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Upper bounds of r-hued colorings of planar graphs

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ABSTRACT

For positive integers k and r, a *(k, r)-coloring* of a graph G is a proper k-coloring of the vertices such that every vertex of degree d is adjacent to vertices with at least $\min\{d, r\}$ different colors. The r-hued chromatic number of a graph G, denoted by $\chi_r(G)$, is the smallest integer k such that G has a (k, r)-coloring. In Song et al. (2014), it is conjectured that if $r \geq 8$, then every planar graph G satisfies $\chi_r(G) \leq \lceil \frac{3r}{2} \rceil + 1$. Wegner in 1977 conjectured that the above-mentioned conjecture holds when $r = \Delta(G)$. This conjecture, if valid, would be best possible in some sense. In this paper, we prove that, if G is a planar graph and $r \geq 8$, then $\chi_r(G) \leq 2r + 16$.

1. Introduction

Graphs in this paper are simple and finite. Undefined terminology and notation are referred to [1]. For $v \in V(G)$, let $N_G(v)$ denote the set of vertices adjacent to v in G, and $d_G(v) = |N_G(v)|$, the degree of v in G. A vertex v with $d_G(v) = h'$, a cycle of length h'' and a face of degree h''' are often referred to as an h'-**vertex**, an h''-**cycle** and an h'''-**face**, respectively. Let k, r be positive integer. Throughout this paper, define $\bar{k} = \{1, 2, \ldots, k\}$. If $c : V(G) \mapsto \bar{k}$, and if $S \subseteq V(G)$, then define $c(S) = \{c(u)|u \in S\}$. A (k, r)*-coloring* of a graph G is a map $c : V(G) \mapsto \bar{k}$ satisfying both the following.

(C1) $c(u) \neq c(v)$ for every edge $uv \in E(G)$;

(C2) $|c(N_G(v))| \geq \min\{d_G(v), r\}$ for any $v \in V(G)$.

For a fixed integer $r > 0$, the *r-hued chromatic number* of G, denoted by $\chi_r(G)$, is the smallest k such that G has a (k, r)-coloring. As observed in [5], we have

$$\chi(G) \leq \chi_2(G) \leq \cdots \chi_{r-1}(G) \leq \chi_r(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \chi_{\Delta(G)+1}(G) = \cdots$$

The notion was first introduced in [10] and [6]. When $r = 2$, $\chi_2(G)$ is often called *the dynamic chromatic number* of G. In [7], it was shown that (3,2)-colorability remains NP-complete even when restricted to planar bipartite graphs with maximum degree at most 3 and with arbitrarily high girth. This differs considerably from the well-known result that the classical 3-colorability is polynomially solvable for graphs with maximum degree at most 3. Nevertheless, there have been quite a few studies on the upper bounds of r-hued chromatic number of planar graphs. For any planar graph G, it is proved that $\chi_2(G) \leq 5$ in [2] without using the 4-Color Theorem. Utilizing the 4-Color Theorem, Kim et al. in [4] showed that 5-cycle is the only planar graph with 2-hued chromatic number being 5. More recently, Loeb et al. in [9] proved that $\chi_3(G) \leq 10$. In [12], Song et al. proved that any planar graph G with girth at least 6 satisfies $\chi_r(G) \leq r + 5$ when $r \geq 3$. A conjecture on the upper bound of r-hued-chromatic number of planar graphs is stated below. Wegner [13] conjectured the case when

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$r = \Delta(G)$ in Conjecture 1.1. Song et al. generalized Wegner's conjecture in [11]. As commented in [11], the conjecture below is best possible in some sense.

Conjecture 1.1. *Let G be a planar graph. Then*

$$\chi_r(G) \le \begin{cases} r+3, & \text{if } 1 \le r \le 2 \\ r+5, & \text{if } 3 \le r \le 7; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \ge 8. \end{cases}$$

A graph H is a **minor** of a graph G if G has a subgraph contractible to H; G is said to be H-*minor free* if G does not have H as a minor. Define

$$K(r) = \begin{cases} r+3, & \text{if } 2 \le r \le 3; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \ge 4. \end{cases}$$

Lih et al. [8] proved that, for any K_4-minor free graph G, $\chi_\Delta(G) \le K(\Delta(G))$. Song et al. extended it to $\chi_r(G) \le K(r)$ for any K_4-minor free graph G in [11]. Heuvel et al. [3] proved that, for any planar graph G, $\chi_\Delta(G) \le 2\Delta(G) + 25$. The main result of this paper are the following.

Theorem 1.2. *If G is a planar graph and* $r \ge 8$, *then* $\chi_r(G) \le 2r + 16$.

In the next section, we derive some structural properties of planar graphs. These properties will be applied in Section 3 to prove Theorem 1.2.

2. The structure of planar graphs

Throughout this section, we assume that G is a simple plane graph with a fixed embedding on the plane. If $v \in V(G)$, define

$$E_G(v) = \{e \in E(G) \,|e \text{ is incident with } v \text{ in } G\}.$$

When the graph G is understood from the context, we often use E_v for $E_G(v)$. Let $F(G)$ denote the collection of faces of G. For each face $f \in F(G)$, let $d(f)$ denote the number of edges belonging to f, where cut-edges are counted twice. For any element $x \in V(G) \cup E(G)$, define $\epsilon(x)$ be the number of 3-faces incident with x. If $v \in V(G)$ is an h-vertex, we assign an ordering $\preceq$, called the **difficulty-increasing order** on the set $N_G(v) = \{v_1, v_2, \ldots, v_h\}$ as follows: define $v_i \preceq v_j$ if either $d(v_i) < d(v_j)$, or both $d(v_i) = d(v_j)$ and $\epsilon(vv_i) \ge \epsilon(vv_j)$. The main result of this section is the following structural lemma.

Lemma 2.1. *Let G be a simple plane graph. Then there exists a k-vertex v with its neighbors* $v_1 \preceq v_2 \preceq, \ldots, \preceq v_k$ *such that one of the following is true:*
(i) $k \le 2$.
(ii) $k = 3$ *with*

$$\begin{cases} d(v_1) \le 5 & \text{if } \epsilon(v) = 0, \\ d(v_1) \le 8 & \text{if } \epsilon(v) = 1, \\ d(v_1) \le 11 & \text{if } \epsilon(v) \ge 2. \end{cases}$$

(iii) $k = 4$ *with*

$$\begin{cases} d(v_1) \le 6 \text{ and } d(v_1) + d(v_2) \le 13 & \text{if } \epsilon(v) \le 1, \\ d(v_1) \le 6 \text{ and } d(v_1) + d(v_2) \le 17 & \text{if } \epsilon(v) \ge 2 \text{ and } \epsilon(vv_1) = 0, \\ d(v_1) \le 7 \text{ and } d(v_1) + d(v_2) \le 17 & \text{if } \epsilon(v) \ge 2 \text{ and } \epsilon(vv_1) \ge 1. \end{cases}$$

(iv) $k = 5$ *with*

$$\begin{cases} d(v_1) + d(v_2) \le 9 \text{ and } d(v_1) + d(v_2) + d(v_3) \le 14 & \text{if } \epsilon(v) \le 1, \\ d(v_1) + d(v_2) \le 11 \text{ and } d(v_1) + d(v_2) + d(v_3) \le 17 & \text{if } \epsilon(v) = 2, \\ d(v_1) \le 5, d(v_1) + d(v_2) \le 13 \text{ and } d(v_1) + d(v_2) + d(v_3) \le 21 & \text{if } \epsilon(v) \ge 3 \text{ and } \epsilon(vv_1) = 0, \\ d(v_1) + d(v_2) \le 13 \text{ and } d(v_1) + d(v_2) + d(v_3) \le 21 & \text{if } \epsilon(v) \ge 3 \text{ and } \epsilon(vv_1) \ge 1. \end{cases}$$

Proof. By contradiction, we assume that there exists a simple plane graph such that none of the conclusions listed above holds. We assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{1}$$

As G is a counterexample,

$$\text{none of the conclusions in Lemma 2.1 (i)–(iv) holds.} \tag{2}$$

In particular, Lemma 2.1(i) does not hold. Hence by (1), we observed that

$$G \text{ is connected, } \delta(G) \geq 3 \text{ and } k \geq 3. \tag{3}$$

We first assign to each edge e of G an initial charge $ch_0(e)$. We then will apply certain recharge rules to obtain new charges $ch_2(e)$ to reach a contradiction.

For each edge $e = uv \in E(G)$ incident with two distinct faces $f, g \in F(G)$, we define the initial charge of e by

$$ch_0(e) = \frac{d(u)-4}{d(u)} + \frac{d(v)-4}{d(v)} + \frac{d(f)-4}{d(f)} + \frac{d(g)-4}{d(g)}. \tag{4}$$

If e is incident with only one face $f \in F(G)$, then we view $f = g$ and use (4) again to define $ch_0(e)$. By Euler's formula, we have

$$\sum_{v\in V(G)} (d(v)-4) + \sum_{f\in F(G)} (d(f)-4) = -8. \tag{5}$$

It follows from (4) and (5) that

$$\sum_{v\in V(G)} \sum_{e\in E_v} ch_0(e) = 2\left[\sum_{v\in V(G)} (d(v)-4) + \sum_{f\in F(G)} (d(f)-4)\right] = -16. \tag{6}$$

We now recharge the edges of G in the following order:

(Step 1) For each 3-face with vertices u, x, y satisfying $3 \leq d(u) \leq 5$, $d(x) \geq 6$ and $d(y) \geq 6$, transfer a charge of $\frac{1}{2}(\frac{d(y)-4}{d(y)} - \frac{1}{3})$ from xy to ux, and a charge of $\frac{1}{2}(\frac{d(x)-4}{d(x)} - \frac{1}{3})$ from xy to uy.

(Step 2) For each triple of vertices u, v, v' in $V(G)$ satisfying $uv, uv' \in E_u$, $d(u) = 5$, $d(v) \geq 6$, $d(v') \geq 6$, $\epsilon(uv) = 2$ and $\epsilon(uv') = 0$, transfer a charge of $\frac{1}{6}$ from uv' to uv.

For each edge $e \in E(G)$, we denote the charge of e after Step 1 by $ch_1(e)$ and the charge of e after Step 2 by $ch_2(e)$. As no new charges is introduced and no charge is lost, it follows by (6) that

$$\sum_{v\in V(G)} \sum_{e\in E_v} ch_2(e) = \sum_{v\in V(G)} \sum_{e\in E_v} ch_1(e) = \sum_{v\in V(G)} \sum_{e\in E_v} ch_0(e) = -16. \tag{7}$$

The proposition below, follows immediately from the definition of the charge rules.

Proposition 2.2. *Let $e = uv \in E(G)$ be an edge of G.*

(i) If $ch_0(e) < 0$, then $ch_2(e) \geq ch_0(e)$ and either $d(u) \leq 5$ or $d(v) \leq 5$;

(ii) If $ch_0(e) \geq 0$, then $ch_2(e) \geq 0$;

(iii) If $d(v) \leq 5$, $\sum_{e\in E_v} ch_2(e)$ is not increasing when $\epsilon(v)$ increases;

(iv) If $ch_2(e) < 0$, then $ch_2(e) \geq ch_0(e)$.

For each vertex $u \in V(G)$ and a specified vertex $x^1 \in N_G(u)$, the fixed planar embedding of G places edges in $E_G(u)$ clockwise around u, yielding an ordering $ux^1, ux^2, \ldots, ux^{d(u)}$. For each $w \in N_G(u)$, if $w = x^i$ for some i, then define $w^- = x^{i-1}$ and $w^+ = x^{i+1}$, where the superscripts are counted modulo $d_G(u)$.

By (7), there exists such a vertex $v \in V(G)$ with

$$\sum_{e\in E_v} ch_2(e) < 0. \tag{8}$$

To obtain a contradiction, we start with a vertex $v \in V(G)$ satisfying (8) with $d(v) = k$ and $N(v) = \{v_1, \ldots, v_k\}$ such that $v_1 \preceq v_2 \preceq, \ldots, \preceq v_k$.

Claim 2.3. $k \neq 3$.

Proof. Suppose $k = 3$. As Lemma 2.1(ii) does not hold, we must have either $d(v_1) \geq 6$ when $\epsilon(v) = 0$, or $d(v_1) \geq 9$ when $\epsilon(v) = 1$, or $d(v_1) \geq 12$ when $\epsilon(v) \geq 2$. But when any one of these cases occurs, by the recharging rules, we have

$$\sum_{e\in E_v} ch_2(e) \geq d(v)-4+\sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)})-\epsilon(v)\times\frac{2}{3}+\epsilon(v)(\frac{d(v_1)-4}{d(v_1)}-\frac{1}{3})$$
$$\geq -1-\epsilon(v)+(3+\epsilon(v))\times\frac{d(v_1)-4}{d(v_1)}\geq 0,$$

contrary to (8). Thus if $k=3$, then Lemma 2.1(ii) must hold, contrary to (2). □

By (3) and Claim 2.3, $d(v)=k>3$. If $d(v)\in\{4,5\}$ and $d(v_1)\leq 3$, then Lemma 2.1(i) or (ii) holds, violating (2). Hence we assume that

$$\text{if } d(v)\in\{4,5\}, \text{ then } d(v_1)\geq 4. \tag{9}$$

Claim 2.4. $k\neq 4$.

Proof. Suppose that $k=4$. By (9), $d(v_1)\geq 4$. By (2), Lemma 2.1(iii) does not hold. We first show that

$$\text{If } \epsilon(v)\leq 2, \text{ then } d(v_1)\leq 5; \text{ if } \epsilon(v)=3, \text{ then } d(v_1)\leq 6 \text{ and if } \epsilon(v)=4, \text{ then } d(v_1)\leq 7. \tag{10}$$

If (10) fails, then we may assume that $d(v_1)\geq 6$ when $\epsilon(v)\leq 2$, or $d(v_1)\geq 7$ when $\epsilon(v)=3$, or $d(v_1)\geq 8$ when $\epsilon(v)=4$. By the recharging rules, we have

$$\sum_{e\in E_v} ch_2(e) \geq d(v)-4+\sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)})-\epsilon(v)\times\frac{2}{3}+\epsilon(v)(\frac{d(v_1)-4}{d(v_1)}-\frac{1}{3})$$
$$\geq 0-\epsilon(v)+(4+\epsilon(v))\times\frac{d(v_1)-4}{d(v_1)}\geq 0,$$

contrary to (8). Hence (10) must hold. We now justify Lemma 2.1(iii).

A. If $\epsilon(v)\leq 2$, then by (9) and (10), we have $4\leq d(v_1)\leq 5$. As Lemma 2.1(iii) does not hold, we must have $d(v_2)\geq 9$, and so $\sum_{e\in E_v} ch_2(e)\geq d(v)-4+\frac{d(v_1)-4}{d(v_1)}+\sum_{i=2}^{k}(\frac{d(v_i)-4}{d(v_i)})-2\times\frac{2}{3}\geq 0+0+3\times\frac{5}{9}-2\times\frac{2}{3}\geq 0$, contrary to (8).

B. If $\epsilon(v)=3$, then $\epsilon(vv_1)\geq 1$. As Lemma 2.1(iii) does not hold, by (10), we have $d(v_2)\geq 12$. This, together with (9), we observe that $\sum_{e\in E_v} ch_2(e)\geq d(v)-4+\frac{d(v_1)-4}{d(v_1)}+\sum_{i=2}^{k}(\frac{d(v_i)-4}{d(v_i)})-3\times\frac{2}{3}+(\frac{d(v_2)-4}{d(v_2)}-\frac{1}{3})\geq 0+0+3\times\frac{8}{12}-3\times\frac{2}{3}+\frac{1}{3}\geq 0$, contrary to (8).

C. If $\epsilon(v)=4$, then $\epsilon(vv_1)=2$. As Lemma 2.1(iii) does not hold, by (9) and (10), we note that either $d(v_2)\geq 13$ when $4\leq d(v_1)\leq 5$; or $d(v_2)\geq 11$ when $6\leq d(v_1)\leq 7$.

Suppose first that $4\leq d(v_1)\leq 5$ and $d(v_2)\geq 13$. Direct computation yields $\sum_{e\in E_v} ch_2(e)\geq d(v)-4+\sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)})-4\times\frac{2}{3}+2\times(\frac{d(v_2)-4}{d(v_2)}-\frac{1}{3})\geq 0+0+3\times\frac{9}{13}-\frac{8}{3}+2\times(\frac{9}{13}-\frac{1}{3})\geq 0$, contrary to (8).

Therefore, we assume that $6\leq d(v_1)\leq 7$ and $d(v_2)\geq 11$. Direct computation yields $\sum_{e\in E_v} ch_2(e)\geq d(v)-4+\sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)})-4\times\frac{2}{3}+\sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)}-\frac{1}{3})\geq 0+2\times\frac{2}{6}+6\times\frac{7}{11}-4\geq 0$, contrary to (8). This justifies the claim. □

Claim 2.5. $k\neq 5$.

Proof. Suppose $k=5$. We first show that

$$d(v_1)=4 \text{ when } \epsilon(v)\leq 3;\ 4\leq d(v_1)\leq 5 \text{ when } \epsilon(v)=4;\ 4\leq d(v_1)\leq 6 \text{ when } \epsilon(v)=5. \tag{11}$$

By (9), $d(v_1)\geq 4$. If $d(v_1)\geq 5$ when $\epsilon(v)\leq 3$, then $\sum_{e\in E_v} ch_2(e)\geq d(v)-4+\sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)})-\epsilon(v)\times\frac{2}{3}\geq 1+5\times\frac{1}{5}-3\times\frac{2}{3}=0$; if $d(v_1)\geq 6$ when $\epsilon(v)=4$, then $\sum_{e\in E_v} ch_2(e)\geq d(v)-4+\sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)})-\epsilon(v)\times\frac{2}{3}\geq 1+5\times\frac{2}{6}-4\times\frac{2}{3}=0$; if $d(v_1)\geq 7$ when $\epsilon(v)=5$, then $\sum_{e\in E_v} ch_2(e)\geq d(v)-4+\sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)})-\epsilon(v)\times\frac{2}{3}+\epsilon(v)(\frac{d(v_1)-4}{d(v_1)}-\frac{1}{3})\geq 1-\epsilon(v)+(5+\epsilon(v))\times\frac{d(v_1)-4}{d(v_1)}\geq 0$. So if (11) fails, we always have $\sum_{e\in E_v} ch_2(e)\geq 0$, contrary to (8).

Subclaim 2.5.1. Each of the following holds.
(A) If $\epsilon(v)\leq 3$, then $d(v_1)+d(v_2)\leq 9$.
(B) If $\epsilon(v)=4$, then $d(v_1)+d(v_2)\leq 11$.
(C) If $\epsilon(v)=5$, then $d(v_1)+d(v_2)\leq 12$.

By (11), it suffices to check that each of the following cases which will always lead to a contradiction to (8).

A. Suppose that $\epsilon(v)\leq 3$. By (11), $d(v_1)=4$. If Subclaim 2.5.1(A) does not hold, then $d(v_2)\geq 6$. In this case, we have $\sum_{e\in E_v} ch_2(e)\geq d(v)-4+\sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)})-\epsilon(v)\times\frac{2}{3}\geq 1+0+4\times\frac{2}{6}-3\times\frac{2}{3}\geq 0$.

B. Suppose that $\epsilon(v) = 4$. By (11), $4 \le d(v_1) \le 5$. If Subclaim 2.5A(B) does not hold, then $d(v_2) \ge 7$. Thus we have $\sum_{e\in E_v} ch_2(e) \ge d(v) - 4 + \sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)}) - \epsilon(v) \times \frac{2}{3} + 2 \times (\frac{d(v_2)-4}{d(v_2)} - \frac{1}{3}) \ge 1 + 0 + 4 \times \frac{3}{7} - 4 \times \frac{2}{3} + 2 \times (\frac{3}{7} - \frac{1}{3}) \ge 0$.

C. Suppose that $\epsilon(v) = 5$. By (11), $4 \le d(v_1) \le 6$. Assume that Subclaim 2.5.1(C) does not hold. If $4 \le d(v_1) \le 5$, then $d(v_2) \ge 8$, whence we have $\sum_{e\in E_v} ch_2(e) \ge d(v) - 4 + \sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)}) - \epsilon(v) \times \frac{2}{3} + 3 \times (\frac{d(v_2)-4}{d(v_2)} - \frac{1}{3}) \ge 1 + 0 + 4 \times \frac{4}{8} - 5 \times \frac{2}{3} + 3 \times (\frac{4}{8} - \frac{1}{3}) \ge 0$. Hence we have $d(v_1) = 6$, forcing $d(v_2) \ge 7$. In this case, $\sum_{e\in E_v} ch_2(e) \ge d(v) - 4 + \sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)}) - \epsilon(v) \times \frac{2}{3} + \sum_{i=2}^{k}(\frac{d(v_i)-4}{d(v_i)} - \frac{1}{3}) \ge 1 + \frac{2}{6} + 4 \times \frac{3}{7} - 5 \times \frac{2}{3} + 4 \times (\frac{3}{7} - \frac{1}{3}) \ge 0$. This completes the proof of Subclaim 2.5.1.

We are now to show that Lemma 2.1(iv) must hold to complete the proof of Claim 2.5. This will be done after we justify the following subclaim.

Subclaim 2.5.2. Each of the following holds.
(A) If $\epsilon(v) \le 3$, then $d(v_3) \le 5$, and so $d(v_1) + d(v_2) + d(v_3) \le 14$.
(B) If $\epsilon(v) = 4$, then $d(v_3) \le 10$, and so $d(v_1) + d(v_2) + d(v_3) \le 21$.
(C) If $\epsilon(v) = 5$, then either $d(v_1) + d(v_2) \le 10$ and $d(v_3) \le 11$; or $11 \le d(v_1) + d(v_2) \le 12$ and $d(v_3) \le 9$. In either case, we have $d(v_1) + d(v_2) + d(v_3) \le 21$.

We assume that Subclaim 2.5.2 does not hold to show that each case of a violation to Subclaim 2.5.2 will lead to a contradiction to (8).

A. Suppose that $\epsilon(v) \le 3$ and $d(v_3) \ge 6$. By Subclaim 2.5.1, $d(v_1) + d(v_2) \le 9$. Then we have $\sum_{e\in E_v} ch_2(e) \ge d(v) - 4 + \sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)}) - \epsilon(v) \times \frac{2}{3} \ge 1 + 0 + 0 + 3 \times \frac{2}{6} - 3 \times \frac{2}{3} = 0$.

B. Suppose that $\epsilon(v) = 4$ and $d(v_3) \ge 11$. By Subclaim 2.5.1, $d(v_1) + d(v_2) \le 11$. It follows that then $\sum_{e\in E_v} ch_2(e) \ge d(v) - 4 + \sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)}) - \epsilon(v) \times \frac{2}{3} \ge 1 + 0 + 0 + 3 \times \frac{7}{11} - \frac{8}{3} \ge 0$.

C. Suppose that $\epsilon(v) = 5$. By Subclaim 2.5.1, $d(v_1) + d(v_2) \le 12$. Assume first that $d(v_1) + d(v_2) \le 10$ but $d(v_3) \ge 12$. Then we have $\sum_{e\in E_v} ch_2(e) \ge d(v) - 4 + \sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)}) - \epsilon(v) \times \frac{2}{3} + (\frac{d(v_3)-4}{d(v_3)} - \frac{1}{3}) \ge 1 + 0 + 0 + 3 \times \frac{8}{12} - \frac{10}{3} + (\frac{8}{12} - \frac{1}{3}) \ge 0$.

Now assume that $11 \le d(v_1) + d(v_2) \le 12$ but $d(v_3) \ge 10$. By (11), we must have $d(v_2) \ge 6$. Hence $\sum_{e\in E_v} ch_2(e) \ge d(v) - 4 + \sum_{i=1}^{k}(\frac{d(v_i)-4}{d(v_i)}) - \epsilon(v) \times \frac{2}{3} + 2 \times (\frac{d(v_3)-4}{d(v_3)} - \frac{1}{3}) \ge 1 + 0 + \frac{2}{6} + 3 \times \frac{6}{10} - \frac{10}{3} + 2 \times (\frac{6}{10} - \frac{1}{3}) \ge 0$.

Thus Subclaim 2.5.2 is justified, and so Claim 2.5 is proved. □

Claim 2.6. $k \ge 8$.

Proof. By contradiction, we assume that $k \le 7$. By the previous claims, we may assume that $k \in \{6, 7\}$. Since G is a counterexample to Lemma 2.1, Lemma 2.1 (i) does not hold, and so $d(v_i) \ge 3$ for any $1 \le i \le k$. To justify the claim, we shall show that for each edge $vv_i \in E(v)$, we always have $ch_2(vv_i) \ge 0$, which leads to a contradiction to (8). To this aim, we suppose that for some i with $1 \le i \le k$, we have $ch_2(vv_i) < 0$. Note that, by Proposition 2.2 (iv), $ch_2(vv_i) \ge ch_0(vv_i)$ when $ch_2(vv_i) < 0$. Then we fix this index i and make the following claims.

A. $4 \le d_G(v_i) \le 5$.

If $d(v_i) \ge 6$, then $0 > ch_2(vv_i) \ge ch_0(vv_i) \ge 2 \times \frac{2}{6} - 2 \times \frac{1}{3} = 0$, a contradiction. Next, assume that $d(v_i) = 3$. If $\epsilon(v_i) > 0$, then Lemma 2.1(ii) would hold with v_i takes the place of v. Hence $\epsilon(v_i) = 0$, and so $0 > ch_2(vv_i) \ge ch_0(vv_i) \ge \frac{2}{6} - \frac{1}{3} + 0 + 0 \ge 0$, another contradiction again.

B. $d_G(v_i) \ne 4$.

Assume that $d(v_i) = 4$. If $\epsilon(vv_i) \le 1$, then by the recharging rules, $0 > ch_2(vv_i) \ge ch_0(vv_i) \ge \frac{2}{6} + 0 - \frac{1}{3} + 0 \ge 0$, a contradiction. Thus as each edge has two faces, we assume that $\epsilon(vv_i) = 2$.

If v is not of the least difficulty-increasing order in $N(v_i)$, then there exists a vertex $w \in N(v_i)$ such that $d(w) < d(v)$. So $d(v) + d(w) \le 13$ and Lemma 2.1(iii) must hold with v_i taking the place of v. Hence v must be of the least difficulty-increasing order in $N(v_i)$. As Lemma 2.1(iii) does not hold with v_i replacing v, we must have

$$\min\{d(v_i^-), d(v_i^+)\} \ge \begin{cases} 12 & \text{if } k = 6 \\ 11 & \text{if } k = 7 \end{cases},$$

and accordingly,

$$ch_2(vv_i) \ge \begin{cases} \frac{2}{6} + 0 - \frac{2}{3} + \frac{1}{2}(\frac{d(v_i^-)-4}{d(v_i^-)} - \frac{1}{3}) + \frac{1}{2}(\frac{d(v_i^+)-4}{d(v_i^+)} - \frac{1}{3}) \ge 0 & \text{if } k = 6 \\ \frac{3}{7} + 0 - \frac{2}{3} + \frac{1}{2}(\frac{d(v_i^-)-4}{d(v_i^-)} - \frac{1}{3}) + \frac{1}{2}(\frac{d(v_i^+)-4}{d(v_i^+)} - \frac{1}{3}) \ge 0 & \text{if } k = 7 \end{cases},$$

contrary to the choice of vv_i.

C. $d_G(v_i) \neq 5$.

Assume that $d(v_i) = 5$. If $\epsilon(vv_i) \leq 1$, then by the recharging rules, $0 > ch_2(vv_i) \geq ch_0(vv_i) \geq \frac{2}{6} + \frac{1}{5} - \frac{1}{3} + 0 \geq 0$, a contradiction. Thus we assume that $\epsilon(vv_i) = 2$.

If there exists a vertex $v_i' \in N(v_i)$ with $d(v_i') \geq 6$ and $\epsilon(v_iv_i') = 0$, then by the recharging rules Step 2, vv_i will receive a charge of $\frac{1}{6}$ from v_iv_i', and so $ch_2(vv_i) \geq \frac{k-4}{k} + \frac{1}{5} - \frac{2}{3} + \frac{1}{6} \geq 0$, contrary to the choice of vv_i.

Therefore, we assume that for every $v_i' \in N(v_i) \setminus \{v_i^-, v, v_i^+\}$ with $\epsilon(v_iv_i') = 0$, we must have $d(v_i') \leq 5$. If $\epsilon(v_i) = 2$, suppose $\{v_i', v_i''\} \in N(v_i) \setminus \{v_i^-, v, v_i^+\}$, then $d(v_i') \leq 5$ and $d(v_i'') \leq 5$. As $k \in \{6, 7\}$, we have $d(v) + d(v_i') + d(v_i'') \leq 17$, and so Lemma 2.1(iv) must hold with v_i replacing v, contrary to (2). We now assume that $\epsilon(v_i) \geq 3$. By (2), Lemma 2.1(iv) does not hold with v_i in place of v. This implies that either $\min\{d(v_i^-), d(v_i^+)\} \geq 14 - k$, whence by the recharging rules,

$$ch_2(vv_i) \geq \frac{k-4}{k} + \frac{1}{5} - \frac{2}{3} + (\frac{10-k}{14-k} - \frac{1}{3}) \geq 0;$$

or both $\min\{d(v_i^-), d(v_i^+)\} \leq 13 - k$ and $\max\{d(v_i^-), d(v_i^+)\} \geq 22 - \min\{d(v_i^-), d(v_i^+)\} - k$, whence

$$ch_2(vv_i) \geq \frac{k-4}{k} + \frac{1}{5} - \frac{2}{3} + \max\left\{\frac{1}{2}(\frac{\min\{d(v_i^-), d(v_i^+)\} - 4}{\min\{d(v_i^-), d(v_i^+)\}} - \frac{1}{3}), 0\right\} + \frac{1}{2}\left(\frac{\max\{d(v_i^-), d(v_i^+)\} - 4}{\max\{d(v_i^-), d(v_i^+)\}} - \frac{1}{3}\right) \geq 0.$$

Thus a contradiction to the choice of vv_i is obtained. This proves the claim. □

Claim 2.7. $k \geq 12$.

Suppose that $k \leq 12$. By Claim 2.6, we have $8 \leq k \leq 11$. By (2), Lemma 2.1(i) does not hold. Hence we have $d(v_j) \geq 3$ for any $1 \leq j \leq k$. Similar to the argument in the proof of Claim 2.6, we shall show that $ch_2(vv_i) \geq 0$ holds for any $1 \leq i \leq k$ to obtain a contradiction to (8). Observe that for any $1 \leq j \leq k$, if $\epsilon(vv_j) = 0$, then by Proposition 2.2, we have $ch_2(vv_j) \geq \min\{ch_0(vv_j), 0\} \geq 0$ where $ch_0(vv_j) \geq \frac{4}{8} - \frac{1}{3} + 0 + 0 \geq 0$. Thus we assume that for some fixed i with $1 \leq i \leq k$, we have $\epsilon(vv_i) \geq 1$ and $ch_2(vv_i) < 0$. We make the following claims.

A. $d(v_i) \leq 4$.

Assume that $d(v_i) \geq 5$. Then by Proposition 2.2(iv), $0 > ch_2(vv_i) \geq ch_0(vv_i) \geq \frac{4}{8} + \frac{1}{5} - \frac{2}{3} \geq 0$, a contradiction.

B. $d(v_i) \neq 3$.

Assume that $d(v_i) = 3$. By (2), Lemma 2.1(ii) does not hold with v_i in place of v. Hence

$$\epsilon(v_i) \begin{cases} = 0 & \text{if } k = 8 \\ \leq 1 & k \geq 9 \end{cases}.$$

By the choice of vv_i, we have $\epsilon(v_i) \geq \epsilon(vv_i) \geq 1$. Thus we assume that $k \geq 9$ and $\epsilon(v_i) = \epsilon(vv_i) = 1$. Let v_i' be a vertex in the 3-face containing v_i. If $d(v_i') \leq 8$, then Lemma 2.1(ii) holds with v_i replacing v, contrary to (2). Hence we must have $d(v_i') \geq 9$, and so by the recharge rules,

$$ch_2(vv_i) \geq \frac{5}{9} - \frac{2}{3} + 0 + \frac{1}{2}(\frac{d(v_i') - 4}{d(v_i')} - \frac{1}{3}) \geq 0.$$

This violation to the choice of vv_i implies that $d(v_i) = 4$.

If $\epsilon(vv_i) \leq 1$, then by the recharge rules and Proposition 2.2, $ch_2(vv_i) \geq ch_0(vv_i) \geq \frac{4}{8} + 0 - \frac{1}{3} + 0 \geq 0$, contrary to the choice of vv_i, and so we must have $\epsilon(vv_i) = 2$. Note that $\epsilon(v_i) \geq 2$ and $\epsilon(v) \geq 2$.

Let v_i' denote the vertex with the least difficulty-increasing order in $N(v_i)$. If $d(v_i') \geq 8$, then $\min\{d(v_i^-), k, d(v_i^+)\} \geq d(v_i') \geq 8$; if $d(v_i') = 7$ and $\epsilon(v_iv_i') = 0$, then $\min\{d(v_i^-), k, d(v_i^+)\} \geq 8$, otherwise one vertex $w \in \{v_i^-, v_i^+\}$ with $d(w) = d(v_i')$ must be of the least difficulty-increasing order in $N(v_i)$ instead of v_i', for $\epsilon(v_iw) > \epsilon(v_iv_i') = 0$; if $d(v_i') = 7$ and $\epsilon(v_iv_i') \geq 1$, or $d(v_i') \leq 6$, then $\max\{d(v_i^-), d(v_i^+)\} \geq 11$ and $k \geq 11$, otherwise Lemma 2.1(iii) could hold for v_i. In all cases above, we can check that $ch_2(vv_i) \geq 0$, which contradicts to the choice of vv_i. □

We are to complete the proof of Lemma 2.1. By the claims above, we may assume that $k \geq 12$. For any i with $1 \leq i \leq k$, by (2), Lemma 2.1(i) would not hold with v_i replacing v, and so we have $d(v_i) \geq 3$. We again will show that for any j with $1 \leq j \leq k$, $ch_2(vv_i) \geq 0$ to obtain a contradiction to (8). If this does not hold, then there will be an i with $ch_2(vv_i) < 0$. We fix this i in the argument below.

If $d(v_i) \geq 4$, then by the recharge rules and Proposition 2.2, we have $0 > ch_2(vv_i) \geq ch_0(vv_i) \geq \frac{8}{12} + 0 - \frac{2}{3} = 0$, a contradiction. Hence we must have $d(v_i) = 3$.

If $\epsilon(vv_i) \leq 1$, then the recharge rules and Proposition 2.2, we have $ch_2(vv_i) \geq ch_0(vv_i) \geq \frac{8}{12} - \frac{1}{3} - \frac{1}{3} = 0$, contrary to the choice of vv_i. Now assume that $\epsilon(vv_i) = 2$, then by (2), Lemma 2.1(ii) does not hold with v_i taking the place of v, implying that $\min\{d(v_i^-), d(v_i^+)\} \geq 12$. This, together with the recharge rules, leads to $ch_2(vv_i) \geq \frac{8}{12} - \frac{1}{3} - \frac{2}{3} + (\frac{8}{12} - \frac{1}{3}) = 0$, contrary to the choice of vv_i. This completes the proof of the lemma. □

3. Proof of Theorem 1.2

Let H be a subgraph of a graph G. A (k, r)-coloring c' of H is called a **partial** (k, r)-**coloring** of G with $V(H)$ being the **domain of** c'. A (k, r)-coloring c of G is an **extension** of c' if for any $v \in V(H)$, $c(v) = c'(v)$.

If H is a graph and if X is a set of edges joining vertices in $V(H)$, then in this section, we use $H + X$ to denote the graph with vertex set $V(H)$ and edge set $E(H) \cup X$. As an example, suppose that $u, v \in V(H)$. If $uv \notin E(H)$, then $H + uv$ is the graph obtained from H by adding a new edge uv to H; if $uv \in E(H)$, then $H + uv = H$.

Throughout this section, let $k = 2r + 16$. We argue by contradiction to prove Theorem 1.2. Assume that

$$G \text{ is a counterexample to Theorem 1.2 with } |V(G)| + |E(G)| \text{ minimized.} \tag{12}$$

For each different case in the arguments below, we will obtain a new planar graph G' (called a modified graph of G) by making local modifications of G such that $|V(G')|+|E(G')| < |V(G)|+|E(G)|$. By (12), G' has an (k, r)-coloring c' using colors in $\bar{k}$. Using the relationship between G' and G, we will show that this (k, r)-coloring c' of G' gives rise to a partial (k, r)-coloring c of G. We then shall extend c to a (k, r)-coloring of G to obtain a contradiction to prove the theorem.

For every vertex $v \in V(G)$, define $c[v]$ as follows.

$$c[v] = \begin{cases} \{c(v)\}, & \text{if } |c(N_G(v))| \geq r; \\ \{c(v)\} \cup c(N_G(v)), & \text{otherwise.} \end{cases} \tag{13}$$

Thus $c[v]$ consists of the set of forbidden colors for uncolored neighbors of v after $c(v)$ is chosen. By (13), $|c[v]| \leq r$ for any v; and if v has an uncolored neighbor, then $|c[v]| \leq d(v)$.

By Lemma 2.1, there exists a k-vertex v with its neighbors $v_1 \preceq v_2 \preceq, \ldots, \preceq v_k$ satisfying the property described in Lemma 2.1. We shall justify the theorem by examining each possible values of k.

Case 1. $k \leq 2$.

Suppose first that $k = 1$ and $N_G(v) = \{x\}$. Let $G' = G - v$. By (12), G' has a (k, r)-coloring c which is also a partial (k, r)-coloring of G. As $|c[x]| \leq r < k$, we extend c by including v in the domain of c with $c(v) \in \bar{k} \backslash c[x]$. As c is a partial (k, r)-coloring of G and by the choice of $c(v)$, the extended c is a (k, r)-coloring of G, contrary to (12).

Hence we assume that $k = 2$ and denote $N_G(v) = \{x, y\}$. Define $G' = G - v + xy$. By (12), G' has a (k, r)-coloring c, which can also be viewed as a partial coloring of G with domain $V(G - v)$. Since $|c[x] \cup c[y]| \leq 2r < k$, we extend the domain of c to $V(G)$ by choosing $c(v) \in \bar{k} \backslash (c[x] \cup c[y])$. It is routine to verify that c is a (k, r)-coloring of G, contrary to (12). This proves Case 1.

Case 2. $k = 3$.

By Lemma 2.1(ii), we have

$$d(v_1) \leq \begin{cases} 5 & \text{if } \epsilon(v) = 0 \\ 8 & \text{if } \epsilon(v) = 1 \\ 11 & \text{if } \epsilon(v) \geq 2. \end{cases}$$

We define the modified graph G' according to different conditions on $\epsilon(v)$, as below.
(A) If $\epsilon(v) = 0$, then $d(v_1) \leq 5$. In this case we define $G' = G - v + v_1v_2 + v_1v_3$.
(B) Assume that $\epsilon(v) = 1$. If $\epsilon(vv_1) = 1$, then we may assume that $v_1v_2 \in E(G)$ (similarly the case of $v_1v_3 \in E(G)$). In this case we define $G' = G - v + v_1v_3$. If $\epsilon(vv_1) = 0$, then vv_2v_3 is a triangle in G and $d(v_3) \geq d(v_2) > d(v_1)$. In this case we define

$$G' = \begin{cases} G - v + v_1v_2 + v_1v_3 & \text{if } d(v_1) \leq r - 1 \\ G - v_2v_3 & \text{if } d(v_1) \geq r. \end{cases}$$

(C) Assume that $\epsilon(v) = 2$. If $\{v_1, v_2, v_3\}$ contains two nonadjacent vertices in G, then obtain G' from $G - v$ by adding an edge joining these two nonadjacent vertices; if $G[\{v_1, v_2, v_3\}]$ is a 3-cycle, then define $G' = G - v$.
(D) If $\epsilon(v) = 3$, let $G' = G - v$.

Once G' is defined, we argue by (12) to obtain a (k, r)-coloring c of G'. By the way we define G', in any cases, $c(v_1), c(v_2), c(v_3)$ are mutually distinct. Observe that in (B) when $G' = G - v_2v_3$, c is also a (k, r)-coloring of G. In other cases, as c is a partial (k, r)-coloring of G, and as we have $|c[v_1] \cup c[v_2] \cup c[v_3]| \leq 2r + 11 < k$. c can be extended to a (k, r)-coloring of G by defining $c(v) \in \bar{k} \backslash (c[v_1] \cup c[v_2] \cup c[v_3])$, contrary to (12).

Case 3. $k = 4$.

Thus Lemma 2.1(iii) holds. Define $G' = G - v + v_1v_2 + v_1v_3 + v_1v_4$. As $r \geq 8$, we have $d_{G'}(v_1) \leq 8 \leq r$. By (12), G' has a (k, r)-coloring c, which is also a partial coloring of G. Since $d_{G'}(v_1) \leq r$ and since $v_2, v_3, v_4 \in N_{G'}(v_1)$, we have $|c(\{v_1, v_2, v_3, v_4\})| = 4$. Since $|c[v_1] \cup c[v_2] \cup c[v_3] \cup c[v_4]| \leq 2r + d(v_1) + d(v_2) - 2\epsilon(v) \leq 2r + 13 < k$, we can define $c(v) \in \bar{k} \backslash (c[v_1] \cup c[v_2] \cup c[v_3] \cup c[v_4])$, and so the extended c is a (k, r)-coloring of G, contrary to (12).

Case 4. $k = 5$.

Then Lemma 2.1(iv) holds. Therefore, as $r \geq 8$. By Lemma 2.1(iv), we have either $d(v_1) \leq 6$ or both $d(v_1) \leq 5$ and $\epsilon(vv_1) = 0$. Define $G' = G - v + v_1v_2 + v_1v_3 + v_1v_4 + v_1v_5$, we can check that $d_{G'}(v_1) \leq 8 \leq r$. By (12), G' has a (k, r)-coloring c.

Since $d_{G'}(v_1) \leq r$, we have $|c(\{v_1, \ldots, v_5\})| = 5$ in G'. Since $|\bigcup_{w \in N_G(v)} c[w]| \leq 2r + d(v_1) + d(v_2) + d(v_3) - 2\epsilon(v) \leq 2r + 15 < k$, we extend c by defining $c(v) \in \bar{k} \setminus (\bigcup_{w \in N_G(v)} c[w])$, to result in a (k, r)-coloring of G, contrary to (12).

As each case leads to a contradiction, the theorem is proved. □

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