Graphs and Combinatorics (2018) 34:1713–1721
https://doi.org/10.1007/s00373-018-1963-z

ORIGINAL PAPER

Constructing Graphs Which are Permanental Cospectral and Adjacency Cospectral

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Received: 19 April 2018 / Revised: 19 September 2018 / Published online: 12 October 2018

Abstract

Two graphs are adjacency cospectral (respectively, permanental cospectral) if they have the same adjacency spectrum (respectively, permanental spectrum). In this paper, we present a new method to construct new adjacency cospectral and permanental cospetral pairs of graphs from smaller ones. As an application, we obtain an infinite family of pairs of Cartesian product graphs which are adjacency cospectral and permanental cospetral.

Keywords Permanental polynomial · Characteristic polynomial · Permanental cospectral · Adjacency cospectral

Mathematics Subject Classification 05C31 · 05C50 · 15A15

1 Introduction

The *permanent* of an $n \times n$ matrix M with entries m_{ij} $(i, j = 1, 2, \ldots, n)$ is defined by

$$\mathrm{per}(M) = \sum_{\sigma} \prod_{i=1}^{n} m_{i\sigma(i)},$$

where the sum is taken over all permutations σ of $\{1, 2, \ldots, n\}$. Permanent plays an important role in combinatorics. As an example, the permanent of a (0,1)-matrix

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enumerates perfect matchings in a related bipartite graph [19]. However, Valiant [25] has shown that computing a permanent is #P-complete even when restricted to (0, 1)-matrices.

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. We use $v_i \sim v_j$ to denote the fact that v_i and v_j are adjacent in G. An *acyclic* graph is one that has no cycles, which is also called a *forest*. A *tree* is a connected acyclic graph. Let G and H be two vertex disjoint graphs. The *Cartesian product* $G\square H$ is the graph with vertex set $V(G) \times V(H)$, where for $v, v' \in V(G)$ and $w, w' \in V(H)$, $(v, w) \sim (v', w')$ in $G\square H$ if and only if either $v = v'$ and $w \sim w'$ or $w = w'$ and $v \sim v'$. The union of two graphs G and H, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Let G be a graph of order n, $I = I_n$ be the identity matrix of order n, and $A(G)$ be the (0,1)-*adjacency matrix* of G. The (adjacency) characteristic polynomial of G is $\phi(G, x) = \det(xI - A(G))$. The collection of the eigenvalues (together with their multiplicities) of $A(G)$ is the *adjacency spectrum* of graph G.

The *permanental polynomial* of G, denoted by $\pi(G, x)$, is defined as $\pi(G, x) = \mathrm{per}(xI - A(G))$. The *permanental spectrum* (or just per-spectrum) of G is the collection of all roots (together with their multiplicities) of $\pi(G, x)$.

Characteristic polynomials of graphs and their applications have been intensively studied, as seen in [10–12], among others. As far as we know, permanental polynomials of graphs were first considered by Turner [24]. Subsequently, Borowiecki and Jóźwiak [3], Kasum et al. [15] and Merris et al. [21] systematically introduced permanental polynomials of graphs in mathematical and chemical studies, respectively. For a period of time, the study of permanental polynomials did not seem to be comparable to that of characteristic polynomials, see [5,6,20,22], among others. This may be due to the difficulty of computing $\mathrm{per}(xI - A(G))$. However, permanental polynomials and their applications have received a lot of attention from researchers in recent years. Zhang et al. [32] showed that the per-spectra of bipartite graphs are symmetric with respect to the real and imaginary axes. In [26], the per-nullity of graphs was introduced, and graphs of order n with per-nullity $n-2$, $n-3$, $n-4$ and $n-5$, are respectively determined. Moreover, some edge-deleted subgraphs of a complete graph are shown to be uniquely determined by their per-spectra [27,33]. Liu and Zhang [17,18] showed that complete graphs, stars, regular complete bipartite graphs, odd cycles and odd lollipop graphs are determined by their per-spectra. For more studies see [1,7–9,13,14,16,23,28–31], and the references therein.

Two graphs G and H are *per-cospectral* (resp. *adjacency cospectral*) if G and H have the same per-spectrum (resp. adjacency spectrum). Which graph pairs are adjacency cospectral as well as per-cospectral? Borowiecki and Jóźwiak [3] first considered the problem and showed that for two non-isomorphic trees T_1 and T_2, they are per-cospectral if and only if T_1 and T_2 are adjacency cospectral. This result was extended by Borowiecki in [4]. He proved that if G_1 and G_2 are bipartite graphs without cycles of length k, where $k \equiv 0 \pmod 4$, then G_1 and G_2 are per-cospectral if and only if G_1 and G_2 are adjacency cospectral. Furthermore, Borowiecki and Jóźwiak [3] presented an ingenious way to construct graph pairs which are both adjacency cospectral and per-cospectral. Let G and G' be adjacency cospectral (resp. per-cospectral) graphs and let $u \in V(G)$ and $v \in V(G')$ be vertices of G and G', respectively. Let

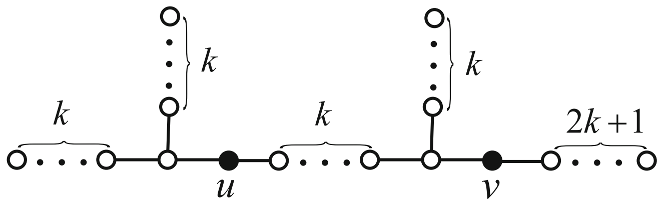


Fig. 1 Graph G

H be an arbitrary graph with a fixed vertex w and let $G_u \cdot H$ denote the coalescence of G and H with respect to u and w, which is the graph obtained from $G \cup H$ by identifying u and w. Similarly, we define $G'_v \cdot H$. Borowiecki and Jóźwiak [3] showed that if both $G - u$ and $G' - v$ are adjacency cospectral (resp. per-cospectral), then both $G_u \cdot H$ and $G'_v \cdot H$ are also adjacency cospectral (resp. per-cospectral). As an example, let $k > 0$ be a integer, and let $G = G'$ be the graph depicted in Fig. 1. As $G - u$ and $G - v$ are isomorphic, they are adjacency cospectral and per-cospectral. By the above mentioned result of Borowiecki and Jóźwiak [3], for any graph H, both $G_u \cdot H$ and $G_v \cdot H$ are per-cospectral and adjacency cospectral.

The main purpose of this paper is to seek new methods to construct new per-cospectral and adjacency cospectral graph pairs from smaller ones. The rest of this paper is organized as follows. In Sect. 2, we present some basic properties of characteristic polynomial and permanental polynomial of a graph. In Sect. 3, a new method to construct per-cospectral and adjacency cospectral forest pairs will be presented; and the characteristic polynomial of $G \square K_2$ will be computed. This allows us to obtain a new result that $G \square K_2$ and $H \square K_2$ are adjacency cospectral if G and H are adjacency cospectral. The permanental polynomial of $F \square K_2$ will also be computed by utilizing a result by Yan and Zhang in [30]. Consequently, we obtain that for any two non-isomorphic forests F_1 and F_2, if F_1 and F_2 are both per-cospectral and adjacency cospectral, then $F_1 \square K_2$ and $F_2 \square K_2$ are both per-cospectral and adjacency cospectral.

2 Some Lemmas

In this section, we develop some lemmas which are useful in the proofs of the main results. Let G be a graph and let $v \in V(G)$. Let $\mathscr{C}(v)$ denote the set of cycles containing v and $N(v)$ denote the neighborhood of v in G. For an edge $e \in E(G)$, let $\mathscr{C}(e)$ be the set of cycles containing e.

Lemma 2.1 (Cvetković, Doob and Sachs [10]) *The characteristic polynomial of a graph G satisfies the following identities:*

(i) $\phi(G \cup H, x) = \phi(G, x)\phi(H, x)$.
(ii) *If $v \in V(G)$, then*

$$\phi(G, x) = x\phi(G - v, x) - \sum_{u \in N(v)} \phi(G - u - v, x) - 2 \sum_{C \in \mathscr{C}(v)} \phi(G - V(C), x).$$

(iii) *If* $e = uv \in E(G)$*, then*

$$\phi(G, x) = \phi(G - e, x) - \phi(G - u - v, x) - 2 \sum_{C \in \mathscr{C}(e)} \phi(G - V(C), x).$$

Borowiecki and Jóźwiak [4] extended Lemma 2.1 to compute the permanental polynomial of a graph, as follows.

Lemma 2.2 (Borowiecki and Jóźwiak [4]) *Let G be a graph.*

(i) *If* $v \in V(G)$*, then*

$$\pi(G, x) = x\pi(G - v, x) + \sum_{u \in N(v)} \pi(G - u - v, x) + 2 \sum_{C \in \mathscr{C}(v)} (-1)^{|V(C)|} \pi(G - V(C), x).$$

(ii) *Let* $e = uv \in E(G)$*, then*

$$\pi(G, x) = \pi(G - e, x) + \pi(G - u - v, x) + 2 \sum_{C \in \mathscr{C}(e)} (-1)^{|V(C)|} \pi(G - V(C), x).$$

By Lemma 2.2, the following result is obvious.

Corollary 2.1 (i) *Let T be a tree. If* $v \in V(T)$*, then* $\pi(T, x) = x\pi(T - v, x) + \sum_{v \in N(u)} \pi(T - u - v, x)$*, where* $N(v)$ *is the neighborhood of* v*.*
(ii) *Let* $e = uv \in E(G)$ *be a cut edge of G, then* $\pi(G, x) = \pi(G - e, x) + \pi(G - u - v, x)$*.*

Lemma 2.3 *Let G be a graph with components* $G_1, G_2, \ldots, G_t$*. Then* $\pi(G, x) = \prod_{i=1}^{t} \pi(G_i, x)$*.*

3 Main Results

For any positive integer ℓ and for a graph G, let ℓG denote the union of ℓ disjoint copies of G. In this section, we will present new methods to construct larger per-cospectral and adjacency cospectral pairs of forests from smaller ones.

Definition 3.1 Let m be a positive integer and let T_1 and T_2 be two trees with $u \in V(T_1)$ and $v \in V(T_2)$. Define $(T_{1u} \wedge T_{2v}^m) \cup (m-1)T_1$ to be the forest obtained from $mT_1 \cup mT_2$ by fixing one tree T_1 in $mT_1 \cup mT_2$, and connecting the vertex u in this fixed T_1 to all the vertices v in each of the T_2's in $mT_1 \cup mT_2$, as depicted in Fig. 2. The forest $(T_{2v} \wedge T_{1u}^m) \cup (m-1)T_2$ is similarly defined, see Fig. 2.

Theorem 3.2 *For any nontrivial trees* T_1 *and* T_2*, and for any integer* $m \geq 2$*, each of the following holds.*

(i) $\pi((T_{1u} \wedge T_{2v}^m) \cup (m-1)T_1, x) = \pi((T_{2v} \wedge T_{1u}^m) \cup (m-1)T_2, x)$.
(ii) $\phi((T_{1u} \wedge T_{2v}^m) \cup (m-1)T_1, x) = \phi((T_{2v} \wedge T_{1u}^m) \cup (m-1)T_2, x)$.

Proof By Lemma 2.3 and Corollary 2.1, we have

$$\begin{aligned}\pi(T_{1u} \wedge T_{2v}^m, x) &= x\pi(T_1 - u, x)[\pi(T_2, x)]^m \\ &\quad + \sum_{i \in V(T_1),\ i \in N(u)} \pi\left(T_1 - u - i, x\right)\left[\pi(T_2, x)\right]^m \\ &\quad + m\pi(T_1 - u, x)\pi(T_2 - v, x)[\pi(T_2, x)]^{m-1} \\ &= [\pi(T_2, x)]^m \pi(T_1, x) + m[\pi(T_2, x)]^{m-1}\pi(T_1 - u, x)\pi(T_2 - v, x).\end{aligned}$$

Hence,

$$\begin{aligned}&\pi\left((T_{1u} \wedge T_{2v}^m) \cup (m-1)T_1, x\right) \\ &\quad = [\pi(T_1, x)\pi(T_2, x)]^m + m[\pi(T_1, x)\pi(T_2, x)]^{m-1}\pi(T_1 - u, x)\pi(T_2 - v, x) \\ &\quad = [\pi(T_1, x)\pi(T_2, x)]^{m-1}[\pi(T_1, x)\pi(T_2, x) + m\pi(T_1 - u, x)\pi(T_2 - v, x)].\end{aligned}$$

Similarly,

$$\begin{aligned}\pi\left(T_{2v} \wedge T_{1u}^m, x\right) &= x\pi(T_2 - v, x)[\pi(T_1, x)]^m \\ &\quad + \sum_{i \in V(T_2),\ i \in N(v)} \pi(T_2 - v - i, x)[\pi(T_1, x)]^m \\ &\quad + m\pi(T_2 - v, x)\pi(T_1 - u, x)[\pi(T_1, x)]^{m-1} \\ &= [\pi(T_1, x)]^m \pi(T_2, x) + m[\pi(T_1, x)]^{m-1}\pi(T_2 - v, x)\pi(T_1 - u, x)\end{aligned}$$

Hence,

$$\begin{aligned}&\pi\left((T_{2v} \wedge T_{1u}^m) \cup (m-1)T_2, x\right) \\ &\quad = [\pi(T_2, x)\pi(T_1, x)]^m + m[\pi(T_2, x)\pi(T_1, x)]^{m-1}\pi(T_2 - v, x)\pi(T_1 - u, x) \\ &\quad = [\pi(T_2, x)\pi(T_1, x)]^{m-1}[\pi(T_2, x)\pi(T_1, x) + m\pi(T_2 - v, x)\pi(T_1 - u, x)].\end{aligned}$$

It follows that we have

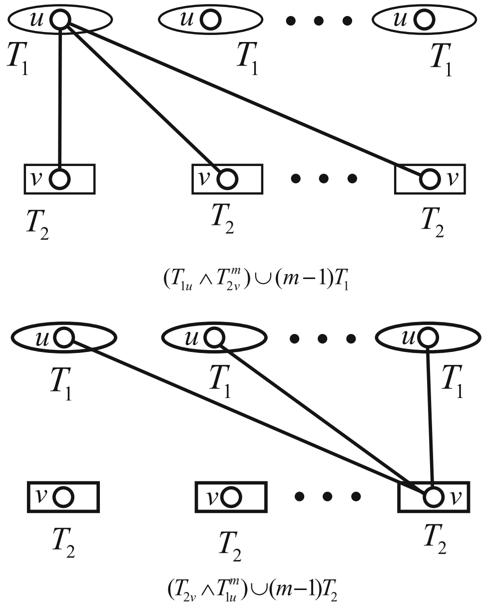
$$\pi\left(T_{1u} \wedge T_{2v}^m \cup (m-1)T_1, x\right) = \pi\left(T_{2v} \wedge T_{1u}^m \cup (m-1)T_2, x\right)$$

For the characteristic polynomial ϕ, we argue similarly, using Lemma 2.1 instead of Lemma 2.3. This also leads to

$$\phi\left((T_{1u} \wedge T_{2v}^m) \cup (m-1)T_1, x\right) = \phi\left((T_{2v} \wedge T_{1u}^m) \cup (m-1)T_2, x\right).$$

□

Fig. 2 Forests $(T_{1u} \wedge T_{2v}^m) \cup (m-1)T_1$ and $(T_{2v} \wedge T_{1u}^m) \cup (m-1)T_2$



By Theorem 3.2, we have the following corollary, which justifies that the construction in Definition 3.1 can produce larger per-cospectral and adjacency cospectral pairs of forests from smaller ones.

Corollary 3.3 *Let T_1 and T_2 be a pair of non-isomorphic trees. For positive integer $m > 1$, then forests $(T_{1u} \wedge T_{2v}^m) \cup (m-1)T_1$ and $(T_{2v} \wedge T_{1u}^m) \cup (m-1)T_2$ are per-cospectral and adjacency cospectral.*

Theorem 3.4 *Let G be a graph with n vertices, and let $G \Box K_2$ be a Cartesian product of G and K_2. Then the characteristic polynomial of $G \Box K_2$ is*

$$\phi(G \Box K_2, x) = \prod_{\lambda} [(x-\lambda)^2 - 1],$$

where the product ranges over all eigenvalues λ of G.

Proof Let $A(G)$ be the adjacency matrix of G, $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A(G)$ and let O_n denote the $n \times n$ zero matrix. By definition, we have

$$\begin{aligned}\phi(G\Box K_2, x) &= \mathbf{det}\begin{bmatrix} xI_n - A(G) & -I_n \\ -I_n & xI_n - A(G)\end{bmatrix}\\ &= \mathbf{det}\begin{bmatrix} I_n & xI_n - A(G) \\ O_n & I_n\end{bmatrix}\begin{bmatrix} xI_n - A(G) & -I_n \\ -I_n & xI_n - A(G)\end{bmatrix}\begin{bmatrix} I_n & xI_n - A(G) \\ O_n & I_n\end{bmatrix}\\ &= \mathbf{det}\begin{bmatrix} O_n & \left(xI_n - A(G)\right)^2 - I_n \\ -I_n & O_n\end{bmatrix}\\ &= (-1)^{n^2+n}\mathbf{det}\left[\left(xI_n - A(G)\right)^2 - I_n\right]\\ &= \mathbf{det}\left[(x^2-1)I_n - \left(2xA(G) - \left(A(G)\right)^2\right)\right] = \prod_{k=1}^{n}[(x-\lambda_k)^2 - 1].\end{aligned}$$

The proof of the theorem is complete. □

Corollary 3.5 is a natural consequence of Theorem 3.4.

Corollary 3.5 *Let G and H be a pair of adjacency cospectral graphs. Then $G\Box K_2$ and $H\Box K_2$ are adjacency cospectral.*

Theorem 3.6 (Yan and Zhang [30]) *Let T be an arbitrary tree with n vertices, and let $T\Box K_2$ be a Cartesian product graph of T and K_2. Then the permanental polynomial of $T\Box K_2$ is*

$$\pi(T\Box K_2, x) = \prod_{\alpha}(x^2 + 1 + \alpha^2),$$

where the product ranges over all eigenvalues α of T.

By Theorem 3.6 and Lemma 2.3, we have the following theorem.

Theorem 3.7 *Let F be a forest with components $T_1, T_2, \ldots, T_t$. Then*

$$\pi(F\Box K_2, x) = \prod_{i=1}^{t}\prod_{\alpha}(x^2 + 1 + \alpha^2),$$

where the first product takes over all components $T_1, T_2, \ldots, T_t$, and the second product ranges over all eigenvalues α of T_i ($i = 1, 2, \ldots, t$).

By Theorem 3.7, we have the following corollary.

Corollary 3.8 *Let F_1 and F_2 be a pair of per-cospectral forests. Then $F_1\Box K_2$ and $F_2\Box K_2$ are per-cospectral.*

By Theorem 3.2 and Corollaries 3.5 and 3.8, we can obtain the following theorem.

Theorem 3.9 *Let $F_1 = (T_{1u} \wedge T_{2v}^m) \cup (m-1)T_1$ and $F_2 = (T_{2v} \wedge T_{1u}^m) \cup (m-1)T_2$ be two forests as defined in Definition 3.1. Then $F_1\Box K_2$ and $F_2\Box K_2$ are per-cospectral and adjacency cospectral.*

Acknowledgements We would like to thank the referees for their helpful recommendations. The research is supported by the National Natural Science Foundation of China (No. 11761056), the Natural Science Foundation of Qinghai Province (2016-ZJ-947Q), the Ministry of Education Chunhui Project (No. Z2017047) and High-level Personnel of Scientific Research Project of QHMU(2016XJG07).

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