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Spectral analogues of Erdős' theorem on Hamilton-connected graphs

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ABSTRACT

A graph G is Hamilton-connected if for any pair of vertices v and w, G has a spanning (v, w)-path. Extending theorems of Dirac and Ore, Erdős prove a sufficient condition in terms of minimum degree and the size of G to assure G to be Hamiltonian. We investigate the spectral analogous of Erdős' theorem for a Hamilton-connected graph with given minimum degree, and prove that there exist two graphs $\{L_n^k, M_n^k\}$ such that each of the following holds for an integer $k \geq 3$ and a simple graph G on n vertices.

(i) If $n \geq 6k$, $\delta(G) \geq k$, and $|E(G)| > \binom{n-k}{2} + k(k+1)$, then G is Hamilton-connected if and only if $C_{n+1}(G) \notin \{L_n^k, M_n^k\}$.

(ii) If $n \geq \max\{6k, \frac{1}{2}k^3 - \frac{1}{2}k^2 + k + 4\}$, $\delta(G) \geq k$ and spectral radius $\lambda(G) \geq n - k$, then G is Hamilton–connected if and only if $G \notin \{L_n^k, M_n^k\}$.

1. Introduction

We consider finite and simple graphs, with undefined notation and term following [3]. We normally use $e(G)$, n, $\delta(G)$ and $A(G)$ to denote $|E(G)|$, $|V(G)|$, the minimum degree and the adjacency matrix of a graph G, respectively. The largest eigenvalue of $A(G)$, called *the spectral radius* of G, is denoted by $\lambda(G)$. Let H be a subgraph of a graph G, and let $u \in V(G)$. The set of neighbors of a vertex u in H is denoted by $N_H(u)$. Thus

$$N_H(u) = \{v \in V(H) : uv \in E(G)\}.$$

Define $d_H(u) = |N_H(u)|$. A *clique* is a subset of vertices of an undirected graph whose induced subgraph is a complete graph. The maximum size of a clique of a graph is called *clique number*, denoted by $\omega(G)$. For $S \subseteq V(G)$, the *induced subgraph* $G[S]$ is the graph with vertex set S and edge set $\{uv \in E(G) \mid u, v \in S\}$.

The disjoint union of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is the graph with the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The disjoint union of k copies of a graph G is denoted by kG. *The join* of G_1 and G_2, denoted by $G_1 \vee G_2$, has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{xy | x \in V(G_1), y \in V(G_2)\}$.

A path (or a cycle, respectively) of a graph G is called a *Hamilton path (or Hamilton cycle, respectively)* if it passes through all the vertices of G. A graph is *Hamilton-connected* if any two vertices are connected by a Hamilton path. The investigation of hamiltonian graphs has a long history. Dirac and Ore proved the following.

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Theorem 1.1. *Let G be a graph of order n.*

(i) *(Dirac [6]) If $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.*
(ii) *(Ore [14]) If $e(G) > \binom{n-1}{2} + 1$, then G is Hamiltonian.*

Motivated by these results, Erdős [7] later extended Theorem 1.1 (ii) by utilizing the minimum degree as a new parameter.

Theorem 1.2. *(Erdős [7]) Let G be a graph of order n and the minimum degree δ and k be an integer with $1 \leq k \leq \delta \leq \frac{n-1}{2}$. If*

$$e(G) > \max\left\{\binom{n-k}{2} + k^2, \binom{\lceil \frac{n+1}{2} \rceil}{2} + \left\lfloor \frac{n+1}{2} \right\rfloor^2\right\},$$

then G is Hamiltonian.

How many edges can ensure a graph to be Hamilton–connected with a given number of vertices? In 1963, Ore [15] answered the question.

Theorem 1.3. *[15] Let G be a graph of order n, if*

$$e(G) \geq \binom{n-1}{2} + 3,$$

then G is Hamilton-connected.

Theorem 1.4. *([16], Theorem 1.8) Let G be a graph of order $n \geq 6k^2 - 8k + 5$ with $\delta(G) \geq k \geq 2$. If $e(G) \geq \frac{n^2-(2k-1)n+2k-2}{2}$, then G is Hamilton-connected unless $cl_{n+1}(G) = K_2 \vee (K_{n-k-1} \cup K_{k-1})$ or $cl_{n+1}(G) = K_k \vee (K_{n-2k-1} \cup \overline{K}_{k-1})$.*

Theorem 1.5. *([16], Corollary 1.10) Let G be a graph of order $n \geq \max\{6k^2 - 8k + 5, \frac{k^3-k^2+4k-1}{2}\}$ with $\delta(G) \geq k \geq 2$. If $\rho(G) \geq n-k$, then G is Hamilton-connected unless $G = K_2 \vee (K_{n-k-1} \cup K_{k-1})$ or $G = K_k \vee (K_{n-2k+1} \cup \overline{K}_{k-1})$.*

The results above, as well as the recent advances in [9,13,16], motivate the current research. In this paper, we present a spectral analogous of Erdős theorem for a Hamilton-connected graph with a given minimum degree. For a graph G, notice that $\delta(G) \geq 3$ is a necessary condition for G to be Hamilton-connected. A sufficient condition for a Hamilton-connected graph in terms of spectral radius is also justified. This paper is independently research work with Chen and Zhang's ([16]) results.

Throughout this paper, for $2 \leq k \leq \frac{n}{2}$, let

$$L_n^k = K_2 \vee (K_{n-k-1} + K_{k-1}) \text{ and } M_n^k = K_k \vee (K_{n-2k+1} + (k-1)K_1).$$

In Section 2, extremal sizes of graphs to ensure Hamilton-connectedness are investigated. These will be applied in Section 3 to find an optimal spectral sufficient condition for a graph G to be Hamilton-connected.

2. Extremal sizes of Hamilton-connected graphs

Let X, Y be vertex subsets of a graph G. Following [3], we adopt these notation: $e(X) = |E(G[X])|$,

$$E_G[X, Y] = \{xy \in E(G) : x \in X \text{ and } y \in Y\}, \text{ and } e(X, Y) = |E_G[X, Y]|.$$

Throughout this section, if J is a subgraph of G and $v \in V(G) - V(J)$, define $d_J(v) = |E_G[\{v\}, V(J)]|$.

The purpose of this section is to prove two extremal results, namely, Theorems 2.2 and 2.5 in this section, on the optimal sizes to assure a graph to be Hamilton-connected . We state some known results as our tools.

Theorem 2.1. *(Erdős, Gallai, [8]) Let G be a graph of order $n \geq 3$, and u, v are any pair distinct and nonadjacent vertices. If*

$$d_G(u) + d_G(v) \geq n + 1,$$

then G is Hamilton-connected.

Lemma 2.1. *[1] Let G be a graph of order $n \geq 3$ with the degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no integer $2 \leq t \leq \frac{n}{2}$ such that $d_{t-1} \leq t$ and $d_{n-t} \leq n-t$, then G is Hamilton-connected.*

Theorem 2.2. *Let G be a graph with order n and the minimum degree δ, and let k be an integer with $2 \leq k \leq \delta$. If*

$$e(G) > \max\left\{\binom{n-k+1}{2} + k(k-1), \binom{\lceil \frac{n}{2} \rceil + 1}{2} + \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\right\}, \tag{2.1}$$

then G is Hamilton-connected.

Proof. Suppose that G is not Hamilton-connected. By Lemma 2.1, there exists an integer t such that $d_{t-1} \leq t$, where $k \leq t \leq \frac{n}{2}$. Without loss of generality, let $d(v_i) = d_i$ for $1 \leq i \leq t-1$. The number of edges which are not incident to any vertex in

$\{v_1, v_2, \ldots, v_{t-1}\}$ does not exceed $\binom{n-t+1}{2}$, and the number of edges incident to any vertex in $\{v_1, v_2, \ldots, v_{t-1}\}$ is at most $t(t-1)$. It follows that

$$e(G) \le \binom{n-t+1}{2} + t(t-1). \tag{2.2}$$

The bound in (2.2) is best possible in the sense that the graph $M_n^t = K_t \vee (K_{n-2t+1} + (t-1)K_1)$ is not Hamilton-connected.

For $k \le t \le \frac{n}{2}$, by (2.1),

$$\begin{aligned} e(G) &> \max\left\{\binom{n-k+1}{2} + k(k-1), \binom{\lceil \frac{n}{2} \rceil + 1}{2} + \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\right\} \\ &\ge \binom{n-t+1}{2} + t(t-1), \end{aligned}$$

contrary to (2.2). Hence G must be Hamilton-connected. □

In [2], Bondy and Chvátal introduced the closure concept which plays an important role in cycle theory. For a graph G of order n and an integer $k = k(n) > 0$, the *k–closure* of G, denoted by $C_k(G)$, is obtained from G by sequentially joining pairs of nonadjacent vertices whose degree sum is at least k until no such vertex pairs exist.

Theorem 2.3. *(Bondy and Chvátal [2]) A graph G is Hamilton-connected if and only if $C_{n+1}(G)$ is Hamilton-connected.*

Lemma 2.2. *Let $k \ge 2$ be an integer, G be a graph of order $n \ge 6k$, and $G = C_{n+1}(G)$. Let $\omega(G)$ denote the clique number of G. If*

$$e(G) > \binom{n-k}{2} + k(k+1),$$

then $\omega(G) \ge n-k+1$.

Proof. It suffices to show that G contains a clique C with $|C| \ge n-k+1$. Define

$$F = \left\{u \in V(G) : d_G(u) \ge \frac{n+1}{2}\right\}.$$

As $G = C_{n+1}(G)$, F is a clique, and so there exists a maximal clique C of G with $F \subseteq C$. Let $s = |C|$ and $H = G - C$. As C is a maximal clique and as $F \subseteq C$, for any $v \in V(H)$, we have

$$d_C(v) \le s-1 \text{ and } d_G(v) \le \frac{n}{2}. \tag{2.3}$$

Claim 1. $s \ge \frac{n}{3} + k + 1$.

By contradiction, we assume that $s < \frac{n}{3} + k + 1$. It follows by $|V(H)| = n - s$ and by (2.3) that

$$\begin{aligned} e(H) + e(V(H), C) &= \frac{\sum_{v \in V(H)} d_G(v) + \sum_{v \in V(H)} |N_C(v)|}{2} = \frac{\sum_{v \in V(H)} d_G(v) + \sum_{v \in V(H)} d_C(v)}{2} \\ &\le \frac{(n-s)\frac{n}{2} + (n-s)(s-1)}{2} = \frac{(n-s)(n+2s-2)}{4}. \end{aligned} \tag{2.4}$$

As C is a clique, $e(G[C]) = \binom{s}{2}$ and so by (2.4) and by $s < \frac{n}{3} + k + 1$, we have

$$\begin{aligned} e(G) &= e(G[C]) + e(H) + e(V(H), C) \le \binom{s}{2} + \frac{(n-s)(n+2s-2)}{4} \\ &= \frac{n(n+s-2)}{4} < \frac{n(n + \frac{n}{3} + k + 1 - 2)}{4} \\ &= \frac{1}{3}n^2 + \frac{k-1}{4}n \le \binom{n-k}{2} + k(k+1) < e(G), \end{aligned}$$

a contradiction. Hence Claim 1 must hold.

Claim 2. $s \ge n-k+1$.

By contradiction, we assume that $s \le n-k$. Since $G = C_{n+1}(G)$, if $uv \notin E(G)$, then $d_G(u) + d_G(v) \le n$. As C is a clique, every vertex $u \in C$ satisfies $d_G(u) \ge s-1$. For each $v \in V(H)$, as $v \notin C$, we have $d_G(v) + d_G(u) \le n$, and so $d_G(v) \le n - d_G(u) \le n - s + 1$. As $H = G - C$, we have $\sum_{v \in V(H)} d_G(v) = 2e(H) + e(V(H), C)$. Thus

$$e(H)+e(V(H),C)=\frac{\sum_{v\in V(H)}d_H(v)}{2}+\sum_{v\in V(H)}d_C(v)\le\sum_{v\in V(H)}d_G(v)\le(n-s)(n-s+1),$$

and so

$$e(G)=e(G[C])+e(H)+e(V(H),C)\le\binom{s}{2}+(n-s)(n-s+1)=\frac{3}{2}s^2-\left(2n+\frac{3}{2}\right)s+n^2+n.$$

Let $f(x)=\frac{3}{2}x^2-(2n+\frac{3}{2})x+n^2+n$. It is routine to show that $f(x)$ is increasing on x for $x\ge\frac{2}{3}n+\frac{1}{2}$ and decreasing on x for $x\le\frac{2}{3}n+\frac{1}{2}$. As $f(n-k)=f(\frac{n}{3}+k+1)=\binom{n-k}{2}+k(k+1)$. it follows that

$$e(G)=\frac{3}{2}s^2-(2n+\frac{3}{2})s+n^2+n\le\binom{n-k}{2}+k(k+1)<e(G),$$

a contradiction. Hence Claim 2 holds and so $\omega(G)\ge s\ge n-k+1$. □

Theorem 2.4. *(Dirac [6]) If a simple graph G has minimum degree $d>1$, then G contains a cycle of length at least $d+1$.*

Let H be a subgraph of a graph G. Define the **vertices of attachment** of H in G to be the vertex set:

$$A_G(H)=\{v\in V(H):\exists u\in V(G)-V(H)\text{ such that }uv\in E(G)\}.$$

Theorem 2.5. *Let $k\ge3$ be an integer, G be a graph with order $n\ge6k$ and $\delta(G)\ge k$. Suppose that*

$$e(G)>\binom{n-k}{2}+k(k+1).$$

Then G is Hamilton-connected if and only if $C_{n+1}(G)\notin\{L_n^k,M_n^k\}$.

Proof. Let $G'=C_{n+1}(G)$. By Theorem 2.3, G is Hamilton-connected if and only if G' is Hamilton-connected. It is routine to verify that neither L_n^k nor M_n^k is Hamilton-connected, and so it remains to assume that $G'\notin\{L_n^k,M_n^k\}$ to prove that G' is Hamilton-connected.

We argue by contradiction and assume that $G'\notin\{L_n^k,M_n^k\}$ and G' is not Hamilton-connected. Since $\delta(G')\ge\delta(G)\ge k$ and $e(G')\ge e(G)$, it follows by Lemma 2.2 that $\omega(G')\ge n-k+1$. Let C be a maximum clique of G', $w=\omega(G')$ and $H=G'-C$.

Claim 1: Each of the following holds:

(i) If $v\in C$ satisfying $d_{G'}(v)\ge\omega(G')$, then for any $u\in V(H)$, $uv\in E(G')$.
(ii) There is no vertex $u\in V(H)$ satisfying $d_{G'}(u)\ge n-\omega(G')+2$.

Let $v\in C$ be a vertex with $d_{G'}(v)\ge\omega(G')$. For any $u\in V(H)$, by $\delta(G')\ge\delta(G)\ge k$ and by Lemma 2.2, we have $d_{G'}(v)+d_{G'}(u)\ge\omega(G')+k\ge n-k+1+k=n+1$, and so as $G'=C_{n+1}(G)$, we have $uv\in E(G')$. This proves (i).

We argue by contradiction to prove (ii) and assume that there exists a vertex $u\in V(H)$ satisfying $d_{G'}(u)\ge n-\omega(G')+2$, then as C is a clique, for any vertex $v\in C$, $d_{G'}(v)\ge\omega(G')-1$. Hence $d_{G'}(v)+d_{G'}(u)\ge\omega(G')-1+n-\omega(G')+2=n+1$. It follows by $G'=C_{n+1}(G)$ that $uv\in E(G')$, and so every vertex in H is adjacent to every vertex in C, contrary to the fact that C is a maximum clique. This verifies Claim 1(ii), and so Claim 1 is justified.

Claim 2: $\omega(G')=n-k+1$ and for any vertex $u\in V(H)$, $d_{G'}(u)=k$.

If $\omega(G')\ge n-k+2$, then for any vertex $v\in C$, $d_{G'}(v)\ge n-k+1$. Since G' is not Hamilton-connected, G' is not a clique, and so $V(H)\ne\emptyset$. For any vertex $u\in V(H)$, as $d_{G'}(u)\ge\delta(G)\ge k$, we obtain a contradiction to Claim 1. Hence by Lemma 2.2, we have $\omega(G')=n-k+1$. As $\omega(G')=n-k+1$ and $\delta(G)\ge k$, it follows by Claim 1 that for any vertex $u\in V(H)$, $d_{G'}(u)=k$. This justifies Claim 2.

Denote $F=A_{G'}(C)=\{u_1,u_2,\ldots,u_s\}$. As C is a maximum clique of G', it follows from Claim 2 that $d_{G'}(u_i)\ge n-k+1=\omega(G')$, and so by Claim 1(i) that $d_{G'}(u_i)=n-1$. This implies that for any $u\in V(H)$, $A_{G'}(C)\subseteq N_{G'}(u)$, and so by Claim 2, $s\le k$. As $\omega(G')=n-k+1$, for any $u\in H$, $d_H(u)\le k-2$. Hence $2\le s\le k$.

By inspection, if $s=2$, then $G'=K_2\vee(K_{k-1}+K_{n-k-1})=L_n^k$; and if $s=k$, then $G'=K_k\vee(K_{n-2k+1}+(k-1)K_1)=M_n^k$. As we assume that $G'\notin\{L_n^k,M_n^k\}$, we must have $3\le s\le k-1$.

For any vertex $u\in V(H)$, since $|A_{G'}(C)|=s$ and $A_{G'}(C)\subseteq N_{G'}(u)$, we have $d_H(u)=k-s$. By Theorem 2.4, H has a cycle C_1 with $q=|C_1|\ge k-s+1$. Let $C_1=x_1x_2\cdots x_qx_1$. For any pair of distinct vertices x_i and x_j on C_1 with $i\ne j$, we use $x_iC_1x_j$ ($x_i\overleftarrow{C_1}x_j$) to denote the subpath $x_ix_{i+1}\cdots x_j$ ($x_ix_{i-1}\cdots x_{j+1}x_j$) on C_1, where the subscripts are taken modulo q.

In the rest of the proof of this theorem, we denote $V(H-C_1)=\{y_1,y_2,\cdots,y_{k-q-1}\}$ and $C-F=\{u_{s+1},u_{s+2},\cdots u_{n-k+1}\}$.

To obtain a contradiction, we are to show that a Hamiltonian (v,w)-path always exists in G' for any $v,w\in V(G')$. If $v,w\in F$, then without loss of generality, we assume that $v=u_1$ and $w=u_s$. If $|C_1|=q\ge k-s+2$, then $k-q\le s-2$, and so $u_1y_1u_2y_2\cdots u_{k-q-1}y_{k-q-1}u_{k-q}x_1Px_q\ u_{k-q+1}\cdots u_{s-1}u_{s+1}\cdots u_{n-k+1}u_s$ is a Hamiltonian (v,w)-path.

Hence we may assume that $|C_1|=q=k-s+1$, or equivalently, $k-q=s-1$. If there exists no edge in G' linking a vertex in $V(C_1)$ to a vertex $V(H-C_1)$, then $E(H-C_1)\ne\emptyset$. By symmetry, we assume that $y_1y_2\in E(H-C_1)$. In this case,

Table 1
The existence of a Hamiltonian (v, w)-path.

Cases	v	w	Hamiltonian (v, w)-path in G'
$v, w \in V(C_1)$	x_i	x_j	$x_iC_1x_{j-1}u_1y1\cdots u_{k-q-1}u_{k-q}\cdots u_{s-1}$ $u_{s+1}\cdots u_{n-k+1}u_sx_{i-1}\overleftarrow{C_1}x_j$
$v \in V(C_1),\ w \in F$	x_1	u_1	$x_1C_1x_qu_su_{s+1}\cdots u_{n-k+1}u_{k-q}\cdots u_{s-1}y_{k-q-1}$ $u_{k-q-1}\cdots y_2u_2y_1u_1$
$v \in V(C_1),\ w \in V(H-C_1)$	x_1	y_1	$x_1C_1x_qu_su_{s+1}\cdots u_{n-k+1}u_{k-q}\cdots u_{s-1}u_{k-q-1}$ $y_{k-q-1}\cdots u_2y_2u_1y_1$
$v \in V(C_1),\ w \in C-F$	x_1	u_{n-k+1}	$x_1C_1x_qu_1y_1u_2y_2\cdots u_{k-q-1}y_{k-q-1}u_{k-q}\cdots u_{s-1}u_s$ $u_{s+1}\cdots u_{n-k+1}$
$v \in F,\ w \in C-F$	u_1	u_{n-k+1}	$u_1y_1u_2y_2\cdots u_{k-q-1}y_{k-q-1}u_{k-q}\cdots u_{s-1}x_1C_1x_q$ $u_su_{s+1}\cdots u_{n-k+1}$
$v \in F,\ w \in V(H-C_1)$	u_1	y_1	$y_1u_2y_2u_3y_3\cdots u_{k-q-1}y_{k-q-1}u_{k-q}\cdots u_{s-1}x_1C_1x_q$ $u_su_{s+1}\cdots u_{n-k+1}u_1$
$v \in V(H-C_1),\ w \in V(H-C_1)$	y_1	y_{k-q-1}	$y_1u_1y_2u_2\cdots y_{k-q-2}u_{k-q-2}u_{k-q}\cdots u_{s-1}x_1C_1x_q$ $u_s\cdots u_{n-k+1}\ u_{k-q-1}y_{k-q-1}$
$v \in V(H-C_1),\ w \in C-F$	y_1	u_{n-k+1}	$y_1u_1y_2u_2\cdots y_{k-q-1}u_{k-q-1}u_{k-q}\cdots u_{s-1}x_1C_1x_q$ $u_s\cdots u_{n-k+1}$
$v \in C-F,\ w \in C-F$	u_{s+1}	u_{n-k+1}	$u_{s+1}u_1y_1u_2y_2\cdots u_{k-q-1}y_{k-q-1}u_{k-q}\cdots u_{s-1}$ $x_1Px_qu_s\ u_{s+2}\cdots u_{n-k+1}$

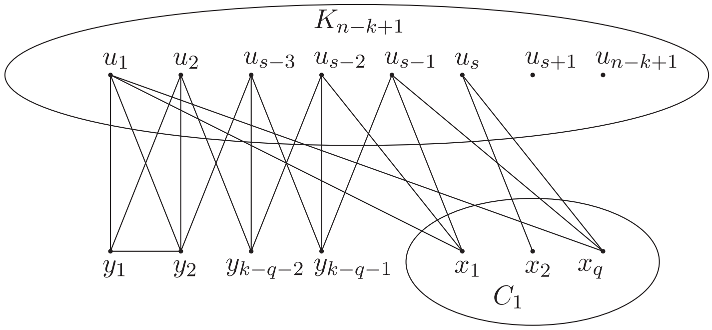

Fig. 1. The underlying graph to constructing P_i $(i = 1, \ldots, 10)$.

$u_s\cdots u_{n-k+1}u_{s-1}x_1C_1x_qu_{k-q-1}\ y_{k-q-1}\cdots u_2y_2y_1u_1$ is a Hamiltonian (v, w)-path. Hence we assume that there exists an edge x_1y_1 (say) linking $V(C_1)$ to $V(H-C_1)$. Then $u_s\cdots u_{n-k+1}u_{s-1}y_1x_1C_1x_qu_{k-q-1}y_{k-q-1}\cdots u_2y_2u_1$ is a Hamiltonian (v, w)-path.

Therefore, in the discussions below, we assume that $|\{v, w\} \cap F| \le 1$. As $V(G)$ is partitioned into F, $C-F$, $V(C_1)$ and $V(H-C_1)$, Table 1 indicates that for any other choices of $v, w \in V(G')$, by symmetry, G' always have a Hamiltonian (v, w)-path (Fig. 1).

As for any $v, w \in V(G')$, we have shown that G' always has a Hamiltonian (v, w)-path, leading to contradiction to the assumption that G' is not Hamilton-connected. This contradiction completes the proof of the theorem. □

3. Spectral radius and Hamilton-connected graphs

The goal of this section is to show a relationship between the spectral radius of a graph G and the Hamilton-connectedness of G.

Given two distinct vertices u, v in a graph G, obtain a new graph $G' = G'(u, v)$ by replacing all edges vw by uw for each $w \in N_G(v) \backslash (N_G(u) \cup \{u\})$. This operation is called the *Kelmans transformation*[11]. We start with some lemmas.

Lemma 3.1. *(Hong et al. [10], and Nikiforov [12]) Let G be a graph of order n with the minimum degree $\delta \ge k$. Then*

$$\lambda(G) \le \frac{k-1+\sqrt{(k+1)^2+4(2e(G)-nk)}}{2}.$$

Lemma 3.2. *(Csikvári [4]) Let G be a graph and G' be the graph obtained from G by some Kelmans transformation. Then*

$$\lambda(G) \le \lambda(G').$$

Since K_{n-k+1} is a proper subgraph of both L_n^k and M_n^k, it follows that $\lambda(L_n^k) > \lambda(K_{n-k+1}) = n-k$ and $\lambda(M_n^k) > \lambda(K_{n-k+1}) = n-k$. Motivated by the ideas in [13] and [9], we establish the following theorem.

Theorem 3.1. *Let G be a graph of order $n \ge \max\{6k, \frac{1}{2}k^3 - \frac{1}{2}k^2 + k + 4\}$ with the minimum degree $\delta \ge k \ge 3$.*

(i) If G is a subgraph of L_n^k, then $\lambda(G) < n-k$ if and only if $G \ne L_n^k$.

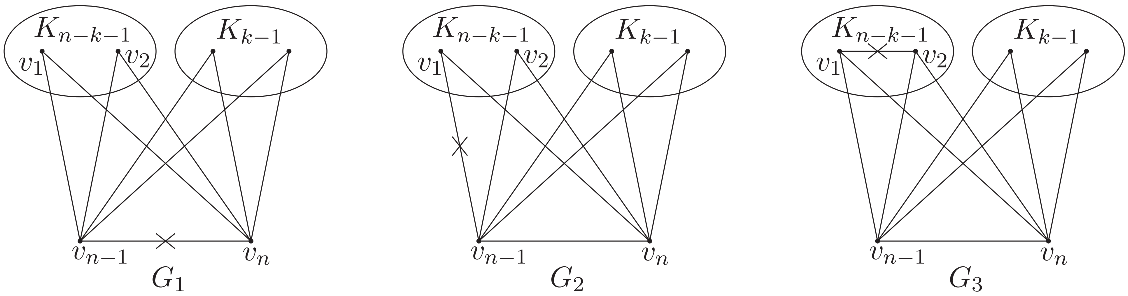

Fig. 2. The graphs obtained from L_n^k by deleting one edge.

(ii) If G is a subgraph of M_n^k, then $\lambda(G) < n-k$ if and only if $G \neq M_n^k$.

Proof. As we have observed above, both $\lambda(L_n^k) > \lambda(K_{n-k+1}) = n-k$ and $\lambda(M_n^k) > \lambda(K_{n-k+1}) = n-k$, it suffices to assume that $G \neq L_n^k$ to prove $\lambda(G) < n-k$ in (i); and that $G \neq M_n^k$ to prove $\lambda(G) < n-k$ in (ii).

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be a positive unit eigenvector of $\lambda(G)$. By Rayleigh's quotient inequality [5],

$$\lambda(G) = \mathbf{x}^{\mathrm{T}} A(G)\mathbf{x} = \langle A(G)\mathbf{x}, \mathbf{x}\rangle.$$

Throughout the rest of the proof, we often use λ for $\lambda(G)$, when G is understood from the context.

We argue by contradiction to prove (i), and assume that

$$\lambda(G) \geq n-k \text{ and } G \neq L_n^k. \tag{3.5}$$

Then G is a proper subgraph of L_n^k. Clearly, we only need to consider G with the maximum spectral radius which can be obtained from L_n^k by deleting one edge. By symmetry, there are only three such graphs: $G_1 = L_n^k - v_{n-1}v_n$, $G_2 = L_n^k - v_1 v_{n-1}$ and $G_3 = L_n^k - v_1 v_2$.

We claim that $\lambda(G_1) \leq \lambda(G_2) \leq \lambda(G_3)$. Using the notation in Fig. 2, let $u = v_n, v = v_1$ in G_1. Thus $N_G(v)\backslash(N_G(u) \cup \{u\}) = \{v_{n-1}\}$. Let $G_1' = G_1'(u, v)$ be a Kelmans transformation of G_1. Then $G_1' = G_2$. By Lemma 3.2, $\lambda(G_1) \leq \lambda(G_1') = \lambda(G_2)$. Now let $u = v_{n-1}, v = v_2$ in G_2. We have $N_{G_2}(v)\backslash(N_{G_2}(u) \cup \{u\}) = \{v_1\}$. Then the Kelmans transformation $G_2' = G_2'(u, v)$ is isomorphic to G_3, and so Lemma 3.2, $\lambda(G_2) \leq \lambda(G_2') = \lambda(G_3)$. This justifies the claim.

Define $Z = \{v \in V(L_n^k) : d_{L_n^k}(v) = n-1\}$, $X = \{v \in V(L_n^k) : d_{L_n^k}(v) = n-k\}$, and $Y = \{v \in V(L_n^k) : d_{L_n^k}(v) = k\}$. Hence, it suffices to assume $G = L_n^k - uv$ for some edge uv with $\{u, v\} \subset X$ to prove (i). Therefore, in the rest of the proof for (i), we shall have such an assumption. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ denote a positive unit eigenvector of $\lambda(G)$, and define

$$\begin{aligned}
&x = x_i, v_i \in X \setminus \{u, v\},\\
&y = x_j, v_j \in Y,\\
&z = x_k, v_k \in Z,\\
&s = x_u = x_v.
\end{aligned}$$

Then the n eigenequations of G can be reduced to the following four equations:

$$\begin{aligned}
\lambda x &= (n-k-4)x + 2z + 2s,\\
\lambda y &= (k-2)y + 2z,\\
\lambda z &= (n-k-3)x + (k-1)y + z + 2s,\\
\lambda s &= (n-k-3)x + 2z.
\end{aligned}$$

It follows from algebraic manipulations that

$$\begin{aligned}
y &= \frac{2}{\lambda - k + 2} z,\\
x &= \left(1 - \frac{2(k-1)}{(\lambda+1)(\lambda-k+2)}\right) z,\\
s &= \frac{\lambda+1}{\lambda+2}\left(1 - \frac{2(k-1)}{(\lambda+1)(\lambda-k+2)}\right) z.
\end{aligned}$$

By the definition of G, we have

$$G - \{yz : y \in Y \text{ and } z \in Z\} + uv \cong K_{n-k+1} + K_{k-1}.$$

Let $\mathbf{x}'$ be the restriction of $\mathbf{x}$ to K_{n-k+1}, then

$$\begin{aligned}\langle A(K_{n-k+1})\mathbf{x}', \mathbf{x}'\rangle &= \langle A(G)\mathbf{x}, \mathbf{x}\rangle + 2s^2 - 4(k-1)yz - (k-1)(k-2)y^2 \\ &= \lambda + 2s^2 - 4(k-1)yz - (k-1)(k-2)y^2.\end{aligned}$$

By Rayleigh's quotient inequality,

$$\frac{\langle A(K_{n-k+1})\mathbf{x}', \mathbf{x}'\rangle}{\|\mathbf{x}'\|^2} < \lambda(K_{n-k+1}) = n-k.$$

By (3.5), $\lambda \geq n-k$. This, together with $\|\mathbf{x}'\|^2 = \|\mathbf{x}\|^2 - (k-1)y^2$, implies that

$$2s^2 + \lambda(k-1)y^2 < 4(k-1)yz + (k-1)(k-2)y^2. \tag{3.6}$$

Since $k \geq 3$ and $\lambda \geq n-k \geq 5k > k-2$, we have

$$\begin{aligned}s^2 &= \left(\frac{\lambda+1}{\lambda+2}\right)^2\left(1 - \frac{2(k-1)}{(\lambda+1)(\lambda-k+2)}\right)^2 z^2 \\ &> \left(1 - \frac{2}{\lambda+2}\right)\left(1 - \frac{4(k-1)}{(\lambda+1)(\lambda-k+2)}\right)z^2 \\ &> \left(1 - \frac{2}{\lambda+2} - \frac{4(k-1)}{(\lambda+1)(\lambda-k+2)}\right)z^2 \\ &> \left(1 - \frac{2}{\lambda-k+2} - \frac{1}{\lambda-k+2}\right)z^2 = \left(\frac{\lambda-k-1}{\lambda-k+2}\right)z^2 \\ &> \left(\frac{4k-4}{\lambda-k+2}\right)z^2 = 2(k-1)yz,\end{aligned}$$

and $\lambda(k-1)y^2 > (k-1)(k-2)y^2$. It follows that

$$2s^2 + \lambda(k-1)y^2 > 4(k-1)yz + (k-1)(k-2)y^2,$$

contrary to (3.6). This completes the proof of (i).

The proof for (ii) follows a similar proving strategy as in that of (i), and so we also argue by contradiction. Assume that

$$\lambda(G) \geq n-k \text{ and } G \neq M_n^k. \tag{3.7}$$

Then G is a proper subgraph of M_n^k.

Define $Z = \{v \in V(M_n^k) : d_{M_n^k}(v) = n-1\}$, $X = \{v \in V(M_n^k) : d_{M_n^k}(v) = n-k\}$, and $Y = \{v \in V(M_n^k) : d_{M_n^k}(v) = k\}$.

As in the proof of (i), we only need to consider the case that $G = M_n^k - uv$ for an edge uv with $\{u, v\} \subset X$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ denote a positive unit eigenvector of $\lambda(G)$, and define

$$\begin{aligned}&x = x_i, v_i \in X \setminus \{u, v\}, \\ &y = x_j, v_j \in Y, \\ &z = x_k, v_k \in Z, \\ &s = x_u = x_v.\end{aligned}$$

Then the n eigenequations of G can be reduced to the following four equations:

$$\begin{aligned}&\lambda x = (n-2k-2)x + kz + 2s, \\ &\lambda y = kz, \\ &\lambda z = (n-2k-1)x + (k-1)y + (k-1)z + 2s, \\ &\lambda s = (n-2k-1)x + kz.\end{aligned}$$

It follows by algebraic manipulations that

$$\begin{aligned}&y = \frac{k}{\lambda}z, \\ &x = \left(1 - \frac{k(k-1)}{\lambda(\lambda+1)}\right)z, \\ &s = \frac{\lambda+1}{\lambda+2}\left(1 - \frac{k(k-1)}{\lambda(\lambda+1)}\right)z.\end{aligned}$$

By the definition of G, we have

$$G - \{yz : y \in Y \text{ and } z \in Z\} + uv \cong K_{n-k+1} + (k-1)K_1.$$

Let $\mathbf{x}'$ be the restriction of $\mathbf{x}$ to K_{n-k+1}, then $\|\mathbf{x}'\|^2 = \|\mathbf{x}\|^2 - (k-1)y^2 = 1 - (k-1)y^2$. By Rayleigh's quotient inequality,

$$\frac{\langle A(K_{n-k+1})\mathbf{x}', \mathbf{x}'\rangle}{\|\mathbf{x}'\|^2} < \lambda(K_{n-k+1}) = n-k.$$

By (3.7), $\lambda \geq n-k$, and so,

$$\langle A(K_{n-k+1})\mathbf{x}', \mathbf{x}'\rangle < \lambda\big(1-(k-1)y^2\big). \tag{3.8}$$

Since

$$\langle A(K_{n-k+1})\mathbf{x}', \mathbf{x}'\rangle = \langle A(G)\mathbf{x}, \mathbf{x}\rangle + 2s^2 - 2k(k-1)yz = \lambda + 2s^2 - 2k(k-1)yz, \tag{3.9}$$

by (3.8) and (3.9),

$$s^2 + \frac{k-1}{2}\lambda y^2 - k(k-1)yz < 0. \tag{3.10}$$

Since $\lambda \geq n-k \geq \frac{1}{2}k^3 - \frac{1}{2}k^2 + 4$,

$$\begin{aligned}
&\lambda\left(s^2 + \frac{k-1}{2}\lambda y^2 - k(k-1)yz\right)\\
&= \left(\frac{\lambda+1}{\lambda+2}\right)^2\left(1 - \frac{k(k-1)}{\lambda(\lambda+1)}\right)^2\lambda z^2 + \frac{k^2(k-1)}{2}z^2 - k^2(k-1)z^2\\
&> \left(1 - \frac{2}{\lambda+2}\right)\left(1 - \frac{2k(k-1)}{\lambda(\lambda+1)}\right)\lambda z^2 - \frac{k^2(k-1)}{2}z^2\\
&> \left(\lambda - \frac{2\lambda}{\lambda+2} - \frac{2k(k-1)}{\lambda+1} - \frac{k^2(k-1)}{2}\right)z^2\\
&> \left(\lambda - 2 - \frac{2k^2}{\frac{1}{2}k^3 - \frac{1}{2}k^2 + 5} - \frac{k^2(k-1)}{2}\right)z^2\\
&> \left(\frac{1}{2}k^3 - \frac{1}{2}k^2 + 2 - \frac{2k^2}{\frac{1}{2}k^3 - \frac{1}{2}k^2} - \frac{k^3}{2} + \frac{k^2}{2}\right)z^2\\
&= (2 - \frac{4}{k-1})z^2.
\end{aligned}$$

Let $f(x) = 2 - \dfrac{4}{x-1}$. It is routine to show that $f(x)$ is increasing on x for all real numbers $x \geq 2$. If $k \geq 3$ and k is an integer, then we have $f(k) \geq f(3) = 0$. It follows that

$$s^2 + \frac{k-1}{2}\lambda y^2 - k(k-1)yz > 0,$$

contrary to (3.10). This completes the proof of (ii). □

Theorem 3.2. *Let $k \geq 3$ be an integer, and let G be a graph with order $n \geq \max\{6k, \frac{1}{2}k^3 - \frac{1}{2}k^2 + k + 4\}$, $\delta(G) \geq k$ and spectral radius $\lambda(G) \geq n-k$. Then G is Hamilton-connected if and only if $G \notin \{L_n^k, M_n^k\}$.*

Proof. It routine to verify that if $G \in \{L_n^k, M_n^k\}$, then G is not Hamilton-connected. Therefore, in the rest of the proof, we assume that $G \in \{L_n^k, M_n^k\}$ to prove that G is Hamilton-connected. To the end of this section, we set $\lambda = \lambda(G)$ and $\delta = \delta(G)$.

Since $\lambda \geq n-k$ and by Lemma 3.1, we have

$$n-k \leq \lambda \leq \frac{k-1+\sqrt{(k+1)^2 + 4(2e(G) - nk)}}{2},$$

which implies that

$$e(G) \geq \frac{n^2 - (2k-1)n + 2k^2 - 2k}{2}. \tag{3.11}$$

Since $n \geq \max\{6k, \dfrac{1}{2}k^3 - \dfrac{1}{2}k^2 + k + 4\} \geq \dfrac{k^2+5k+2}{2}$, it follows from (3.11) that

$$e(G) \geq \frac{n^2 - (2k-1)n + 2k^2 - 2k}{2} \geq \binom{n-k}{2} + k(k+1) + 1, \tag{3.12}$$

where the equality in (3.12) holds if and only if $n = \frac{k^2+5k+2}{2}$.

Since $n \geq \max\{6k, \frac{1}{2}k^3 - \frac{1}{2}k^2 + k + 4\}$ and since $G \notin \{L_n^k, M_n^k\}$, it follows by Theorem 2.5 (when $n \geq \max\{6k, \frac{1}{2}k^3 - \frac{1}{2}k^2 + k + 4\}$) or by Theorem 3.1 (when $\lambda(G) \geq n - k$) then G is Hamilton-connected. The proof is complete. □

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