



Vertex-connectivity and eigenvalues of graphs with fixed girth

Ruifang Liu^a, Hong-Jian Lai^{b,*}, Yingzhi Tian^c, Yang Wu^b^aSchool of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan 450001, China^bDepartment of Mathematics, West Virginia University, Morgantown, WV 26506, USA^cCollege of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, China

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ABSTRACT

Let $\kappa(G)$, $g(G)$, $\delta(G)$ and $\Delta(G)$ denote the vertex-connectivity, the girth, the minimum degree and the maximum degree of a simple graph G , and let $\lambda_i(G)$, $\mu_i(G)$ and $q_i(G)$ denote the i th largest adjacency eigenvalue, Laplacian eigenvalue and signless Laplacian eigenvalue of G . We investigate functions $f(\delta, \Delta, g, k)$ with $\Delta \geq \delta \geq k \geq 2$ and $g \geq 3$ such that any graph G satisfying $\lambda_2(G) < f(\delta(G), \Delta(G), g(G), k)$ has connectivity $\kappa(G) \geq k$. Analogous results involving the Laplacian eigenvalues and the signless Laplacian eigenvalues to describe connectivity of a graph are also presented. As corollaries, we show that for an integer $k \geq 2$ and a simple graph G with $n = |V(G)|$, maximum degree Δ and minimum degree $\delta \geq k$, the connectivity $\kappa(G) \geq k$ if one of the following holds.

- (i) $\lambda_2(G) < \delta - \frac{(k-1)\Delta n}{2(\delta-k+2)(n-\delta+k-2)}$, or
 (ii) $\mu_{n-1}(G) > \frac{(k-1)\Delta n}{2(\delta-k+2)(n-\delta+k-2)}$, or
 (iii) $q_2(G) < 2\delta - \frac{(k-1)\Delta n}{2(\delta-k+2)(n-\delta+k-2)}$.

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1. Introduction

We consider finite and simple graphs and follow Refs. [3,4] for undefined terms and notation. In particular, $\Delta(G)$, $\delta(G)$ and $\kappa(G)$ denote the maximum degree, the minimum degree and the vertex-connectivity of a graph G , respectively. The girth of a graph G , is defined as

$$g(G) = \begin{cases} \min\{|E(C)| : C \text{ is a cycle of } G\} & \text{if } G \text{ is not acyclic,} \\ \infty & \text{if } G \text{ is acyclic.} \end{cases}$$

Let $\bar{d}(G)$ be the average degree of G . As in [3], for a vertex subset $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S .

Let G be a simple graph of vertex set $\{v_1, \dots, v_n\}$. The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent and otherwise $a_{ij} = 0$. As G is a simple and undirected graph, $A(G)$ is a symmetric $(0,1)$ -matrix. Throughout this paper, we use $\lambda_i(G)$ to denote the i th largest adjacency eigenvalue of G . Let $P_G(\lambda)$ be the characteristic polynomial of G . Let $D(G)$ be the diagonal degree matrix of G . The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are known as the Laplacian matrix and the signless Laplacian matrix of G , respectively. We use $\mu_i(G)$ and $q_i(G)$ to denote the i th largest eigenvalue of $L(G)$ and $Q(G)$, respectively. In [1], Abiad et al. raised the following research problem.

* Corresponding author.

E-mail addresses: rliu@zzu.edu.cn (R. Liu), hjlai@math.wvu.edu (H.-J. Lai), tianyzhxj@163.com (Y. Tian), ygwu@mix.wvu.edu (Y. Wu).

Problem 1.1 (Abiad et al. [1]). For a d -regular simple graph G and an integer k with $2 \leq k \leq d$, what is the best upper bound for $\lambda_2(G)$ which guarantees $\kappa'(G) \geq k$ or $\kappa(G) \geq k$?

The edge-connectivity problem was earlier investigated by Cioabă [6], and has been intensively studied by many others in [1,5,7,9,10,12–15,17,18], among others. For the vertex-connectivity, the following results have been proved.

Theorem 1.2. Let d and k be integers with $d \geq k \geq 2$ and G be a d -regular multigraph. Each of the following holds.

- (i) (O [18]) If $|V(G)| \geq 3$ and $\lambda_2(G) < \frac{3d}{4}$, then $\kappa(G) \geq 2$.
- (ii) (Abiad et al. [1]) Suppose G is not spanned by a complete graph on at most k vertices, and let

$$f(d, k) = \begin{cases} 2 & \text{if } G \text{ is a multigraph and } k = 2, \\ 1 & \text{if } G \text{ is a multigraph and } k \geq 3, \\ d + 1 & \text{if } G \text{ is a simple graph and } k = 2, \\ d - k + 2 & \text{if } G \text{ is a simple graph and } k \geq 3. \end{cases}$$

If $\lambda_2(G) < d - \frac{(k-1)d}{2f(d,k)} - \frac{(k-1)d}{2(n-f(d,k))}$, then $\kappa(G) \geq k$.

These former results motivate the current research. The purpose of this study is to investigate ranges of $\lambda_2(G)$, $\mu_{n-1}(G)$ and $q_2(G)$ which assure that $\kappa(G) \geq k$ for a simple graph G . The results can also be routinely extended to multigraphs. We first introduce some of the functions that will appear in our discussions.

Definition 1.3. For integers Δ, δ, k and g with $\Delta \geq \delta \geq k \geq 2, g \geq 3$ and $1 \leq c \leq k - 1$, we have the following definitions.

- (i) Define $t = \lfloor \frac{g-1}{2} \rfloor$, and

$$\nu(\delta, g, c) = \begin{cases} 1 + (\delta - c) \sum_{i=0}^{t-1} (\delta - 1)^i & \text{if } g = 2t + 1 \text{ and } c \leq \delta - 1, \\ 2 + (2\delta - 2 - c) \sum_{i=0}^{t-1} (\delta - 1)^i & \text{if } g = 2t + 2 \text{ and } \delta \geq 3, \\ 2t + 1 & \text{if } g = 2t + 2 \text{ and } \delta = 2. \end{cases} \tag{1}$$

- (ii) Define $\alpha = \lceil \frac{\delta+1+\sqrt{(\delta+1)^2-2(k-1)\Delta}}{2} \rceil$, and

$$\phi(\delta, \Delta, k) = \begin{cases} (\delta - k + 2)(n - \delta + k - 2) & \text{if } \Delta \geq 2(\delta - k + 2), \\ \alpha(n - \alpha) & \text{if } \delta \leq \Delta < 2(\delta - k + 2). \end{cases} \tag{2}$$

- (iii) Define $\beta = \lceil \frac{d+1+\sqrt{(d+1)^2-2(k-1)d}}{2} \rceil$, and

$$\varphi(d, k) = \begin{cases} (d+1)(n-d-1) & \text{if } k = 2, \\ (d-k+2)(n-d+k-2) & \text{if } k \geq 3 \text{ and } d \leq 2k-4, \\ \beta(n-\beta) & \text{if } k \geq 3 \text{ and } d > 2k-4. \end{cases} \tag{3}$$

- (iv) Define $\gamma = \lceil \delta + \sqrt{\delta^2 - (k-1)\Delta} \rceil$, and

$$\psi(\delta, \Delta, k) = \begin{cases} (2\delta - k + 1)(n - 2\delta + k - 1) & \text{if } \Delta \geq 2\delta - k + 1, \\ \gamma(n - \gamma) & \text{if } \delta \leq \Delta < 2\delta - k + 1. \end{cases} \tag{4}$$

Throughout this paper, for any graph G with the adjacency matrix A and the diagonal degree matrix D , we define $\lambda_i(G, a)$ to be the i th largest eigenvalue of $aD + A$, where $a \geq -1$ is a real number. The main results are presented as Theorems 1.4–1.8 below. Throughout the rest of this paper, unless otherwise is stated, k always denotes an integer with $k \geq 2$; and for a given graph G , we use the notation

$$n = |V(G)|, g = g(G), \Delta = \Delta(G) \text{ and } \delta = \delta(G) \geq k.$$

Theorem 1.4. Each of the following holds.

- (i) If $\lambda_2(G, a) < (a+1)\delta - \frac{(k-1)\Delta n}{2\nu(\delta, g, k-1)(n-\nu(\delta, g, k-1))}$, then $\kappa(G) \geq k$.
- (ii) If $\lambda_2(G, a) < (a+1)\delta - \frac{(k-1)\Delta}{\nu(\delta, g, k-1)}$, then $\kappa(G) \geq k$.

Example 1.5. To illustrate Theorem 1.4(i), we present an application with $a = 0$. Let k, s be integers with $\delta \geq k, s \geq 2, G_1$ and G_2 be two disjoint copies of $K_{\delta+1}$ with $V(G_i) = \{v_1^i, v_2^i, \dots, v_{\delta+1}^i\}$. Define H to be the graph obtained from the disjoint union of G_1 and G_2 by adding all possible edges between $\{v_1^1, v_2^1, \dots, v_s^1\}$ and $\{v_1^2, v_2^2, \dots, v_s^2\}$ (see Fig. 1). Direct computation yields

$$P_H(\lambda) = (\lambda + 1)^{2\delta-2} (\lambda^2 - (\delta - 1 + s)\lambda - \delta + s(\delta - s)) (\lambda^2 - (\delta - 1 - s)\lambda - \delta - s(\delta - s)).$$

Hence $\lambda_2(H) = \frac{\delta-1-s+\sqrt{(\delta+1+s)^2-4s^2}}{2}$.

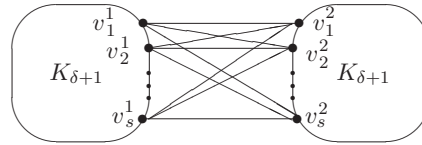


Fig. 1. Graph H.

As \$|V(H)| = 2\delta + 2\$, \$\delta(H) = \delta\$, \$\Delta(H) = \delta + s\$ and \$g(H) = 3\$, it follows by (1) that \$\nu(\delta, g, k - 1) = \nu(\delta, 3, k - 1) = \delta - k + 2\$. Thus the upper bound in Theorem 1.4(i) becomes \$\delta - \frac{(k-1)\Delta n}{2(\delta-k+2)(n-\delta+k-2)} = \delta - \frac{(k-1)(\delta+s)(\delta+1)}{(\delta-k+2)(\delta+k)}\$. It is routine to show that there exist integers \$s, k\$ and \$\delta\$ satisfying \$\delta \ge k, s \ge 2\$ and \$\frac{\delta-1-s+\sqrt{(\delta+1+s)^2-4s^2}}{2} < \delta - \frac{(k-1)(\delta+s)(\delta+1)}{(\delta-k+2)(\delta+k)}\$. Hence Theorem 1.4(i) explains that \$\kappa(H) \ge k\$. As concrete examples, by taking \$s = k - 1\$ and \$\delta \ge 6\$, Theorem 1.4(i) shows that \$\kappa(H) \ge \delta - 1\$; and by setting \$s = k - 2\$ and \$\delta \ge 9\$, Theorem 1.4(i) shows that \$\kappa(H) \ge \delta - 2\$.

For simple graphs, we have girth \$g \ge 3\$. By Definition 1.3(i), \$\nu(\delta, 3, k - 1) = \delta - k + 2\$, and so \$\nu(\delta, 3, k - 1)(n - \nu(\delta, 3, k - 1)) = (\delta - k + 2)(n - \delta + k - 2)\$. With a taking values in \$\{0, 1, -1\}\$, Corollary 3.5 in Section 3 is obtained as an immediate consequence of Theorem 1.4(i). However, arguing with a different technique, a slightly improved result for simple graphs can be obtained as follows.

Theorem 1.6. Let \$G\$ be a simple graph. Each of the following holds.

- (i) If \$\lambda_2(G) < \delta - \frac{(k-1)\Delta n}{2\phi(\delta, \Delta, k)}\$, then \$\kappa(G) \ge k\$.
- (ii) If \$\mu_{n-1}(G) > \frac{(k-1)\Delta n}{2\phi(\delta, \Delta, k)}\$, then \$\kappa(G) \ge k\$.
- (iii) If \$q_2(G) < 2\delta - \frac{(k-1)\Delta n}{2\phi(\delta, \Delta, k)}\$, then \$\kappa(G) \ge k\$.

Improved sufficient conditions to guarantee \$\kappa(G) \ge k\$ for regular graphs are stated as follows.

Theorem 1.7. Let \$G\$ be a \$d\$-regular simple graph with \$d \ge k\$. If \$\lambda_2(G) < d - \frac{(k-1)dn}{2\phi(d, k)}\$, then \$\kappa(G) \ge k\$.

For simple bipartite graphs, we have \$g \ge 4\$. By Definition 1.3(i), \$\nu(\delta, 4, k - 1) = 2\delta - k + 1\$, and so \$\nu(\delta, 4, k - 1)(n - \nu(\delta, 4, k - 1)) = (2\delta - k + 1)(n - 2\delta + k - 1)\$. With a taking values in \$\{0, 1, -1\}\$, Corollary 3.7 can be obtained as a direct consequence of Theorem 1.4(i). However, arguing with a different technique, a slightly improved result for bipartite graphs can be obtained as follows.

Theorem 1.8. Let \$G\$ be a simple bipartite graph. Each of the following holds.

- (i) If \$\lambda_2(G) < \delta - \frac{(k-1)\Delta n}{2\psi(\delta, \Delta, k)}\$, then \$\kappa(G) \ge k\$.
- (ii) If \$\mu_{n-1}(G) > \frac{(k-1)\Delta n}{2\psi(\delta, \Delta, k)}\$, then \$\kappa(G) \ge k\$.
- (iii) If \$q_2(G) < 2\delta - \frac{(k-1)\Delta n}{2\psi(\delta, \Delta, k)}\$, then \$\kappa(G) \ge k\$.

In the next section, we display some tools to be deployed in our arguments. The proofs of the main results are in the subsequent sections.

2. Preliminaries

The main tool in our paper is the eigenvalue interlacing technique described below. Given two non-increasing real sequences \$\theta_1 \ge \theta_2 \ge \dots \ge \theta_n\$ and \$\eta_1 \ge \eta_2 \ge \dots \ge \eta_m\$ with \$n > m\$, the second sequence is said to *interlace* the first one if \$\theta_i \ge \eta_i \ge \theta_{n-m+i}\$ for \$i = 1, 2, \dots, m\$. The interlacing is *tight* if exists an integer \$k \in [0, m]\$ such that \$\theta_i = \eta_i\$ for \$1 \le i \le k\$ and \$\theta_{n-m+i} = \eta_i\$ for \$k + 1 \le i \le m\$.

Lemma 2.1. (Cauchy Interlacing [2]) Let \$A\$ be a real symmetric matrix and \$B\$ be a principal submatrix of \$A\$. Then the eigenvalues of \$B\$ interlace the eigenvalues of \$A\$.

Consider an \$n \times n\$ real symmetric matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

whose rows and columns are partitioned according to a partitioning X_1, X_2, \dots, X_m of $\{1, 2, \dots, n\}$. The *quotient matrix* $R(M)$ of the matrix M is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M . The partition is *equitable* if each block $M_{i,j}$ of M has constant row (and column) sum.

Lemma 2.2. (Brouwer and Haemers [2, 11]) *Let M be a real symmetric matrix. Then the eigenvalues of every quotient matrix of M interlace the ones of M . Furthermore, if the interlacing is tight, then the partition is equitable.*

3. Proof of the main results

Following Bondy and Murty [3], for disjoint subsets A and B of $V(G)$, let $E(A, B)$ be the set of edges with one end in A and the other end in B , and

$$e(A, B) = |E(A, B)|, \text{ and } d(A) = e(A, V(G) - A).$$

Tutte [19] initiated the cage problem, which seeks, for any given integers d and g with $d \geq 2$ and $g \geq 3$, the smallest possible number $n(d, g)$ such that there exists a d -regular simple graph with girth g . A tight lower bound (often referred as the Moore bound) on $n(d, g)$ can be found in [8].

Lemma 3.1. (Exoo and Jajcay [8]) *For given integers $d \geq 2$ and $g \geq 3$, let $t = \lfloor \frac{g-1}{2} \rfloor$. Then*

$$n(d, g) \geq \begin{cases} 1 + d \sum_{i=0}^{t-1} (d-1)^i & g = 2t + 1, \\ 2 \sum_{i=0}^t (d-1)^i & g = 2t + 2. \end{cases}$$

Motivated by Lemma 3.1, we start our arguments with a technical lemma. For a subset $A \subseteq V(G)$, define $\bar{A} = V(G) - A$, and $N_G(A) = \{u \in \bar{A} : \exists v \in A \text{ such that } uv \in E(G)\}$. If $A = \{v\}$, then we use $N_G(v)$ for $N_G(\{v\})$.

Lemma 3.2. *Let G be a simple connected graph with minimum degree $\delta = \delta(G) \geq k \geq 2$ and girth $g = g(G) \geq 3$. Let C be a minimum vertex cut of G with $|C| = c$ and A be a connected component of $G - C$, and let $\nu(\delta, g, c)$ be defined as in (1). If $c \leq k - 1 < \delta$, then each of the following holds.*

(i) *For any integer s with $g \geq s \geq 3$, A contains a path $P = v_0v_1v_2 \dots v_{s-3}$ such that*

$$|N_G(v_j) - V(P)| \geq \delta - 1, \forall j \in \{0, s-3\} \text{ and } |N_G(v_i) - V(P)| \geq \delta - 2, \forall i \in \{1, 2, \dots, s-4\}. \tag{5}$$

(ii) $|A| \geq \nu(\delta, g, c)$.

Proof. Let $n_1 = |A|$ and $\nu = \nu(\delta, s, c)$. We shall prove (i) by induction on s .

Pick any $v_0 \in V(A)$. As $|N_G(v_0)| \geq \delta$, we have that A contains a path $P = v_0$ such that $|N_G(v_0) - V(P)| \geq \delta - 1$. Hence (i) holds if $s = 3$. Assume that $s = 4$. Note that $|C| \leq k - 1 < \delta$, we have $|N_G(v_0) - C| \geq 1$. Therefore, there exists a vertex $v_1 \in N_G(v_0) - C$, and hence A contains a path $P = v_0v_1$ such that $|N_G(v_0) - V(P)| \geq \delta - 1$ and $|N_G(v_1) - V(P)| \geq \delta - 1$. Therefore, (i) holds for $s = 4$.

Assume that $s \geq 5$ and Lemma 3.2(i) holds for smaller values of s . Thus A contains a path $P' = v_0v_1 \dots v_{s-4}$ satisfying (5). Note that $|C| < \delta$, we distinguish the following two cases to prove. If $|C \cap (N_G(v_{s-4}) - V(P'))| \leq \delta - 2$, it follows by $|N_G(v_{s-4}) - V(P')| \geq \delta - 1 \geq 1$ that there exists a vertex $v_{s-3} \in N_G(v_{s-4}) - V(P') - C$. Let $P = v_0v_1v_2 \dots v_{s-3}$. Then A contains a path $P = v_0v_1v_2 \dots v_{s-3}$, $|N_G(v_{s-4}) - V(P)| \geq \delta - 2$ and $|N_G(v_{s-3}) - V(P)| \geq \delta - 1$, and hence (i) holds by induction in this case. If $|C \cap (N_G(v_{s-4}) - V(P'))| = \delta - 1$, then $C \subseteq N_G(v_{s-4}) - V(P')$. Noting that $|N_G(v_0) - V(P')| \geq \delta - 1 \geq 1$, then there must be a vertex $v_{-1} \in N_G(v_0) - V(P')$ with $|N_G(v_{-1}) - V(P')| \geq \delta - 1$. This implies that, letting $u_i = v_{i-1}$ for $0 \leq i \leq s-3$, then A contains a path $P = u_0u_1 \dots u_{s-3}$ satisfying (5). Hence (i) is proved by induction.

To prove (ii), we start letting $t = \lfloor \frac{g-1}{2} \rfloor$. By (i), there exists a path $P = v_0v_1 \dots v_{g-3} \subseteq A$ satisfying (5). For any $X \subseteq V(G)$, define $N_G(X) = \{v \in V(G) - X : \exists x \in X, vx \in E(G)\}$. Recall that G is a simple connected graph with minimum degree $\delta \geq 2$ and girth $g \geq 3$, and let C be a minimum vertex cut of G .

Case 1. $g = 2t + 1$ is odd.

Choose $v_{t-1} = v_{\lfloor \frac{g-1}{2} \rfloor - 1} \in V(P)$ as a root vertex. Define $N_0 = \{v_{t-1}\}$, $N_1 = N(v_{t-1})$, $c_1 = |N_1 \cap C|$. For each i with $2 \leq i \leq t$, define $N_i = N(N_{i-1} - C) - N_{i-2}$ and $c_i = |N_i \cap C|$. Thus for each i, j with $1 \leq i < j \leq t$, by definition, $N_i \cap N_j = \emptyset$, and $\sum_{i=1}^t c_i \leq |C| = c$.

Clearly $|N_1 - C| \geq \delta - c_1$. As $g(G) = g$, for every $i \geq 1$, and each $v \in N_i$, we have

$$|N_G(v) \cap N_{i-1}| = 1 \text{ and so } |N_G(v) \cap N_{i+1}| \geq \delta - 1. \tag{6}$$

It follows that $|N_2 - C| \geq (\delta - c_1)(\delta - 1) - c_2$. Inductively, assume that $3 \leq i \leq t$ and $|N_{i-1} - C| \geq (\delta - c_1)(\delta - 1)^{i-2} - c_2(\delta - 1)^{i-3} - \dots - c_{i-2}(\delta - 1) - c_{i-1}$. Then by (6) that $|N_i - C| \geq (\delta - 1)|N_{i-1} - C| - c_i \geq (\delta - c_1)(\delta - 1)^{i-1} - c_2(\delta - 1)^{i-2} - \dots - c_{i-1}(\delta - 1) - c_i$.

Note that $c \leq k - 1 \leq \delta - 1$. As $N_0, N_1 - C, N_2 - C, \dots, N_t - C$ are mutually disjoint subsets of A , we have

$$\begin{aligned} |A| &\geq |N_0 \cup (N_1 - C) \cup (N_2 - C) \cup \dots \cup (N_t - C)| \\ &= |N_0| + |N_1 - C| + |N_2 - C| + \dots + |N_t - C| \end{aligned}$$

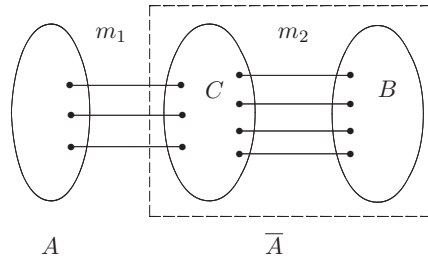


Fig. 2. The partition of $V(G)$ into A and \bar{A} .

$$\begin{aligned}
 &\geq 1 + (\delta - c_1) + [(\delta - c_1)(\delta - 1) - c_2] + \dots + [(\delta - c_1)(\delta - 1)^{i-1} - c_2(\delta - 1)^{i-2} - \dots - c_i] \\
 &\quad + \dots + [(\delta - c_1)(\delta - 1)^{t-1} - c_2(\delta - 1)^{t-2} - \dots - c_t] \\
 &= 1 + (\delta - c_1 - c_2 - \dots - c_t) + (\delta - c_1 - c_2 - \dots - c_{t-1})(\delta - 1) \\
 &\quad + \dots + (\delta - c_1 - c_2)(\delta - 1)^{t-2} + (\delta - c_1)(\delta - 1)^{t-1} \\
 &\geq 1 + \left(\delta - \sum_{i=1}^t c_i\right) + \left(\delta - \sum_{i=1}^t c_i\right)(\delta - 1) + \dots + \left(\delta - \sum_{i=1}^t c_i\right)(\delta - 1)^{t-2} + \left(\delta - \sum_{i=1}^t c_i\right)(\delta - 1)^{t-1} \\
 &\geq 1 + (\delta - c) \sum_{i=0}^{t-1} (\delta - 1)^i = \nu(\delta, g, c). \tag{7}
 \end{aligned}$$

Case 2. $g = 2t + 2$ is even.

Choose $v_{t-1}v_t = v_{\lfloor \frac{g-1}{2} \rfloor - 1}v_{\lfloor \frac{g-1}{2} \rfloor} \in E(P)$ as a root edge, and define $N_0 = \{v_{t-1}, v_t\}$, $N_1 = N(v_{t-1}) \cup N(v_t) - \{v_{t-1}, v_t\}$. For each i with $2 \leq i \leq t$, define $N_i = N(N_{i-1} - C) - N_{i-2}$. By the definition of the girth of G , we observe that N_1, N_2, \dots, N_t are mutually disjoint. As before, for each i with $1 \leq i \leq t$, set $c_i = |N_i \cap C|$. Then by definition, $\sum_{i=1}^t c_i \leq |C| = c$, and $|N_1 - C| \geq 2(\delta - 1) - c_1$. Since the girth of G is g , for each $i \in \{1, 2, \dots, t\}$ and for each $v \in N_i$, $|N_G(v) \cap N_{i-1}| = 1$, and so (6) holds again. Once again, using (6), we inductively conclude that for $i \in \{2, 3, \dots, t\}$,

$$|N_i - C| \geq (2\delta - 2 - c_1)(\delta - 1)^{i-1} - c_2(\delta - 1)^{i-2} - \dots - c_{i-1}(\delta - 1) - c_i.$$

If $\delta \geq 3$, then as $N_0, (N_1 - C), (N_2 - C), \dots, (N_t - C)$ are mutually disjoint subsets of A , and as $c \leq k - 1 \leq \delta - 1 \leq 2\delta - 4$, we have

$$\begin{aligned}
 |A| &\geq |N_0 \cup (N_1 - C) \cup (N_2 - C) \cup \dots \cup (N_t - C)| \\
 &= |N_0| + |N_1 - C| + |N_2 - C| + \dots + |N_t - C| \\
 &\geq 2 + (2\delta - 2 - c_1) + [(2\delta - 2 - c_1)(\delta - 1) - c_2] \\
 &\quad + \dots + [(2\delta - 2 - c_1)(\delta - 1)^{i-1} - c_2(\delta - 1)^{i-2} - \dots - c_i] \\
 &\quad + \dots + [(2\delta - 2 - c_1)(\delta - 1)^{t-1} - c_2(\delta - 1)^{t-2} - \dots - c_t] \\
 &= 2 + (2\delta - 2 - c_1 - c_2 - \dots - c_t) + (2\delta - 2 - c_1 - c_2 - \dots - c_{t-1})(\delta - 1) \\
 &\quad + \dots + (2\delta - 2 - c_1 - c_2)(\delta - 1)^{t-2} + (2\delta - 2 - c_1)(\delta - 1)^{t-1} \\
 &\geq 2 + (2\delta - 2 - c) \sum_{i=0}^{t-1} (\delta - 1)^i = \nu(\delta, g, c). \tag{8}
 \end{aligned}$$

If $\delta = 2$, then we have $1 = c = k - 1 = \delta - 1$. By Lemma 3.2(i), A contains the path $P = v_0v_1v_2 \dots v_{t-1}v_t \dots v_{2t-1}$. Hence we have

$$|A| \geq 2t + 2 - 1 = 2t + 1 = \nu(\delta, g, c). \tag{9}$$

This justifies Case 2, and completes the proof of the lemma. \square

3.1. Proof of Theorem 1.4

Suppose that k is an integer with $k \geq 2$. By contradiction, we assume that $\kappa(G) = c \leq k - 1$. Then there exists a minimum vertex cut C of G such that $|C| = c \leq k - 1 \leq \delta - 1$. Let A be a connected component of $G - C$ and let $m_1 = e(A, C) = d(A)$. Then the incidence counting leads to $m_1 \leq \frac{c\Delta}{2} \leq \frac{(k-1)\Delta}{2}$. Let $B = G - (V(A) \cup C)$, $m_2 = e(B, C)$, and $\bar{A} = V(G) - A$. See Fig. 2 for an illustration of the partition.

Throughout this subsection, we let $\nu = \nu(\delta, g, k - 1)$, $|A| = n_1$ and $|B| = n_2$. By Lemma 3.2, we have $\nu \leq \min\{n_1, n_2\} \leq \frac{n}{2} \leq n - \nu$, and so

$$n \geq 2\nu \text{ or } \frac{n}{2(n - \nu)} \leq 1. \tag{10}$$

Since $n_1 + c + n_2 = n$, we also have $\nu \leq n_1 \leq n - c - \nu$. As $1 \leq c \leq k - 1$, it follows from the behavior of quadratic functions that

$$n_1(n_2 + c) = n_1(n - n_1) \geq \nu(n - \nu). \tag{11}$$

Let $\bar{d}_1 = \frac{1}{n_1} \sum_{\nu \in A} d_G(\nu)$ and $\bar{d}_2 = \frac{1}{n_2+c} \sum_{\nu \in \bar{A}} d_G(\nu)$. Then $\min\{\bar{d}_1, \bar{d}_2\} \geq \delta$. With these notation, the quotient matrix $R(aD + A)$ of $aD + A$ corresponding to the partition $(A, C \cup B)$ becomes:

$$R(aD + A) = \begin{pmatrix} (a + 1)\bar{d}_1 - \frac{m_1}{n_1} & \frac{m_1}{n_2} \\ \frac{m_1}{n_2+c} & (a + 1)\bar{d}_2 - \frac{m_1}{n_2+c} \end{pmatrix}.$$

As the characteristic polynomial of $R(aD + A)$ is

$$\lambda^2 - \left[(a + 1)\bar{d}_1 - \frac{m_1}{n_1} + (a + 1)\bar{d}_2 - \frac{m_1}{n_2 + c} \right] \lambda + \left[(a + 1)\bar{d}_1 - \frac{m_1}{n_1} \right] \left[(a + 1)\bar{d}_2 - \frac{m_1}{n_2 + c} \right] - \frac{m_1^2}{n_1(n_2 + c)},$$

we have, by a direct computation,

$$\begin{aligned} & \lambda_2(R(aD + A)) \\ &= \frac{1}{2} \left\{ \left[(a + 1)\bar{d}_1 - \frac{m_1}{n_1} + (a + 1)\bar{d}_2 - \frac{m_1}{n_2 + c} \right] \right. \\ & \quad \left. - \sqrt{\left[(a + 1)\bar{d}_1 - \frac{m_1}{n_1} + (a + 1)\bar{d}_2 - \frac{m_1}{n_2 + c} \right]^2 - 4 \left[(a + 1)\bar{d}_1 - \frac{m_1}{n_1} \right] \left[(a + 1)\bar{d}_2 - \frac{m_1}{n_2 + c} \right] + \frac{4m_1^2}{n_1(n_2 + c)}} \right\} \\ &= \frac{1}{2} \left\{ \left[(a + 1)\bar{d}_1 - \frac{m_1}{n_1} + (a + 1)\bar{d}_2 - \frac{m_1}{n_2 + c} \right] - \sqrt{\left[(a + 1)\bar{d}_1 - \frac{m_1}{n_1} - (a + 1)\bar{d}_2 + \frac{m_1}{n_2 + c} \right]^2 + \frac{4m_1^2}{n_1(n_2 + c)}} \right\} \\ &= \frac{1}{2} \left\{ \left[(a + 1)\bar{d}_1 - \frac{m_1}{n_1} + (a + 1)\bar{d}_2 - \frac{m_1}{n_2 + c} \right] - \sqrt{\left[(a + 1)(\bar{d}_1 - \bar{d}_2) - \left(\frac{m_1}{n_1} - \frac{m_1}{n_2 + c} \right) \right]^2 + \frac{4m_1^2}{n_1(n_2 + c)}} \right\} \\ &= \frac{1}{2} \left\{ \left[(a + 1)\bar{d}_1 - \frac{m_1}{n_1} + (a + 1)\bar{d}_2 - \frac{m_1}{n_2 + c} \right] \right. \\ & \quad \left. - \sqrt{(a + 1)^2(\bar{d}_1 - \bar{d}_2)^2 + \left(\frac{m_1}{n_1} - \frac{m_1}{n_2 + c} \right)^2 - 2(a + 1)(\bar{d}_1 - \bar{d}_2) \left(\frac{m_1}{n_1} - \frac{m_1}{n_2 + c} \right) + \frac{4m_1^2}{n_1(n_2 + c)}} \right\} \\ &= \frac{1}{2} \left\{ \left[(a + 1)(\bar{d}_1 + \bar{d}_2) - \frac{m_1}{n_1} - \frac{m_1}{n_2 + c} \right] \right. \\ & \quad \left. - \sqrt{(a + 1)^2(\bar{d}_1 - \bar{d}_2)^2 + \left(\frac{m_1}{n_1} + \frac{m_1}{n_2 + c} \right)^2 + 2(a + 1)(\bar{d}_1 - \bar{d}_2) \left(\frac{m_1}{n_2 + c} - \frac{m_1}{n_1} \right)} \right\} \\ &\geq \frac{1}{2} \left\{ \left[(a + 1)(\bar{d}_1 + \bar{d}_2) - \frac{m_1}{n_1} - \frac{m_1}{n_2 + c} \right] \right. \\ & \quad \left. - \sqrt{(a + 1)^2(\bar{d}_1 - \bar{d}_2)^2 + \left(\frac{m_1}{n_1} + \frac{m_1}{n_2 + c} \right)^2 + 2(a + 1)|\bar{d}_1 - \bar{d}_2| \left(\frac{m_1}{n_1} + \frac{m_1}{n_2 + c} \right)} \right\} \\ &= \frac{1}{2} \left\{ \left[(a + 1)(\bar{d}_1 + \bar{d}_2) - \frac{m_1}{n_1} - \frac{m_1}{n_2 + c} \right] - \left[(a + 1)|\bar{d}_1 - \bar{d}_2| + \left(\frac{m_1}{n_1} + \frac{m_1}{n_2 + c} \right) \right] \right\} \\ &= \min\left[(a + 1)\bar{d}_1, (a + 1)\bar{d}_2 \right] - \frac{m_1 n}{n_1(n_2 + c)} \\ &\geq (a + 1)\delta - \frac{(k - 1)\Delta n}{2\nu(n - \nu)}. \end{aligned}$$

It follows by Lemma 2.2 that $\lambda_2(G, a) \geq \lambda_2(R(aD + A)) \geq (a + 1)\delta - \frac{(k-1)\Delta n}{2\nu(n-\nu)}$, contrary to the assumption of Theorem 1.4(i). This proves Theorem 1.4(i).

By contradiction, assume that $\kappa(G) = |C| \leq k - 1$. By (10), $\frac{(k-1)\Delta}{\nu} \geq \frac{(k-1)\Delta n}{2\nu(n-\nu)}$. From the proof of Theorem 1.4(i), we have $\lambda_2(G, a) \geq \lambda_2(R(aD + A)) \geq (a + 1)\delta - \frac{(k-1)\Delta n}{2\nu(n-\nu)} \geq (a + 1)\delta - \frac{(k-1)\Delta}{\nu}$, contrary to the assumption of Theorem 1.4(ii), and so Theorem 1.4(ii) follows. This completes the proof of the theorem.

3.2. Corollaries of Theorem 1.4

Throughout this section, a and b are two real numbers satisfying $\frac{a}{b} \geq -1$, and k is an integer with $k \geq 2$. We only consider simple graphs, and we always denote $n = |V(G)|$, $\delta = \delta(G)$, $\Delta = \Delta(G)$ and $g = g(G)$ for a graph G . Let $\lambda_i(G, a, b)$ be the i th largest eigenvalue of the matrix $aD + bA$. Thus $\lambda_i(G, a, 1) = \lambda_i(G, a)$. In Corollaries 3.3 and 3.4 below, we let $\nu = \nu(\delta, g, k - 1)$.

Corollary 3.3. *Suppose that $\delta \geq k$. Then $\kappa(G) \geq k$ if one of the following holds.*

- (i) $b > 0$ and $\lambda_2(G, a, b) < (a + b)\delta - \frac{b(k-1)\Delta n}{2\nu(n-\nu)}$.
- (ii) $b < 0$ and $\lambda_{n-1}(G, a, b) > (a + b)\delta - \frac{b(k-1)\Delta n}{2\nu(n-\nu)}$.

Proof. As $aD + bA = b(\frac{a}{b}D + A)$, it follows by definition that

$$\begin{cases} \text{if } b > 0, & \text{then } \lambda_i(G, a, b) = b\lambda_i(G, \frac{a}{b}); \text{ and} \\ \text{if } b < 0, & \text{then } \lambda_{n-i+1}(G, a, b) = b\lambda_i(G, \frac{a}{b}). \end{cases} \tag{12}$$

Hence Corollary 3.3 follows from Theorem 1.4(i). \square

Choosing $a \in \{0, -1, 1\}$ and $b = 1$ in Corollary 3.3, we have the following special case.

Corollary 3.4. *Suppose that $\delta \geq k$. Each of the following holds.*

- (i) If $\lambda_2(G) < \delta - \frac{(k-1)\Delta n}{2\nu(n-\nu)}$, then $\kappa(G) \geq k$.
- (ii) If $\mu_{n-1}(G) > \frac{(k-1)\Delta n}{2\nu(n-\nu)}$, then $\kappa(G) \geq k$.
- (iii) If $q_2(G) < 2\delta - \frac{(k-1)\Delta n}{2\nu(n-\nu)}$, then $\kappa(G) \geq k$.

Next, we consider three special cases of Theorem 1.4: simple graphs, regular graphs and bipartite graphs. So the proofs of Theorems 1.6–1.8 still base on that of Theorem 1.4 by contradiction, and hence we only write core parts of their proofs in the following.

For simple graphs, we have $g \geq 3$. By Definition 1.3(i), $\nu = \nu(\delta, g, k - 1) \geq \nu(\delta, 3, k - 1) = \delta - k + 2$. Suppose that $\kappa(G) \leq k - 1$, by (11), then $n_1(n - n_1) \geq \nu(n - \nu) \geq (\delta - k + 2)(n - \delta + k - 2)$. By Corollary 3.4, then $\lambda_2(G) \geq \delta - \frac{(k-1)\Delta n}{2\nu(n-\nu)} \geq \delta - \frac{(k-1)\Delta n}{2(\delta-k+2)(n-\delta+k-2)}$. Thus Corollary 3.5 follows.

Corollary 3.5. *Let G be a simple graph with $\delta \geq k$. Each of the following holds.*

- (i) If $\lambda_2(G) < \delta - \frac{(k-1)\Delta n}{2(\delta-k+2)(n-\delta+k-2)}$, then $\kappa(G) \geq k$.
- (ii) If $\mu_{n-1}(G) > \frac{(k-1)\Delta n}{2(\delta-k+2)(n-\delta+k-2)}$, then $\kappa(G) \geq k$.
- (iii) If $q_2(G) < 2\delta - \frac{(k-1)\Delta n}{2(\delta-k+2)(n-\delta+k-2)}$, then $\kappa(G) \geq k$.

In fact, in Corollary 3.5, the lower bound $n_1(n - n_1)$ in (11) can be slightly improved when both δ and Δ are in the play, which leads to Theorem 1.6. Throughout the proofs of Theorems 1.6–1.8 below, we will continue using the notation and similar arguments in the proof of Theorem 1.4.

Let A be a connected component of G . Next we define $d(A) = |E(A, V(G) - A)|$.

Proof of Theorem 1.6. In the proof of Theorem 1.4(i), we have that $d(A) \leq \frac{(k-1)\Delta}{2}$. Counting the degree sum of vertices in A , we have

$$n_1\delta \leq \sum_{v \in A} d_G(v) = 2|E(A)| + d(A) \leq n_1(n_1 - 1) + \frac{(k-1)\Delta}{2}.$$

It follows by algebraic manipulations and by the notation in Definition 1.3(ii) that

$$n_1 \geq \left\lceil \frac{\delta + 1 + \sqrt{(\delta + 1)^2 - 2(k-1)\Delta}}{2} \right\rceil = \alpha \quad \text{or} \quad n_1 \leq \left\lfloor \frac{\delta + 1 - \sqrt{(\delta + 1)^2 - 2(k-1)\Delta}}{2} \right\rfloor = \alpha^*.$$

Case 1. $n_1 \geq \lceil \frac{\delta + 1 + \sqrt{(\delta + 1)^2 - 2(k-1)\Delta}}{2} \rceil = \alpha$.

If $\Delta \geq 2(\delta - k + 2)$, then $n_2 + c \geq \delta + 1 > \delta - k + 2 \geq \alpha$, and so we have $n_1(n - n_1) \geq (\delta - k + 2)(n - \delta + k - 2)$. If $\delta \leq \Delta < 2(\delta - k + 2)$, then $\delta - k + 2 < \alpha \leq \delta + 1 \leq n_2 + c$, and so $n_1(n - n_1) \geq \alpha(n - \alpha)$.

Case 2. $n_1 \leq \lfloor \frac{\delta + 1 - \sqrt{(\delta + 1)^2 - 2(k-1)\Delta}}{2} \rfloor = \alpha^*$.

If $\alpha^* = \alpha$, we can turn to Case 1. Next we assume that $\alpha^* < \alpha$. If $\Delta \geq 2(\delta - k + 2)$, then $\alpha^* < \alpha \leq \delta - k + 2$, contrary to $n_1 \geq \delta - k + 2$. If $\delta \leq \Delta < 2(\delta - k + 2)$, then $\alpha^* < \delta - k + 2$, also a contradiction.

In either case, by Definition 1.3(ii), Theorem 1.6 follows. \square

Setting $\delta = \Delta$ in Corollary 3.5, a direct corollary for regular graph is obtained.

Corollary 3.6. Let k be an integer with $k \geq 2$, and G be a d -regular simple graph with order n and $d \geq k$. If $\lambda_2(G) < d - \frac{(k-1)dn}{2(d-k+2)(n-d+k-2)}$, then $\kappa(G) \geq k$.

In fact, we can do better for regular graphs as shown by Theorem 1.7.

Proof of Theorem 1.7. For a d -regular graph, if $k = 2$, then $c = 1$. By Lemma 3.2, $n_2 \geq d$. If $n_2 = d$, then $d_G(v) = d = m_2$, and so $m_1 = 0$, a contradiction. Hence we must have $n_2 \geq d + 1$. Similarly, we also have $n_1 \geq d + 1$. It follows that $d + 1 \leq \min\{n_1, n_2\} \leq \frac{n}{2} \leq n - d - 1$, which implies that $d + 1 \leq n_1 = n - n_2 - c \leq n - d - 2$. This leads to $n_1(n - n_1) \geq (d + 1)(n - d - 1)$. By Definition 1.3(iii), Theorem 1.7 holds for the case when $k = 2$.

Assume that $k \geq 3$. As shown in the proof of Theorem 1.4(i), we have $d(A) \leq \frac{(k-1)d}{2}$. Counting the degree sum of vertices in A , we have

$$n_1 d = \sum_{v \in A} d_G(v) = 2|E(A)| + d(A) \leq n_1(n_1 - 1) + \frac{(k-1)d}{2},$$

and so by Definition 1.3(iii),

$$n_1 \geq \left\lceil \frac{d+1 + \sqrt{(d+1)^2 - 2(k-1)d}}{2} \right\rceil = \beta \quad \text{or} \quad n_1 \leq \left\lfloor \frac{d+1 - \sqrt{(d+1)^2 - 2(k-1)d}}{2} \right\rfloor = \beta^*.$$

Case 1. $n_1 \geq \lceil \frac{d+1 + \sqrt{(d+1)^2 - 2(k-1)d}}{2} \rceil = \beta$.

If $d \leq 2k - 4$, then $n_2 + c \geq d + 1 > d - k + 2 \geq \beta$, and so we have $n_1(n - n_1) \geq (d - k + 2)(n - d + k - 2)$. If $d > 2k - 4$, then $d - k + 2 < \beta \leq d + 1 \leq n_2 + c$, and so $n_1(n - n_1) \geq \beta(n - \beta)$.

Case 2. $n_1 \leq \lfloor \frac{d+1 - \sqrt{(d+1)^2 - 2(k-1)d}}{2} \rfloor = \beta^*$.

If $\beta^* = \beta$, we can refer to Case 1. Next we consider $\beta^* < \beta$. If $d \leq 2k - 4$, then $\beta^* < \beta \leq d - k + 2$, a contradiction. If $d > 2k - 4$, then $\beta^* < d - k + 2$, also a contradiction.

In either case, by Definition 1.3(iii), Theorem 1.7 follows. \square

For simple bipartite graphs, we have $g \geq 4$. By Definition 1.3(i), we have $\nu(\delta, 4, k - 1) = 2\delta - k + 1$. It follows from (11) that $n_1(n - n_1) \geq \nu(n - \nu) \geq (2\delta - k + 1)(n - 2\delta + k - 1)$. Corollary 3.7 below follows from Corollary 3.4.

Corollary 3.7. Let k be an integer with $k \geq 2$, and G be a simple bipartite graph with $n = |V(G)|$, maximum degree $\Delta = \Delta(G)$ and minimum degree $\delta = \delta(G) \geq k$. Each of the following holds.

- (i) If $\lambda_2(G) < \delta - \frac{(k-1)\Delta n}{2(2\delta-k+1)(n-2\delta+k-1)}$, then $\kappa(G) \geq k$.
- (ii) If $\mu_{n-1}(G) > \frac{(k-1)\Delta n}{2(2\delta-k+1)(n-2\delta+k-1)}$, then $\kappa(G) \geq k$.
- (iii) If $q_2(G) < 2\delta - \frac{(k-1)\Delta n}{2(2\delta-k+1)(n-2\delta+k-1)}$, then $\kappa(G) \geq k$.

We can slightly improve Corollary 3.7 by utilizing the relationship among Δ , δ and k . This effort leads to Theorem 1.8. To prove Theorem 1.8, we need one more lemma below.

Lemma 3.8. (Mantel [16]) Let G be a triangle-free graph with order n . Then $|E(G)| \leq \frac{1}{4}n^2$.

Proof of Theorem 1.8. As in the proof of Theorem 1.4(i), we have $d(A) \leq \frac{(k-1)\Delta}{2}$. Counting the degree sum of vertices in A with $g \geq 4$ in Lemma 3.8, we have

$$n_1 \delta \leq \sum_{v \in A} d_G(v) = 2|E(A)| + d(A) \leq \frac{1}{2}n_1^2 + \frac{(k-1)\Delta}{2}.$$

This leads to, by algebraic manipulations and by Definition 1.3(iv),

$$n_1 \geq \lceil \delta + \sqrt{\delta^2 - (k-1)\Delta} \rceil = \gamma \quad \text{or} \quad n_1 \leq \lfloor \delta - \sqrt{\delta^2 - (k-1)\Delta} \rfloor = \gamma^*.$$

Case 1. $n_1 \geq \lceil \delta + \sqrt{\delta^2 - (k-1)\Delta} \rceil = \gamma$.

If $\Delta \geq 2\delta - k + 1$, then $n_2 + c \geq 2\delta > 2\delta - k + 1 \geq \gamma$, and so we have $n_1(n - n_1) \geq (2\delta - k + 1)(n - 2\delta + k - 1)$. If $\delta \leq \Delta < 2\delta - k + 1$, then $2\delta - k + 1 < \gamma \leq 2\delta \leq n_2 + c$, and so we have $n_1(n - n_1) \geq \gamma(n - \gamma)$.

Case 2. $n_1 \leq \lfloor \delta - \sqrt{\delta^2 - (k-1)\Delta} \rfloor = \gamma^*$.

If $\gamma^* = \gamma$, we can refer to Case 1. Next we assume that $\gamma^* < \gamma$. If $\Delta \geq 2\delta - k + 1$, then $\gamma^* < \gamma \leq 2\delta - k + 1$, contrary to $n_1 \geq 2\delta - k + 1$. If $\delta \leq \Delta < 2\delta - k + 1$, then $\gamma^* < 2\delta - k + 1$, also a contradiction.

In either case, Theorem 1.8 follows by checking Definition 1.3(iv). \square

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