

Nonseparating trees in 2-connected graphs and oriented trees in strongly connected digraphs

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ABSTRACT

Mader (2010) conjectured that for every positive integer k and every finite tree T with order m , every k -connected, finite graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor + m - 1$ contains a subtree T' isomorphic to T such that $G - V(T')$ is k -connected. The conjecture has been verified for paths, trees when $k = 1$, and stars or double-stars when $k = 2$. In this paper we verify the conjecture for two classes of trees when $k = 2$.

For digraphs, Mader (2012) conjectured that every k -connected digraph D with minimum semi-degree $\delta(D) = \min\{\delta^+(D), \delta^-(D)\} \geq 2k + m - 1$ for a positive integer m has a dipath P of order m with $\kappa(D - V(P)) \geq k$. The conjecture has only been verified for the dipath with $m = 1$, and the dipath with $m = 2$ and $k = 1$. In this paper, we prove that every strongly connected digraph with minimum semi-degree $\delta(D) = \min\{\delta^+(D), \delta^-(D)\} \geq m + 1$ contains an oriented tree T isomorphic to some given oriented stars or double-stars with order m such that $D - V(T)$ is still strongly connected.

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1. Introduction

In this paper, all graphs (digraphs) are finite and without multiple edges (parallel arcs) and without loops. For graph-theoretical terminologies and notation not defined here, we follow [1]. A graph (digraph) is k -connected means (strongly) k -vertex-connected. We use $\kappa(G)$ ($\kappa(D)$) to denote the connectivity of the graph G (digraph D). The order of a graph G (digraph D) is the cardinality of its vertex set, denoted by $|G|$ ($|D|$).

In 1972, Chartrand, Kaugars, and Lick proved the following well-known result.

Theorem 1.1 ([2]). *Every k -connected graph G of minimum degree $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor$ has a vertex u with $\kappa(G - u) \geq k$.*

Fujita and Kawarabayashi proved in [4] that every k -connected graph G with minimum degree at least $\lfloor \frac{3}{2}k \rfloor + 2$ has an edge $e = uv$ such that $G - \{u, v\}$ is still k -connected. In the same paper, they stated the following conjecture.

Conjecture 1 ([4]). *For all positive integers k, m , there is a (least) non-negative integer $f_k(m)$ such that every k -connected graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor - 1 + f_k(m)$ contains a connected subgraph W of exact order m such that $G - V(W)$ is still k -connected.*

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The examples given in [4] showed that $f_k(m)$ must be at least m for all positive integers k, m . In [7], Mader confirmed [Conjecture 1](#) and proved that $f_k(m) = m$ holds for all k, m .

Theorem 1.2 ([7]). *Every k -connected graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor + m - 1$ for positive integers k, m contains a path P of order m such that $G - V(P)$ remains k -connected.*

Mader [7] further conjectured that [Theorem 1.2](#) holds for all trees.

Conjecture 2 ([7]). *For every positive integer k and every finite tree T , there is a least non-negative integer $t_k(T)$, such that every k -connected, finite graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor - 1 + t_k(T)$ contains a subgraph $T' \cong T$ with $\kappa(G - V(T')) \geq k$. Furthermore, $t_k(T) = |T|$ holds.*

In [8], Mader showed that $t_k(T)$ exists. Actually, he showed that $t_k(T) \leq 2(k + m - 1)^2 + m - \lfloor \frac{3}{2}k \rfloor$. [Theorem 1.2](#) implied that [Conjecture 2](#) is true when T is a path. Diwan and Tholiya [3] proved that [Conjecture 2](#) holds when $k = 1$. In [9], the authors verified that [Conjecture 2](#) is true when T is a star or double-star and $k = 2$. In Section 3, we will verify [Conjecture 2](#) for two classes of trees when $k = 2$.

The minimum outdegree and the minimum indegree of a digraph D are denoted by $\delta^+(D)$ and $\delta^-(D)$, respectively. The minimum semi-degree of D is $\delta(D) := \min\{\delta^+(D), \delta^-(D)\}$. The following result is a digraph analogue to [Theorem 1.1](#).

Theorem 1.3 ([6]). *Every k -connected digraph D with minimum semi-degree $\delta(D) = \min\{\delta^+(D), \delta^-(D)\} \geq 2k$ has a vertex u with $\kappa(D - u) \geq k$.*

Considering the results for graphs and digraphs, Mader [8] suggested the following conjecture.

Conjecture 3 ([8]). *Every k -connected digraph D with minimum semi-degree $\delta(D) = \min\{\delta^+(D), \delta^-(D)\} \geq 2k + m - 1$ for a positive integer m has a dipath P of order m with $\kappa(D - V(P)) \geq k$.*

Mader remarked that one could conjecture also similar results for trees with special orientations, but he thought even a proof of [Conjecture 3](#) is very difficult. [Conjecture 3](#) has only been verified for the dipath with $m = 1$, and the dipath with $m = 2$ and $k = 1$. In Section 4, we will prove that every strongly connected digraph with minimum semi-degree $\delta(D) = \min\{\delta^+(D), \delta^-(D)\} \geq m + 1$ contains an oriented tree T isomorphic to some given oriented stars or double-stars with order m such that $D - V(T)$ is still strongly connected.

2. Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $u \in G$ for $u \in V(G)$. For a vertex $u \in G$, let $N_G(u)$ be the set of neighbors of u in G and $d_G(u) = |N_G(u)|$ be the degree of u in G . For a vertex subset U of a graph G , $G(U)$ denotes the subgraph induced by U and $G - U$ is the subgraph induced by $V(G) - U$. The neighborhood $N_G(U)$ of U is the set of vertices in $V(G) - U$ which are adjacent to some vertex in U . If $U = \{u\}$, we use $G - u$ for $G - \{u\}$. If H is a subgraph of G , we often use H for $V(H)$. For example, $N_G(H), H \cap G$ and $G(H)$ mean $N_G(V(H)), V(H) \cap V(G)$ and $G(V(H))$, respectively. If there is no confusion, we always delete the subscript, for example, $d(u)$ for $d_G(u)$, $N(u)$ for $N_G(u)$, $N(U)$ for $N_G(U)$ and so on. For $H \subseteq G$, we define $\delta_G(H) := \min_{x \in H} d_G(x)$, whereas $\delta(H)$ is the minimum degree of the graph H . For $H_1, H_2 \subseteq G, H_1 \cup H_2$ is the subgraph of G with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. For a set $S, K(S)$ denotes the complete graph on vertex set S .

A vertex set S is a separating set of a connected graph G if $G - S$ is disconnected, and S is a minimum separating set if $|S| = \kappa(G)$. For a minimum separating set S of G , we call the union F of at least one, but not all components of $G - S$ a fragment F to S , and $\bar{F} := G - (S \cup V(F))$ the complementary fragment. An end of G is a fragment of G which does not contain another fragment of G . An end of G exists if and only if G is not complete, and then, of course, there are at least two. The completion of $S \subseteq V(G)$ in G , denoted by $G[S]$, is the graph $G \cup K(S)$.

Let $\mathcal{K}_k(m)$ denote the class of all pairs (G, C) , where G is a k -connected graph with $|G| \geq k + 1, C$ is a complete subgraph of G with $|C| = k$ and with $\delta_G(G - V(C)) \geq \lfloor \frac{3}{2}k \rfloor + m - 1$. Let $\mathcal{K}_k^+(m)$ consist of all $(G, C) \in \mathcal{K}_k(m)$ with $\kappa(G) \geq k + 1$.

In order to use induction to prove [Theorem 1.2](#), Mader [7] proved the following result.

Theorem 2.1 (Mader [7]). *For all $(G, C) \in \mathcal{K}_k^+(m)$ and $p \in G - V(C)$, there is a path $P \subseteq G - V(C)$ of order m starting from p such that $\kappa(G - V(P)) \geq k$ holds.*

The following Theorem was stated in [8]. A proof of [Theorem 2.2](#) was not given, but it follows from Theorem 2 in [7] in a similar way as that of Theorem 1 in [7].

Theorem 2.2 (Mader [8]). *Let G be a $(k + 1)$ -connected graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor + m - 1$ and let p be a vertex of G . Then there is a path P of order m starting from p such that $\kappa(G - V(P)) \geq k$ holds.*

A tree is a connected graph without cycles. A star is a tree that has exact one vertex with degree greater than one. We call this vertex u with degree greater than one the center-vertex of the star. A double-star is a tree that has exact two vertices with degree greater than one. Those two vertices u and v with degree greater than one must be adjacent in a double-star. We call this edge uv the center-edge of the double-star. The authors in [9] proved the following theorem, which verified [Conjecture 2](#) for stars and double-stars when $k = 2$.

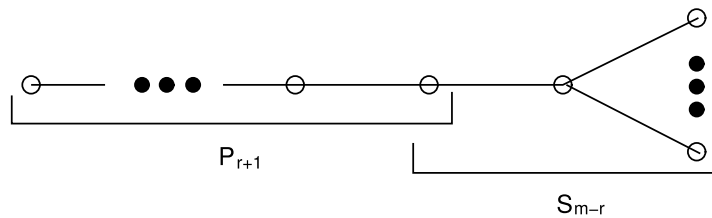


Fig. 1. The path-star graph $PS(r, m - r)$.

Theorem 2.3 ([9]). Let G be a 2-connected graph with minimum degree $\delta(G) \geq m + 2$, where m is a positive integer. Then
 (i) G contains a star T' with order m such that $G - V(T')$ is 2-connected;
 (ii) for every double-star T with order m , G contains a double-star T' isomorphic to T such that $G - V(T')$ is 2-connected.

As indicated in [7], given a $(G, C) \in \mathcal{K}_k(m)$, we can obtain a graph G' from (G, C) such that G' satisfies the condition $\delta(G') \geq \lfloor \frac{3}{2}k \rfloor + m - 1$ by pasting together sufficiently many copies of G at C . Then by applying Theorems 1.2 and 2.3 to G' , we obtain the following results.

Theorem 2.4. (i) [7] Every $(G, C) \in \mathcal{K}_k(m)$ contains a path $P \subseteq G - V(C)$ of order m such that $\kappa(G - V(P)) \geq k$ holds.
 (ii) [9] Every $(G, C) \in \mathcal{K}_2(m)$ contains a star $T' \subseteq G - V(C)$ of order m such that $\kappa(G - V(T')) \geq 2$ holds;
 (iii) [9] For every double-star T with order m , every $(G, C) \in \mathcal{K}_2(m)$ contains a double-star $T' \subseteq G - V(C)$ isomorphic to T such that $\kappa(G - V(T')) \geq 2$ holds.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. An arc (u, v) is considered to be directed from u to v . $u \in D$ means $u \in V(D)$. For a vertex set $S \subseteq V(D)$, define $N_D^+(S) := \{v \in D - S : \text{there is an } s \in S \text{ such that } (s, v) \in A(D)\}$; for a subdigraph $H \subseteq D$ and a vertex $u \in D$, we write $N_D^+(H)$ instead of $N_D^+(V(H))$ and $N_D^+(u)$ instead of $N_D^+(\{u\})$. For $u \in D$, $d_D^+(u) = |N_D^+(u)|$ denotes the outdegree of u . $\delta^+(D)$ denotes the minimum outdegree of D . For the dual concepts, we use the notation N_D^-, d_D^- and $\delta^-(D)$, respectively. The minimum semi-degree $\delta(D)$ of D is $\min\{\delta^+(D), \delta^-(D)\}$. The subdigraph of D induced by $S \subseteq V(D)$ or $S \subseteq D$ is denoted by $D(S)$. We say D is strongly connected if $\kappa(D) \geq 1$.

3. Connectivity keeping trees in 2-connected graphs

The next lemma is widely used in studying connectivity of graphs.

Lemma 3.1 (Hamidoune [5]). Let G be a k -connected graph and let S be a separating set of G with $|S| = k$. Then for every fragment F of G to S , $G[S] - V(\bar{F})$ is k -connected. Furthermore, if F is an end of G with $|F| \geq 2$, then $G[S] - V(\bar{F})$ is $(k + 1)$ -connected.

Proof. Since a separating set of $G[S] - V(\bar{F})$ is one of G , too, Lemma 3.1 follows immediately. \square

Before proving the main results in this section, we need one more lemma.

Lemma 3.2 (Mader [7]). Let G be a k -connected graph and let S be a separating set of G with $|S| = k$. Then the following hold.
 (i) Assume $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor + m - 1$ and let F be a fragment of G to S . If $W \subseteq G - (S \cup V(F))$ has order at most m and $\kappa(G[S] - V(F \cup W)) \geq k$ holds, then also $\kappa(G - V(W)) \geq k$ holds.
 (ii) Assume $(G, C) \in \mathcal{K}_k(m)$ and let F be a fragment of G to S with $C \subseteq G(F \cup S)$. If $W \subseteq G - (S \cup V(F))$ has order at most m and $\kappa(G[S] - V(F \cup W)) \geq k$ holds, then also $\kappa(G - V(W)) \geq k$ holds.

Definition 1. The path-star graph, $PS(r, m - r)$, is obtained from the disjoint union of a path with order $r + 1$ and a star with order $m - r$, by identifying one end vertex of the path with one vertex of degree one in the star. See Fig. 1 for example.

Theorem 3.3. Let $PS(r, m - r)$ be a path-star graph with order m , where $1 \leq r \leq m - 3$. Then every 2-connected graph G with minimum degree $\delta(G) \geq m + 2$ contains a subgraph T isomorphic to $PS(r, m - r)$ such that $G - V(T)$ is 2-connected.

Proof. By Theorem 2.3(i), there is a star T_1 with order $m - r$ such that $G_1 := G - V(T_1)$ is 2-connected. Let $V(T_1) = \{u, v_1, \dots, v_{m-r-1}\}$ and $E(T_1) = \{uv_i | i = 1, \dots, m - r - 1\}$. Since $\delta(G) \geq m + 2$, we have $\delta(G_1) \geq m + 2 - (m - r) = r + 2$ and $|N_{G_1}(v) \cap G_1| \geq m + 2 - (m - r - 1) = r + 3$ for each $v \in V(T_1)$.

Case 1. G_1 is 3-connected.

Let w_1 be a neighbor of v_1 in G_1 . By Theorem 2.2, there is a path P_1 of order r in G_1 starting from w_1 such that $G_1 - V(P_1)$ is 2-connected. Then the subgraph T obtained from the union of T_1 and P_1 by adding an edge $v_1 w_1$ satisfies $T \cong PS(r, m - r)$ and $G - V(T)$ is 2-connected.

Case 2. $\kappa(G_1) = 2$.

Then G_1 has an end F . Let $S = N_{G_1}(F)$ and $S = \{x, y\}$. We have $|F| \geq 2$ by $\delta(G_1) \geq r + 2$ and $|S| = 2$. By Lemma 3.1, $G_1[S] - V(\bar{F})$ is 3-connected. By $\delta(G_1) \geq r + 2$, we have $(G_1[S] - V(\bar{F}), S) \in \mathcal{K}_2^+(r)$.

Case 2.1. G_1 has an end, say F , such that $N_G(T_1) \cap F \neq \emptyset$.

If v_i has a neighbor w_i in F for some $i \in \{1, \dots, m - r - 1\}$, then by Theorem 2.1, there is a path $P_1 \subseteq G_1[S] - V(\bar{F}) - S$ of order r starting from w_i such that $G_1[S] - V(\bar{F}) - V(P_1)$ is 2-connected. Since $\delta(G_1) \geq r + 2$ and $|V(P_1)| = r$, we have $G_1 - V(P_1)$ is 2-connected by Lemma 3.2(i). Thus the subgraph T obtained from the union of T_1 and P_1 by adding an edge $v_i w_i$ satisfies $T \cong PS(r, m - r)$ and $G - V(T)$ is 2-connected.

Assume $N_G(v_i) \cap F = \emptyset$ for any $i \in \{1, \dots, m - r - 1\}$. By $N_G(T_1) \cap F \neq \emptyset$, u must have a neighbor, say w , in F . Let $T_2 = T_1 - v_{m-r-1}$ and $G_2 = G - V(T_2)$. Then F is still an end of G_2 and $\delta(G_2) \geq r + 3$. Thus $(G_2[S] - V(\bar{F}), S) \in \mathcal{K}_2^+(r + 1)$. By Theorem 2.1, there is a path $P_2 \subseteq G_2[S] - V(\bar{F}) - S$ of order $r + 1$ starting from w such that $G_2[S] - V(\bar{F}) - V(P_2)$ is 2-connected. Since $\delta(G_2) \geq r + 3$ and $|V(P_2)| = r + 1$, we have $G_2 - V(P_2)$ is 2-connected by Lemma 3.2(i). Thus the graph T obtained from the union of T_2 and P_2 by adding an edge uw satisfies $T \cong PS(r, m - r)$ and $G - V(T)$ is 2-connected.

Case 2.2. Each end F in G_1 satisfies $N_G(T_1) \cap F = \emptyset$.

As above, assume F is an end of G_1 . Let $S = N_{G_1}(F)$ and $S = \{x, y\}$. Since $N_G(T_1) \cap F = \emptyset$, we know F is also an end of G . If we can find a subgraph $T \subseteq F$ such that $T \cong PS(r, m - r)$ and $\kappa(G[S] - V(\bar{F}) - V(T)) \geq 2$, then, by applying Lemma 3.2(i) to G , we obtain $G - V(T)$ is 2-connected. Thus, in the following, we only need to prove that $G[S] - V(\bar{F})$ contains a subgraph $T' \subseteq G[S] - V(\bar{F}) - S$ such that $T' \cong PS(r, m - r)$ and $\kappa(G[S] - V(\bar{F}) - V(T')) \geq 2$.

Let $G' = G[S] - V(\bar{F})$. By Lemma 3.1, G' is 3-connected. Since $\delta(G) \geq m + 2$, we have $(G', S) \in \mathcal{K}_2^+(m)$. By Theorem 2.4(ii), there is a star $T'_1 \subseteq G' - S$ with order $m - r$ such that $G'_1 := G' - V(T'_1)$ is 2-connected. Let $V(T'_1) = \{u', v'_1, \dots, v'_{m-r-1}\}$ and $E(T'_1) = \{u'v'_i | i = 1, \dots, m - r - 1\}$. Since $\delta(G) \geq m + 2$, we have $\delta_{G'_1}(G'_1 - S) \geq m + 2 - (m - r) = r + 2$ and $|N_{G'}(v') \cap G'_1| \geq m + 2 - (m - r - 1) = r + 3$ for each $v' \in V(T'_1)$.

Case 2.2.1. G'_1 is 3-connected.

Let w'_1 be a neighbor of v'_1 in $G'_1 - S$ (w'_1 exists because $|N_{G'}(v'_1) \cap G'_1| \geq r + 3$ and $|S| = 2$). By Theorem 2.1, there is a path $P'_1 \subseteq G'_1 - S$ of order r starting from w'_1 such that $G'_1 - V(P'_1)$ is 2-connected. Then the graph T' obtained from the union of T'_1 and P'_1 by adding an edge $v'_1 w'_1$ satisfies $T' \cong PS(r, m - r)$ and $G' - V(T')$ is 2-connected.

Case 2.2.2. $\kappa(G'_1) = 2$.

The proof of Case 2.2.2 is similar to Case 2.1. Nevertheless, we also outline the proof for completeness.

We can choose an end F' of G'_1 such that $F' \cap S = \emptyset$. Let $S' = N_{G'_1}(F')$ and $S' = \{x', y'\}$. We have $|F'| \geq 2$ by $\delta_{G'_1}(G'_1 - S) \geq r + 2$ and $|S'| = 2$. By Lemma 3.1, $G'_1[S'] - V(\bar{F}')$ is 3-connected. By $\delta_{G'_1}(G'_1 - S) \geq r + 2$, we have $(G'_1[S'] - V(\bar{F}'), S') \in \mathcal{K}_2^+(r)$.

If v'_i has a neighbor w'_i in F' for some $i \in \{1, \dots, m - r - 1\}$, then by Theorem 2.1, there is a path $P'_i \subseteq G'_1[S'] - V(\bar{F}') - S'$ of order r starting from w'_i such that $G'_1[S'] - V(\bar{F}') - V(P'_i)$ is 2-connected. Since $\delta_{G'_1}(G'_1 - S) \geq r + 2$ and $|V(P'_i)| = r$, we have $G'_1 - V(P'_i)$ is 2-connected by Lemma 3.2(ii). Thus the graph T' obtained from the union of T'_1 and P'_i by adding an edge $v'_i w'_i$ satisfies $T' \cong PS(r, m - r)$ and $G' - V(T')$ is 2-connected.

Assume $N_{G'}(v_i) \cap F' = \emptyset$ for any $i \in \{1, \dots, m - r - 1\}$. By G' is 3-connected, u' must have a neighbor, say w' , in F' . Let $T'_2 = T'_1 - u'_{m-r-1}$ and $G'_2 = G' - V(T'_2)$. Then F' is still an end of G'_2 and $\delta_{G'_2}(G'_2 - S) \geq r + 3$. Thus $(G'_2[S'] - V(\bar{F}'), S') \in \mathcal{K}_2^+(r + 1)$. By Theorem 2.1, there is a path $P'_2 \subseteq G'_2[S'] - V(\bar{F}') - S'$ of order $r + 1$ starting from w' such that $G'_2[S'] - V(\bar{F}') - V(P'_2)$ is 2-connected. Since $\delta_{G'_2}(G'_2 - S) \geq r + 3$ and $|V(P'_2)| = r + 1$, we have $G'_2 - V(P'_2)$ is 2-connected by Lemma 3.2(ii). Thus the graph T' obtained from the union of T'_2 and P'_2 by adding an edge $u'w'$ satisfies $T' \cong PS(r, m - r)$ and $G' - V(T')$ is 2-connected. \square

Definition 2. The **path-double-star** graph, $PDS(r, m - r)$, is obtained from the disjoint union of a path with order $r + 1$ and a double-star with order $m - r$, by identifying one end vertex of the path with one vertex of degree one in the double-star. See Fig. 2 for example.

Specifically, we denote $PDS1(r, m - r)$ the path-double-star graph obtained from the disjoint union of a path with order $r + 1$ and a double-star with order $m - r$, by identifying one end vertex of the path with one pendant vertex which is adjacent to the vertex with maximum degree in the double-star. We denote $PDS2(r, m - r)$ the path-double-star graph obtained from the disjoint union of a path with order $r + 1$ and a double-star with order $m - r$, by identifying one end vertex of the path with one pendant vertex which is adjacent to the vertex with the second maximum degree in the double-star.

By replacing the star with a double-star in the proof of Theorem 3.3, we can obtain the proof of Theorem 3.4 by using almost the same arguments as the proof of Theorem 3.3. Besides, we use Theorem 2.3(ii) instead of Theorem 2.3(i) and Theorem 2.4(iii) instead of Theorem 2.4(ii) in the proof of Theorem 3.4. So we omit the proof here.

Theorem 3.4. For two integers r, m with $1 \leq r \leq m - 4$, every 2-connected graph G with minimum degree $\delta(G) \geq m + 2$ contains a subgraph T isomorphic to $PDS1(r, m - r)$ or $PDS2(r, m - r)$ such that $G - V(T)$ is 2-connected.

Remark. In Definition 2, if the degrees of the two vertices of the center-edge in the double-star are equal, then $PDS1(r, m - r)$ is isomorphic to $PDS2(r, m - r)$. Thus Theorem 3.4 implies that Conjecture 2 is true for this kind of path-double-stars when $k = 2$.

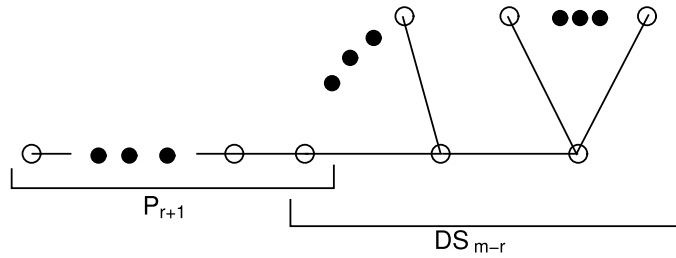


Fig. 2. The path-double-star graph $PDS(r, m - r)$.

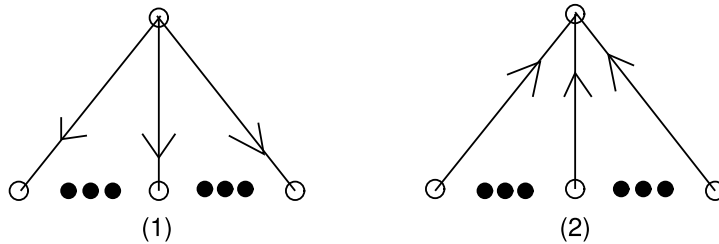


Fig. 3. (1) The out-star; (2) The in-star.

4. Nonseparating oriented stars or double-stars in strongly connected digraphs

Definition 3. The **out-star** OS_m is the digraph obtained from a star of order m by orienting each edge of the star away from the center-vertex. The **in-star** IS_m is the digraph obtained from a star of order m by orienting each edge of the star towards the center-vertex. See Fig. 3 for examples.

Lemma 4.1. Let D be a strongly connected digraph and H be a subdigraph of D with $|V(H)| < |V(D)|$. Then there is a dipath $P := p_0p_1 \cdots p_t$ ($t \geq 2$) in D such that $p_0, p_t \in V(H)$ and $p_1, \dots, p_{t-1} \in V(D) - V(H)$, where p_0 and p_t may be the same vertex.

Proof. Since D is strongly connected, there is an arc, say (p_0, p_1) , from $V(H)$ to $V(D) - V(H)$. By D is strongly connected, there is a dipath, say $P' := p_1 \cdots p_t$, from p_1 to $V(H)$, where $p_1, \dots, p_{t-1} \in V(D) - V(H)$ and $p_t \in V(H)$. Thus the dipath $P := p_0p_1 \cdots p_t$ is just a dipath we needed. \square

Theorem 4.2. Let D be a strongly connected digraph with minimum semi-degree $\delta(D) = \min\{\delta^+(D), \delta^-(D)\} \geq m + 1$. Then D contains a subdigraph T isomorphic to OS_m or IS_m such that $D - V(T)$ remains strongly connected.

Proof. Since $\delta(D) \geq m + 1$, there are subdigraphs in D isomorphic to OS_m or IS_m . Let T be a subdigraph in D isomorphic to OS_m or IS_m . Let $D' = D - T$. If D' is strongly connected, then we are done. Thus we assume that D' is not strongly connected. We order all strong components of D' as C_1, \dots, C_l such that there are no arcs from C_j to C_i for all $1 \leq i < j \leq l$. Let B be a maximum strong component of D' . We choose such a T so that

(1) $|B|$ is as large as possible.

Let $P := p_0p_1 \cdots p_t$ ($t \geq 2$) be a shortest dipath in D such that $p_0, p_t \in B$ and $p_1, \dots, p_{t-1} \in V(D) - B$ (by Lemma 4.1). We consider three cases in the following.

Case 1. $t = 2$.

By $t = 2$, we have $p_1 \in V(T)$. If $B = C_1$, then for any vertex $c_1 \in C_1$, we have $|N_D^+(c_1) \setminus (B \cup \{p_1\})| \geq m + 1 - 1 = m$. Thus we can find an out-star T' rooted at c_1 with order m such that $V(T') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T')$, contrary to (1). If $B \neq C_1$, then for any vertex $c_1 \in C_1$, we have $|N_D^-(c_1) \setminus (B \cup \{p_1\})| \geq m + 1 - 1 = m$. Thus we can find an in-star T'' rooted at c_1 with order m such that $V(T'') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T'')$, contrary to (1).

Case 2. $t = 3$.

By the choice of P , we have $N_D^+(p_1) \cap B = \emptyset$. Then $|N_D^+(p_1) \setminus (B \cup P)| \geq m + 1 - 1 = m$. Let $q \in N_D^+(p_1) \setminus (B \cup P)$. If $N_D^+(q) \cap B = \emptyset$, then $|N_D^+(q) \setminus (B \cup P)| \geq m + 1 - 2 = m - 1$. Thus we can find an out-star T' rooted at q with order m such that $V(T') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T')$, contrary to (1). If $N_D^+(q) \cap B \neq \emptyset$, then $N_D^-(q) \cap B = \emptyset$ (for otherwise, we can find a dipath P' shorter than P). Thus $|N_D^-(q) \setminus (B \cup P)| \geq m + 1 - 2 = m - 1$, and

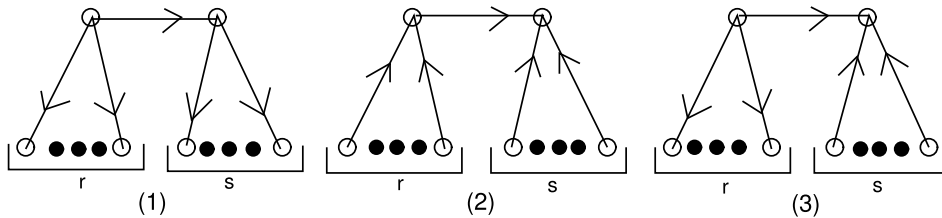


Fig. 4. (1) The out-double-star; (2) The in-double-star; (3) The out-in-double-star.

we can find an in-star T'' rooted at q with order m such that $V(T'') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T'')$, contrary to (1).

Case 3. $t \geq 4$.

By the choice of P , we have $N_D^+(p_1) \cap (B \cup P) = \{p_2\}$. Then $|N_D^+(p_1) \setminus (B \cup P)| \geq m + 1 - 1 = m$. Let $q_1, \dots, q_m \in N_D^+(p_1) \setminus (B \cup P)$. By the choice of P , we have $N_D^+(q_j) \cap (B \cup P) \subseteq \{p_1, p_2, p_3\}$ for each $j \in \{1, \dots, m\}$. Thus $|N_D^+(q_j) \setminus (B \cup P)| \geq m + 1 - 3 = m - 2$ for each $j \in \{1, \dots, m\}$. If there is some $j \in \{1, \dots, m\}$ such that $|N_D^+(q_j) \setminus (B \cup P)| \geq m - 1$, then we can find an out-star T' rooted at q_j with order m such that $V(T') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T')$, contrary to (1). Thus we assume that $|N_D^+(q_j) \setminus (B \cup P)| = m - 2$ and $N_D^+(q_j) \cap (B \cup P) = \{p_1, p_2, p_3\}$ for each $j \in \{1, \dots, m\}$. But then there is an in-star T'' rooted at p_2 with vertex set $\{p_2, q_1, \dots, q_{m-1}\}$ such that $V(T'') \cap (V(B) \cup V(P')) = \emptyset$, where P' is the dipath $p_0 p_1 q_m p_3 \dots p_t$. But then $V(B) \cup V(P')$ is contained in a strong component of $D - V(T'')$, contrary to (1). \square

Definition 4. The **out-double-star** $ODS(m; r, s)$ is the digraph obtained from the disjoint union of two out-stars (one is isomorphic to OS_{r+1} rooted at u and the other is isomorphic to OS_{s+1} rooted at v , where $r + s = m - 2$) by adding an arc from u to v . The **in-double-star** $IDS(m; r, s)$ is the digraph obtained from the disjoint union of two in-stars (one is isomorphic to IS_{r+1} rooted at u and the other is isomorphic to IS_{s+1} rooted at v , where $r + s = m - 2$) by adding an arc from u to v . The **out-in-double-star** $OIDS(m; r, s)$ is the digraph obtained from the disjoint union of one out-star and one in-star (the out-star is isomorphic to OS_{r+1} rooted at u and the in-star is isomorphic to IS_{s+1} rooted at v , where $r + s = m - 2$) by adding an arc from u to v . We often call the arc (u, v) center-arc. See Fig. 4 for examples.

Lemma 4.3. Let D be a digraph, T_1 be an out-double-star with order m , T_2 be an in-double-star with order m , and T_3 be an out-in-double-star with order m .

- (i) If there is an arc $a = (u, v) \in A(D)$ such that $|N_D^+(u) \setminus \{v\}| \geq m - 3$, $|N_D^+(v) \setminus \{u\}| \geq m - 3$ and $|(N_D^+(u) \cup N_D^+(v)) \setminus \{u, v\}| \geq m - 2$, then there is an out-double-star $T \subseteq D$ isomorphic to T_1 .
- (ii) If there is an arc $a = (u, v) \in A(D)$ such that $|N_D^-(u) \setminus \{v\}| \geq m - 3$, $|N_D^-(v) \setminus \{u\}| \geq m - 3$ and $|(N_D^-(u) \cup N_D^-(v)) \setminus \{u, v\}| \geq m - 2$, then there is an in-double-star $T \subseteq D$ isomorphic to T_2 .
- (iii) If there is an arc $a = (u, v) \in A(D)$ such that $|N_D^+(u) \setminus \{v\}| \geq m - 3$, $|N_D^-(v) \setminus \{u\}| \geq m - 3$ and $|(N_D^+(u) \cup N_D^-(v)) \setminus \{u, v\}| \geq m - 2$, then there is an out-in-double-star $T \subseteq D$ isomorphic to T_3 .

Proof. (i) By T_1 is an out-double-star, we have $m \geq 4$. Assume the out-double-star T_1 has the center-arc $a' = (u', v')$, where $|N_{T_1}^+(u') \setminus \{v'\}| = r$, $|N_{T_1}^+(v') \setminus \{u'\}| = s$ and $|(N_{T_1}^+(u') \cup N_{T_1}^+(v')) \setminus \{u', v'\}| = m - 2$ ($1 \leq r, s \leq m - 3$ and $r + s = m - 2$). Since $|N_D^+(u) \setminus \{v\}| \geq m - 3$, $|N_D^+(v) \setminus \{u\}| \geq m - 3$ and $|(N_D^+(u) \cup N_D^+(v)) \setminus \{u, v\}| \geq m - 2$, we can find an out-double-star $T \cong T_1$ in D with center-arc $a = (u, v)$.

(ii) and (iii) follow similarly. \square

While the main idea of the proof of Theorem 4.4 is similar to that of Theorem 4.2, the proof of Theorem 4.4 is somewhat more complicated with different details.

Theorem 4.4. For any integers m, r, s with $1 \leq r, s \leq m - 3$ and $r + s = m - 2$, every strongly connected digraph D with minimum semi-degree $\delta(D) = \min\{\delta^+(D), \delta^-(D)\} \geq m + 1$ contains a subdigraph T isomorphic to a member in $\mathcal{S}(m; r, s) = \{ODS(m; r, s), IDS(m; r, s), OIDS(m; r, s)\}$ such that $D - V(T)$ remains strongly connected.

Proof. Since $\delta(D) \geq m + 1$, there is a subdigraph in D isomorphic to a member in $\mathcal{S}(m; r, s) = \{ODS(m; r, s), IDS(m; r, s), OIDS(m; r, s)\}$ by Lemma 4.3. Let T be a subdigraph in D isomorphic to a member in $\mathcal{S}(m; r, s)$. Let $D' = D - T$. If D' is strongly connected, then we are done. Thus we assume that D' is not strongly connected. We order all strong components of D' as C_1, \dots, C_l such that there are no arcs from C_j to C_i for all $1 \leq i < j \leq l$. Let B be a maximum strong component of D' . We choose such a T so that

- (1) $|B|$ is as large as possible.

Let $P := p_0 p_1 \cdots p_t$ ($t \geq 2$) be a shortest dipath in D such that $p_0, p_t \in B$ and $p_1, \dots, p_{t-1} \in V(D) - B$ (by Lemma 4.1). We consider three cases in the following.

Case 1. $t = 2$.

By $t = 2$, we have $p_1 \in V(T)$. If $B = C_1$, then for any vertex $c_i \in C_1$, we have $|N_D^+(c_i) \setminus (B \cup \{p_1\})| \geq m + 1 - 1 = m$ and $|N_D^+(c_i) \cap C_i| \geq m + 1 - |T| = 1$. Let (c_i, c'_i) be an arc in C_i . Since $|N_D^+(c_i) \setminus (B \cup \{p_1, c'_i\})| \geq m + 1 - 2 = m - 1$ and $|N_D^+(c'_i) \setminus (B \cup \{p_1, c_i\})| \geq m + 1 - 2 = m - 1$, by Lemma 4.3(i), we can find an out-double-star $T' \cong ODS(m; r, s)$ with center-arc (c_i, c'_i) such that $V(T') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T')$, contrary to (1). If $B \neq C_1$, then for any vertex $c_1 \in C_1$, we have $|N_D^-(c_1) \setminus (B \cup \{p_1\})| \geq m + 1 - 1 = m$ and $|N_D^-(c_1) \cap C_1| \geq m + 1 - |T| = 1$. Let (c_1, c'_1) be an arc in C_1 . Since $|N_D^-(c_1) \setminus (B \cup \{p_1, c'_1\})| \geq m + 1 - 2 = m - 1$ and $|N_D^-(c'_1) \setminus (B \cup \{p_1, c_1\})| \geq m + 1 - 2 = m - 1$, by Lemma 4.3(ii), we can find an in-double-star $T'' \cong IDS(m; r, s)$ with center-arc (c_1, c'_1) such that $V(T'') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T'')$, contrary to (1).

Case 2. $t = 3$.

By the choice of P , we have $N_D^+(p_1) \cap B = \emptyset$. Then $|N_D^+(p_1) \setminus (B \cup P)| \geq m + 1 - 1 = m$. Let $q \in N^+(p_1) \setminus (B \cup P)$.

Case 2.1. $N_D^+(q) \cap B = \emptyset$.

By $N_D^+(q) \cap B = \emptyset$, we have $|N_D^+(q) \setminus (B \cup P)| \geq m + 1 - 2 = m - 1$. Let $w \in N_D^+(q) \setminus (B \cup P)$.

If $N_D^+(w) \cap B = \emptyset$, then $|N_D^+(w) \setminus (B \cup P)| \geq m + 1 - 2 = m - 1$. Since $|N_D^+(q) \setminus (B \cup P \cup \{w\})| \geq m + 1 - 3 = m - 2$ and $|N_D^+(w) \setminus (B \cup P \cup \{q\})| \geq m + 1 - 3 = m - 2$, by Lemma 4.3(i), we can find an out-double-star $T' \cong ODS(m; r, s)$ with center-arc (q, w) such that $V(T') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T')$, contrary to (1).

If $N_D^+(w) \cap B \neq \emptyset$, then $N_D^-(w) \cap B = \emptyset$ (for otherwise, we can find a dipath P' shorter than P). Thus $|N_D^-(w) \setminus (B \cup P)| \geq m + 1 - 2 = m - 1$. Since $|N_D^+(q) \setminus (B \cup P \cup \{w\})| \geq m + 1 - 3 = m - 2$ and $|N_D^-(w) \setminus (B \cup P \cup \{q\})| \geq m + 1 - 3 = m - 2$, by Lemma 4.3(iii), we can find an out-in-double-star $T' \cong OIDS(m; r, s)$ with center-arc (q, w) such that $V(T') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T')$, contrary to (1).

Case 2.2. $N_D^+(q) \cap B \neq \emptyset$.

By $N_D^+(q) \cap B \neq \emptyset$, we have $N_D^-(q) \cap B = \emptyset$ (for otherwise, we can find a dipath P' shorter than P), and then $|N_D^-(q) \setminus (B \cup P)| \geq m + 1 - 2 = m - 1$. Let $w' \in N_D^-(q) \setminus (B \cup P)$.

If $N_D^+(w') \cap B = \emptyset$, then $|N_D^+(w') \setminus (B \cup P)| \geq m + 1 - 2 = m - 1$. Since $|N_D^-(q) \setminus (B \cup P \cup \{w'\})| \geq m + 1 - 3 = m - 2$ and $|N_D^+(w') \setminus (B \cup P \cup \{q\})| \geq m + 1 - 3 = m - 2$, by Lemma 4.3(iii), we can find an out-in-double-star $T' \cong OIDS(m; r, s)$ with center-arc (w', q) such that $V(T') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T')$, contrary to (1).

If $N_D^+(w') \cap B \neq \emptyset$, then $N_D^-(w') \cap B = \emptyset$ (for otherwise, we can find a dipath P' shorter than P). Thus $|N_D^-(w') \setminus (B \cup P)| \geq m + 1 - 2 = m - 1$. Since $|N_D^-(q) \setminus (B \cup P \cup \{w'\})| \geq m + 1 - 3 = m - 2$ and $|N_D^-(w') \setminus (B \cup P \cup \{q\})| \geq m + 1 - 3 = m - 2$, by Lemma 4.3(ii), we can find an in-double-star $T' \cong IDS(m; r, s)$ with center-arc (w', q) such that $V(T') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T')$, contrary to (1).

Case 3. $t \geq 4$.

By the choice of P , we have $N_D^+(p_1) \cap (B \cup P) = \{p_2\}$. Then $|N_D^+(p_1) \setminus (B \cup P)| \geq m + 1 - 1 = m$. Let $q_1, \dots, q_m \in N_D^+(p_1) \setminus (B \cup P)$. By the choice of P , we have $N_D^+(q_j) \cap (B \cup P) \subseteq \{p_1, p_2, p_3\}$ for each $j \in \{1, \dots, m\}$. Thus $|N_D^+(q_j) \setminus (B \cup P)| \geq m + 1 - 3 = m - 2$ for each $j \in \{1, \dots, m\}$. Let $w \in N_D^+(q_1) \setminus (B \cup P)$.

If $N_D^-(w) \cap B \neq \emptyset$, then $N_D^-(w) \cap (B \cup P) \subseteq \{p_{t-2}, p_{t-1}\}$ by the choice of P . Thus $|N_D^-(w) \setminus (B \cup P)| \geq m - 1$. Since $|N_D^+(q_1) \setminus (B \cup P)| \geq m - 2$ and $|N_D^-(w) \setminus (B \cup P)| \geq m - 1$, by Lemma 4.3(iii), we can find an out-in-double-star $T' \cong OIDS(m; r, s)$ with center-arc (q_1, w) such that $V(T') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T')$, contrary to (1).

If $N_D^-(w) \cap B = \emptyset$, then $N_D^+(w) \cap (B \cup P) \subseteq \{p_1, p_2\}$ by the choice of P . Thus $|N_D^+(w) \setminus (B \cup P)| \geq m - 1$. Since $|N_D^+(q_1) \setminus (B \cup P)| \geq m - 2$ and $|N_D^+(w) \setminus (B \cup P)| \geq m - 1$, by Lemma 4.3(i), we can find an out-double-star $T' \cong ODS(m; r, s)$ with center-arc (q_1, w) such that $V(T') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T')$, contrary to (1).

In the following, we assume $N_D^+(w) \cap B = \emptyset$ and $N_D^-(w) \cap B = \emptyset$ for every $w \in N_D^+(q_1) \setminus (B \cup P)$. By the choice of P , we have $N_D^+(w) \cap (B \cup P) \subseteq \{p_1, p_2, p_3, p_4\}$ for each $w \in N_D^+(q_1) \setminus (B \cup P)$. Thus $|N_D^+(w) \setminus (B \cup P)| \geq m + 1 - 4 = m - 3$ for each $w \in N_D^+(q_1) \setminus (B \cup P)$.

If there is some $j \in \{1, \dots, m\}$, say $j = 1$, such that $|N_D^+(q_1) \setminus (B \cup P)| \geq m - 1$, let $w_1, \dots, w_{m-1} \in N_D^+(q_1) \setminus (B \cup P)$. If there is some $k \in \{1, \dots, m - 1\}$, say $k = 1$, such that $|N_D^+(w_1) \cap (B \cup P)| \geq m - 2$, then by Lemma 4.3(i), we can find an out-double-star $T' \cong ODS(m; r, s)$ with center-arc (q_1, w_1) such that $V(T') \cap (V(B) \cup V(P)) = \emptyset$. But then $V(B) \cup V(P)$ is contained in a strong component of $D - V(T')$, contrary to (1). So we assume $|N_D^+(w_k) \setminus (B \cup P)| = m - 3$ for each $k \in \{1, \dots, m - 1\}$. Then we have $N_D^+(w_k) \cap (B \cup P) = \{p_1, p_2, p_3, p_4\}$ for each $k \in \{1, \dots, m - 1\}$. By Lemma 4.3(ii), there is an in-double-star $T'' \cong IDS(m; r, s)$ with center-arc (p_2, p_3) and vertex set $\{p_2, p_3, w_1, \dots, w_{m-2}\}$ such that $V(T'') \cap (V(B) \cup V(P)) = \emptyset$, where $P' := p_0 p_1 q_1 w_{m-1} p_4 \cdots p_t$. But then $V(B) \cup V(P')$ is contained in a strong component of $D - V(T'')$, contrary to (1).

Thus we assume that $|N_D^+(q_j) \setminus (B \cup P)| = m - 2$ and $N_D^+(q_j) \cap (B \cup P) = \{p_1, p_2, p_3\}$ for each $j \in \{1, \dots, m\}$. By $|N_D^+(q_j) \setminus (B \cup P)| = m - 2$, there is an integer $j \in \{2, \dots, m\}$, say $j = m$, such that $q_m \notin N_D^+(q_1)$. Since $\{q_2, \dots, q_{m-1}\} \in$

$N_D^-(p_2)$ and $|N_D^+(q_1) \setminus (B \cup P)| = m - 2$, by Lemma 4.3(iii), we can find an out-in-double-star $T' \cong OIDS(m; r, s)$ with center-arc (q_1, p_2) such that $V(T') \cap (V(B) \cup V(P')) = \emptyset$, where $P' := p_0 p_1 q_m p_3 \cdots p_t$. But then $V(B) \cup V(P')$ is contained in a strong component of $D - V(T')$, contrary to (1).

The theorem is established as all cases lead to contradictions. \square

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References

- [1] J.A. Bondy, U.S.R. Murty, Graph Theory, Graph Theory, in: Graduate Texts in Mathematics, vol. 244, Springer, Berlin, 2008.
- [2] G. Chartrand, A. Kaigars, D.R. Lick, Critically n -connected graphs, Proc. Amer. Math. Soc. 32 (1972) 63–68.
- [3] A.A. Diwan, N.P. Tholiya, Non-separating trees in connected graphs, Discrete Math. 309 (2009) 5235–5237.
- [4] S. Fujita, K. Kawarabayashi, Connectivity keeping edges in graphs with large minimum degree, J. Combin. Theory Ser. B 98 (2008) 805–811.
- [5] Y.O. Hamidoune, On critically h -connected simple graphs, Discrete Math. 32 (1980) 257–262.
- [6] W. Mader, Ecken von kleinem Grad in kritisch n -fach zusammenhängenden Digraphen, J. Combin. Theory Ser. B 53 (1991) 260–272.
- [7] W. Mader, Connectivity keeping paths in k -connected graphs, J. Graph Theory 65 (2010) 61–69.
- [8] W. Mader, Connectivity keeping trees in k -connected graphs, J. Graph Theory 69 (2012) 324–329.
- [9] Y.Z. Tian, J.X. Meng, H.J. Lai, L.Q. Xu, Connectivity keeping stars or double-stars in 2-connected graphs, Discrete Math. 341 (2018) 1120–1124.