap (N_G(x_{12}) \cup N_G(x_{13})) = \emptyset$.

Consider the neighborhood of s_4 and let $\{z_9, z_{10}, v_4, s'_4\} \subseteq N_G(s_4)$. If both $z_9v_4 \in E(G)$ and $z_{10}v_4 \in E(G)$, then $z_9, z_{10} \in S_4$ and $z_9z_{10} \in E(G)$. If $z_9s'_4 \notin E(G)$ and $z_{10}s'_4 \notin E(G)$, the subgraph induced by $\{z_9, z_{10}, s_4\} \cup \{s'_4, x_{14}, x_{13}, s'_3, s_3\} \cup V(P(2,3)) \cup \{s_2, s'_2\}$ is $Z_t(t \geq 9)$. Otherwise, assume that $z_9s'_4 \in E(G)$. Then the subgraph induced by $\{z_9, s_4, s'_4\} \cup \{x_{14}, x_{13}, s'_3, s_3\} \cup V(P(2,3)) \cup \{s_2, s'_2\}$ is $Z_t(t \geq 8)$. This contradiction implies that either $z_9v_4 \notin E(G)$ or $z_{10}v_4 \notin E(G)$. Without loss of generality, we assume that $z_9v_4 \notin E(G)$. Then $z_9s'_4 \in E(G)$ and $z_9 \in S'_4$. Thus, $z_9x_{13}, z_9x_{12} \notin E(G)$ (otherwise, $G[\{x_{12}, s'_1, s'_2, z_9\}] = K_{1,3}$ and $G[\{x_{13}, s'_1, s'_3, z_9\}] = K_{1,3}$, a contradiction). Therefore, the subgraph induced by $\{s_4, s'_4, z_9\} \cup V(P(3,4)) \cup \{s_3, s'_3, x_{13}, x_{12}, s'_2, s_2\}$ is $Z_t(t \geq 8)$, a contradiction. □

References

1. Bondy, J.A., Murty, U.S.R.: Graph Theory with Applications. Elsevier, New York (1976)
2. Ferrara, M., Gould, R., Gehrke, S., Magnant, C., Pfender, F.: Pancyclicity of 4-connected claw, generalized net-free graphs **(submitted)** (2010)
3. Ferrara, M., Morris, T., Wenger, P.: Pancyclicity of 4-connected, claw-free, P_{10}-free graphs. J. Graph Theory **71**(4), 435–447 (2012)
4. Gould, R., Łuczak, T., Pfender, F.: Pancyclicity of 3-connected graphs: pairs of forbidden subgraphs. J. Graph Theory **47**(3), 183–202 (2004)
5. Lai, H.-J., Xiong, L., Yan, H., Yan, J.: Every 3-connected claw-free Z_8-free graph is Hamiltonian. J. Graph Theory **64**(1), 1–11 (2010)
6. Matthews, M.M., Sumner, D.P.: Hamiltonian results in $K_{1,3}$-free graphs. J. Graph Theory **8**(1), 139–146 (1984)
7. Ryjáček, Z.: On, : a closure concept in claw-free graphs. J. Combin. Theory Ser. B **70**(2), 217–224 (1997)