Discrete Applied Mathematics 247 (2018) 14–22

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Modulo orientations with bounded independence number

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ARTICLE INFO

Article history:
Received 3 March 2017
Received in revised form 6 January 2018
Accepted 20 January 2018
Available online 19 April 2018

Keywords:
Integer flows
Modulo $(2p+1)$-orientation
Group connectivity
Strongly $\mathbb{Z}_{2p+1}$-connectedness
Independence number

ABSTRACT

A mod $(2p+1)$-orientation D is an orientation of G such that $d_D^+(v) \equiv d_D^-(v) \pmod{2p+1}$ for any vertex $v \in V(G)$. Extending Tutte's integer flow conjectures, it was conjectured by Jaeger that every $4p$-edge-connected graph has a mod $(2p+1)$-orientation. However, this conjecture has been disproved in Han et al. (2018) recently. Infinite families of $4p$-edge-connected graphs (for $p \geq 3$) and $(4p+1)$-edge-connected graphs (for $p \geq 5$) with no mod $(2p+1)$-orientation are constructed in Han et al. (2018). In this paper, we show that every family of graphs with bounded independence number has only finitely many contraction obstacles for admitting mod $(2p+1)$-orientations, contrasting to those infinite families. More precisely, we prove that for any integer $t \geq 2$, there exists a finite family $\mathcal{F} = \mathcal{F}(p, t)$ of graphs that do not have a mod $(2p+1)$-orientation, such that every graph G with independence number at most t either admits a mod $(2p+1)$-orientation or is contractible to a member in $\mathcal{F}$. This indicates that the problem of determining whether every k-edge-connected graph with independence number at most t admits a mod $(2p+1)$-orientation is computationally solvable for fixed k and t. In particular, the graph family $\mathcal{F}(p, 2)$ is determined, and our results imply that every 8-edge-connected graph G with independence number at most two admits a mod 5-orientation.

1. Introduction

In this paper, we consider graphs which are finite and loopless, with possible parallel edges. We follow [1] for undefined terms and notation. Let $\mathbb{Z}$ denote the set of integers. For $k \in \mathbb{Z}$ with $k > 1$, let $[k] = \{1, 2, \ldots, k\}$ and $\mathbb{Z}_k$ denote the set of all integers modulo k, as well as the (additive) cyclic group of order k. Following [1], for a graph G, $\alpha(G)$, $\kappa'(G)$, and $\delta(G)$ denote the independence number, the edge-connectivity, and the minimum degree, respectively. For each edge $e \in E(G)$, let $\mu(e)$ be the maximum number of edges joining the two end vertices of e, and denoted $\mu(G) = \max\{\mu(e) : e \in E(G)\}$ to be the *edge multiplicity* of G. For vertex subsets $U, W \subseteq V(G)$, let $[U, W]_G = \{uw \in E(G) | u \in U, w \in W\}$. When $U = \{u\}$ or $W = \{w\}$, we use $[u, W]_G$ or $[U, w]_G$ for $[U, W]_G$, respectively. For notational convenience, we also denote $E_G(v) = [v, V(G) - \{v\}]$ and $\partial_G(S) = [S, V(G) - S]$ for $v \in V(G)$ and $S \subseteq V(G)$. The subscript G may be omitted when G is understood from the context. For an edge set $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X, and then deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$.

Let $D = D(G)$ denote an orientation of G. For each $v \in V(G)$, let $E_D^+(v)$ ($E_D^-(v)$, respectively) be the set of all arcs directed out from (into, respectively) v. Following [1], $d_D^+(v) = |E_D^+(v)|$ and $d_D^-(v) = |E_D^-(v)|$ denote the out-degree and the in-degree of v under the orientation D, respectively. If a graph G has an orientation D such that $d_D^+(v) - d_D^-(v) \equiv 0 \pmod{k}$ for every

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https://doi.org/10.1016/j.dam.2018.01.016
0166-218X/

vertex $v \in V(G)$, then we say that G admits a **modulo k-orientation**, or a mod k-orientation for short. Let $\mathcal{M}_k$ denote the family of all graphs admitting a mod k-orientation. As a connected graph G has a modulo $2p$-orientation if and only if G is Eulerian, we focus on the case when $k = 2p+1$ is odd in this paper. We shall always assume that p is a positive integer throughout this paper.

The concept of modulo orientation is motivated by the integer flow of graphs introduced by Tutte [17,18]. An *integer flow* of a graph G is an ordered pair (D, f), where D is an orientation and f is a mapping from $E(G)$ to integers such that $\sum_{e\in E_D^+(v)} f(e) - \sum_{e\in E_D^-(v)} f(e) = 0$ for every vertex $v \in V(G)$. An integer flow (D, f) is called a *nowhere-zero k-flow* if $1 \le |f(e)| \le k-1$ for each edge $e \in E(G)$. Jaeger [5] observed that, in a graph G, the existence of a mod $(2p+1)$-orientation is equivalent to the existence of an integer flow (D, f) with $|f(e)| \in \{p, p+1\}$ for each $e \in E(G)$, which is called a *circular $(2+\frac{1}{p})$-flow*. In particular, it is well-known that a graph admits a nowhere-zero 3-flow if and only if it admits a mod 3-orientation (see [17,20,5]). Tutte's 3-flow conjecture (see [1]) can be stated as follows.

Conjecture 1.1 (*Tutte*)**.** *Every 4-edge-connected graph admits a mod 3-orientation.*

In addition, as observed by Jaeger [5], Tutte's famous 5-flow conjecture [18], which asserts that *every bridgeless graph admits a nowhere-zero 5-flow*, is implied by the following conjecture.

Conjecture 1.2 (*Jaeger, [5]*)**.** *Every 9-edge-connected graph admits a mod 5-orientation.*

It was originally conjectured by Jaeger [4] that *every $4p$-edge-connected graph admits a mod $(2p+1)$-orientation*, known as the circular flow conjecture. Thomassen [16] settled the weak version of 3-flow conjecture and the weak version of Jaeger's circular flow conjecture by showing *every 8-edge-connected graph admits a mod 3-orientation and every $(2k^2+k)$-edge-connected graph admits a mod k-orientation*. Lovász, Thomassen, Wu and Zhang [14,19] further refined the method to show that *every $6p$-edge-connected graph admits a mod $(2p+1)$-orientation*. Very recently, infinite families of $4p$-edge-connected graphs with no mod $(2p+1)$-orientation were constructed in [3] for every $p \ge 3$. There exist $(4p+1)$-edge-connected graphs admitting no mod $(2p+1)$-orientation for every $p \ge 5$, as well. A new conjecture on modulo orientation is proposed in [3], that *for every positive integer p, there exists a positive constant $\varepsilon = \varepsilon(p) < \frac{1}{2}$ such that every $\lceil (4+\varepsilon)p \rceil$-edge-connected graph admits a mod $(2p+1)$-orientation*. This suggests that while the connectivity requirement may increase for larger p, the truth of the new conjecture still implies Tutte's 3-flow conjecture and 5-flow conjecture by results of Kochol [7] and Jaeger [5]. The readers are referred to [21] or [10] for a comprehensive introduction on integer flows and modulo orientations.

In this paper, we investigate mod $(2p+1)$-orientations of graphs with bounded independence numbers. It is known that the complete graph K_{4p} does not admit a mod $(2p+1)$-orientation. Since the modulo orientation property is preserved under contraction, it is straightforward to construct an infinite family of graphs of independence number two without mod $(2p+1)$-orientation by replacing a vertex of K_{4p} with a large complete graph. On the other hand, all those graphs have the behavior that each of them is contractible to K_{4p}. So we may expect to characterize mod $(2p+1)$-orientation in the family of graphs with bounded independence number by excluding a list of graphs such that every graph in the family admits a mod $(2p+1)$-orientation if and only if it is not contractible to one of the graphs on the list, such as in Kuratowski's theorem for planar graphs and characterization of graphs embedded on surface by excluding minors. Our first main result asserts that it is indeed the case and such a list contains finitely many graphs only.

Let $t \ge 1$ be an integer, and define a finite graph family $\mathcal{G}_0(t)$ to be

$$\mathcal{G}_0(t) = \{G : G \notin \mathcal{M}_{2p+1}, \alpha(G) \le t, \mu(G) \le 2p-1 \text{ and } |V(G)| \le 6pt - 2p\}.$$

Theorem 1.3. *For any graph G with $\alpha(G) \le t$, G admits a mod $(2p+1)$-orientation if and only if G is not contractible to a member in $\mathcal{G}_0(t)$.*

As a corollary of Theorem 1.3, for a given integer $k > 0$, in order to seek mod $(2p+1)$-orientations for all k-edge-connected graphs with independence number at most t, it suffices to search such graphs on at most $6pt - 2p$ vertices, which consist of only finitely many graphs and is computationally solvable.

Corollary 1.4. *The following are equivalent.*
(i) *Every k-edge-connected graph G with $\alpha(G) \le t$ admits a mod $(2p+1)$-orientation.*
(ii) *Every k-edge-connected graph G with $\alpha(G) \le t$, $\mu(G) \le 2p-1$ and $|V(G)| \le 6pt - 2p$ admits a mod $(2p+1)$-orientation.*

To obtain Theorem 1.3, we need to introduce orientation with boundaries. For a graph G, a function $b : V(G) \to \mathbb{Z}_{2p+1}$ is called a *boundary function* of G, or *boundary* for short, if $\sum_{v\in V(G)} b(v) \equiv 0 \pmod{2p+1}$. Denote $Z(G, \mathbb{Z}_{2p+1})$ to be the set of all boundary functions of G. Motivated by the group connectivity property defined by Jaeger et al. [6], the concept of strongly $\mathbb{Z}_{2p+1}$-connectedness was introduced in [9] (see also [8]), serving as contractible configurations for mod $(2p+1)$-orientations.

Definition 1.5. A graph G is **strongly $\mathbb{Z}_{2p+1}$-connected** if, for every $b \in Z(G, \mathbb{Z}_{2p+1})$, there is an orientation D such that $d_D^+(v) - d_D^-(v) \equiv b(v) \pmod{2p+1}$ for every vertex $v \in V(G)$.

Let $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$ denote the family of all strongly $\mathbb{Z}_{2p+1}$-connected graphs.

Liang et al. [13] proved that the graph family $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$ consists of exactly all *mod* $(2p+1)$*-orientation contractible configurations*, that is, all those graphs G such that for every supergraph Γ containing G as a subgraph, Γ/G has a mod $(2p+1)$-orientation if and only if Γ has a mod $(2p+1)$-orientation.

A subgraph H of G is called a **maximal** $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$**-subgraph** of G if $H \in \langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$ and for any subgraph L of G containing H as a proper subgraph, $L \notin \langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$. Since $K_1 \in \langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$ by definition, every vertex of a graph G lies in a maximal $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-subgraph of G. Let $H_1, H_2, \dots, H_c$ denote the collection of all maximal $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-subgraph of G. Then $G' = G/(\cup_{i=1}^{c} E(H_i))$ is the $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-**reduction** of G, and we also say G is $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced to G'. A graph G is $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-**reduced** if G does not have any nontrivial subgraph in $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$. By definition, the $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction of a graph is always $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced. Since contraction may bring in new parallel edges, even when G is a simple graph, its $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction may have multiple edges. As the counterexamples constructed in [3] are indeed $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced graphs, the following is also obtained in [3]: *there exists infinitely many* $(4p+1)$*-edge-connected* $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$*-reduced graphs for every* $p \geq 5$. While in this paper, we show that there are *finitely many* $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced graph in the family of graphs with bounded independence number (see Corollary 2.4).

Theorem 1.3 is an immediate corollary of the following Theorem 1.6. In Theorem 1.6, the $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction operation, a special contraction which preserves mod $(2p+1)$-orientations, would be used to replace a general contraction operation.

For any integer $t > 0$, define $\mathcal{F}(t)$ and $\mathcal{G}(t)$ to be graph families such that

$$\mathcal{F}(t) = \{G : G \text{ is } \langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle\text{-reduced with } 2 \leq |V(G)| \leq 6pt - 2p \text{ and } \alpha(G) \leq t\} \text{ and}$$
$$\mathcal{G}(t) = \mathcal{F}(t) \setminus \mathcal{M}_{2p+1}.$$

Theorem 1.6. *Let $t > 0$ be an integer. Each of the following holds.*
(i) *A graph G with $\alpha(G) \leq t$ is strongly $\mathbb{Z}_{2p+1}$-connected if and only if the $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction of G is not in $\mathcal{F}(t)$.*
(ii) *A graph G with $\alpha(G) \leq t$ admits a modulo $(2p+1)$-orientation if and only if the $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction of G is not in $\mathcal{G}(t)$.*

More descriptions concerning the graph families $\mathcal{F}(t)$ and $\mathcal{G}(t)$ will be presented below when $t = 2$. In particular, Theorem 1.7 confirms that simple graphs with independence number 2 and large order admit mod $(2p+1)$-orientations under edge-connectivity $4p$.

Let K_n denote a complete graph with $V(K_n) = \{v_1, \dots, v_n\}$. For nonnegative integers $s_1, s_2, \dots, s_{n-1}$, let $K_n(s_1, s_2, \dots, s_{n-1})$ be the graph obtained from K_n by replacing the edge $v_n v_i$ by s_i parallel edges joining v_n and v_i, for each $i \in [n-1]$, and define

$$\mathcal{K}(2p+1) = \{K_n(s_1, s_2, \dots, s_{n-1}) : \ 2 \leq n \leq 4p+1 \text{ and } 0 \leq s_i \leq 2p-1, \forall i \in [n-1]\},$$
$$\mathcal{K}_1(2p+1) = \mathcal{K}(2p+1) \setminus \mathcal{M}_{2p+1} \text{ and } \mathcal{K}_2(2p+1) = \mathcal{K}(2p+1) \setminus \langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle. \tag{1}$$

Theorem 1.7. *Let G be a simple graph of order at least $10p+1$ with $\alpha(G) \leq 2$. Each of the following holds.*
(i) *G admits a mod $(2p+1)$-orientation if and only if the $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction of G is not in $\mathcal{K}_1(2p+1)$.*
(ii) *G is strongly $\mathbb{Z}_{2p+1}$-connected if and only if the $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction of G is not in $\mathcal{K}_2(2p+1)$.*
(iii) *If $\kappa'(G) \geq 2p$ and $\delta(G) \geq 4p$, then G is strongly $\mathbb{Z}_{2p+1}$-connected (and therefore, admits a mod $(2p+1)$-orientation).*

As mod 5-orientation of graphs with multiple edges is related to 5-flow conjecture (see [5,10]), we also show the corresponding Theorem 1.8 for all graphs with independence number two in the mod 5-orientation case. Note that this verifies Conjecture 1.2 for all graphs with order at least 21 and independence number at most two.

Let $\mathcal{K}^*(5)$ be the family of graphs such that $H \in \mathcal{K}^*(5)$ if and only if $H \notin \mathcal{M}_5$, H is $\langle\mathcal{S}\mathbb{Z}_5\rangle$-reduced, and H contains a subgraph isomorphic to $K_{|V(H)|-1}$ with $2 \leq |V(H)| \leq 9$ and $\kappa'(H) \leq 7$.

Theorem 1.8. *Let G be a graph of order at least 21 with $\alpha(G) \leq 2$. Each of the following holds.*
(i) *G admits a mod 5-orientation if and only if the $\langle\mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction of G is not in $\mathcal{K}^*(5)$.*
(ii) *G admits a mod 5-orientation provided it is 8-edge-connected.*

Luo et al. [15] characterized mod 3-orientations of graphs with independence number at most 2, and thus verifies Tutte's 3-flow conjecture for graphs with independence number at most 2. In a consequence paper [11], Li, Luo and Wang adopt a similar idea as in this paper and develop some new reduction method to obtain analogous results for mod 3-orientations. The results in paper [11] further confirm Tutte's 3-flow conjecture for graphs with independence number at most 4.

The remainder of this paper is organized as follows: In Section 2, we introduce some tools and give the proofs of Theorems 1.6 and 1.7. The proof of Theorem 1.8 is presented in Section 3, and we conclude this paper with a few remarks in the last section.

2. Reductions on mod $(2p+1)$-orientations

2.1. Some tools

We first display the needed tools in our proofs of the main results. Lemma 2.1 is a brief summary of certain basic properties from [8,9,12].

Lemma 2.1 ([8,9] and [12]). *Let G be a graph and let $m, p > 0$ be integers. Each of the following holds.*
(i) *If $G \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$ and $e \in E(G)$, then $G/e \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$.*
(ii) *If $H \subseteq G$, and if both $H \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$ and $G/H \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$, then $G \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$.*
(iii) *Let mK_2 denote the loopless graph with two vertices and m parallel edges. Then $mK_2 \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$ if and only if $m \geq 2p$.*
(iv) *The complete graph $K_n \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$ if and only if $n = 1$ or $n \geq 4p + 1$.*
(v) *$G \in \mathcal{M}_{2p+1}$ if and only if its $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction $G' \in \mathcal{M}_{2p+1}$.*
(vi) *$G \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$ if and only if its $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction $G' = K_1$.*

Let G be a graph and $b \in Z(G, \mathbb{Z}_{2p+1})$ be a boundary function. Define an integer valued mapping $\tau : 2^{V(G)} \mapsto \{0, \pm 1, \dots, \pm(2p + 1)\}$ as follows: for each vertex $x \in V(G)$,

$$\tau(x) \equiv \begin{cases} d(x) \pmod{2}; \\ b(x) \pmod{2p+1}. \end{cases} \tag{2}$$

For a vertex set $A \subset V(G)$, let $b(A) \equiv \sum_{v\in A} b(v) \pmod{2p+1}$, $d(A) = |[A, V(G) - A]_G|$ and define $\tau(A)$ to be

$$\tau(A) \equiv \begin{cases} d(A) \pmod{2}; \\ b(A) \pmod{2p+1}. \end{cases} \tag{3}$$

Theorem 2.2 (*Lovász, Thomassen, Wu and Zhang, Theorem 3.1 of* [14]). *Let G be a graph and $b \in Z(G, \mathbb{Z}_{2p+1})$. Let z_0 be a vertex of $V(G)$, and let D_{z_0} be a pre-orientation of $E(z_0)$. Assume that*
(i) *$|V(G)| \geq 3$,*
(ii) *$d(z_0) \leq 4p + |\tau(z_0)|$, and the edges incident with z_0 are pre-directed such that $d^+(z_0) - d^-(z_0) \equiv b(z_0) \pmod{2p+1}$.*
(iii) *$d(A) \geq 4p + |\tau(A)|$ for each nonempty $A \subseteq V(G) \setminus \{z_0\}$ with $|V(G) \setminus A| \geq 2$.*
Then D_{z_0} can be extended to an orientation D of the entire graph G such that, for each vertex $x \in V(G)$,

$$d^+_D(x) - d^-_D(x) \equiv b(x) \pmod{2p+1}.$$

Theorem 2.2 implies that every $6p$-edge-connected graph is strongly $\mathbb{Z}_{2p+1}$-connected. We would further explore more properties concerning $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced graphs below by utilizing Theorem 2.2.

2.2. Proof of Theorem 1.6

Recall that $G \in \mathcal{F}(t)$ if and only if G is $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced with $2 \leq |V(G)| \leq 6pt - 2p$ and $\alpha(G) \leq t$. By Lemma 2.1(iii), every graph in $\mathcal{F}(t)$ has edge multiplicity at most $2p - 1$, and so $\mathcal{F}(t)$ contains finitely many graphs. Note that, by Lemma 2.1(v), Theorem 1.3 is a weak version of Theorem 1.6(ii), and Theorem 1.6(ii) follows from Theorem 1.6(i). We will show a variation of Theorem 1.6(i), as stated in Theorem 2.3.

Theorem 2.3. *For any graph G with $\alpha(G) \leq t$, G is strongly $\mathbb{Z}_{2p+1}$-connected if and only if the $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction of G is not in $\mathcal{F}(t)$.*

Proof. By Lemma 2.1(vi), a graph G is strongly $\mathbb{Z}_{2p+1}$-connected if and only if its $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction is K_1, which is not in $\mathcal{F}(t)$ by definition. So it remains to show that

$$\text{if the } \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle\text{-reduction of } G \text{ is not in } \mathcal{F}(t), \text{ then } G \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle. \tag{4}$$

We shall prove (4) by induction on t. When $t = 1$, (4) follows from Lemma 2.1(iv). Assume that $t \geq 2$ and (4) holds for smaller values of t.

Let Γ be a counterexample to (4) such that $|V(\Gamma)|$ is minimal. Then Γ', the $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction of Γ, satisfies $|V(\Gamma)| \geq 6pt - 2p + 1$ by the definition of $\mathcal{F}(t)$. Hence Γ' itself is a counterexample to (4), and so $|V(\Gamma')| = |V(\Gamma)|$ by the minimality of $|V(\Gamma)|$. Therefore, $\Gamma = \Gamma'$ is a $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced graph.

Claim A. $\delta(\Gamma) \geq 6p$.

Suppose that Γ has minimal degree at most $6p - 1$ and let $z \in V(\Gamma)$ be a vertex with $d_\Gamma(z) = \delta(\Gamma) \leq 6p - 1$. Denote $H = \Gamma - (N_\Gamma(z) \cup \{z\})$. Then $\alpha(H) \leq \alpha(\Gamma) - 1 \leq t - 1$. As H is $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced, we have $|V(H)| \leq 6p(t-1) - 2p$ by (4) with induction hypothesis on $t-1$. It follows that $6pt - 2p + 1 \leq |V(\Gamma)| = |V(H)| + |N_\Gamma(z) \cup \{z\}| \leq 6p(t-1) - 2p + 6p = 6pt - 2p$. This contradiction justifies Claim A.

Now assume $\delta(\Gamma) \geq 6p$. By Theorem 2.2, $\kappa'(\Gamma) < 6p$, and so Γ must have an edge cut of size less than $6p$. For a vertex subset $W \subset V(\Gamma)$, let $W^c = V(\Gamma) - W$. Among all edge-cuts $[W, W^c]$ of size at most $6p - 1$ in Γ, choose one with $|W|$ minimized. As $\delta(\Gamma) \geq 6p$, we have $|W| \geq 2$. Let $G_1 = \Gamma/\Gamma[W^c]$ and z_0 be the vertex in G_1 onto which W^c is contracted. Thus $d_{G_1}(z_0) = |[W, W^c]| \leq 6p - 1$.

Arbitrarily add a set Z of $6p + 1 - d_{G_1}(z_0)$ new edges between z_0 and W in G_1 to form a new graph G. Note that $\Gamma[W] = G_1[W] = G[W] = G - z_0$. We will apply Theorem 2.2 to show the following Claim B, leading a contradiction to the fact that Γ is a $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced graph.

Claim B. *$\Gamma[W] = G - z_0$ is strongly $\mathbb{Z}_{2p+1}$-connected.*

Let D_{z_0} be a fixed orientation of $E_G(z_0)$ such that

$$4p+1 \text{ edges are oriented out of } z_0 \text{ and the rest } 2p \text{ edges are oriented into } z_0. \tag{5}$$

We also use D_{z_0} to denote the digraph induced by the oriented edges of D_{z_0}. Define $b_1(v) = d^+_{D_{z_0}}(v) - d^-_{D_{z_0}}(v)$ for each vertex $v \in N_G(z_0) \cup \{z_0\}$.

For any $b' \in Z(G - z_0, \mathbb{Z}_{2p+1})$, we are to show that there exists an orientation D' of $G - z_0$ such that $d^+_{D'}(v) - d^-_{D'}(v) \equiv b'(v) \pmod{2p+1}$ for any vertex $v \in V(G - z_0)$. Define a mapping $b : V(G) \to \mathbb{Z}_{2p+1}$ as follows. For any $x \in V(G)$,

$$b(x) \equiv \begin{cases} b'(x) + b_1(x) \pmod{2p+1} & \text{if } x \in N_G(z_0); \\ b_1(z_0) \pmod{2p+1} & \text{if } x = z_0; \\ b'(x) \pmod{2p+1} & \text{otherwise.} \end{cases}$$

We are going to show Theorem 2.2 is applicable to this graph G.

As $b_1(z_0) + \sum_{v\in N_G(z_0)} b_1(v) = 0$ and $b' \in Z(G - z_0, \mathbb{Z}_{2p+1})$, we have $\sum_{x\in V(G)} b(x) = b_1(z_0) + \sum_{v\in N_G(z_0)} b_1(v) + \sum_{v\in V(G-z_0)} b'(v) \equiv 0 \pmod{2p+1}$, and so $b \in Z(G, \mathbb{Z}_{2p+1})$. Since $|W| \geq 2$, $|V(G)| \geq 3$. By (5), both $d(z_0) = 6p + 1$ and $b(z_0) = d^+_{D_{z_0}}(z_0) - d^-_{D_{z_0}}(z_0) \equiv 0 \pmod{2p+1}$. This, together with (2), implies $|\tau(z_0)| = 2p + 1$, and so Theorem 2.2(i) and (ii) are satisfied.

By (3) and by the minimality of W, for any $A \subset W$ with $|A| < |W|$, we have $d(A) \geq 6p$, or $d(A) - 4p \geq 2p$. As $d(A) \equiv \tau(A) \pmod 2$ and $|\tau(A)| \leq 2p + 1$, it follows by a parity argument that $d(A) \geq 4p + |\tau(A)|$. Thus Theorem 2.2(iii) holds, and hence it holds also for the graph G.

By Theorem 2.2, there exists an orientation D of G such that $d^+_D(x) - d^-_D(x) \equiv b(x) \pmod{2p+1}$ for each vertex $x \in V(G)$. Let D' be the restriction of D on $G - z_0$. By the definition of b, we have $d^+_{D'}(v) - d^-_{D'}(v) \equiv b'(v) \pmod{2p+1}$ for each vertex $v \in V(G - z_0)$. It follows by definition that $\Gamma[W] = G - z_0$ is strongly $\mathbb{Z}_{2p+1}$-connected, and thus Claim B holds.

Since $|W| \geq 2$, Claim B is contrary to the assumption that Γ is $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced. This proves Theorem 2.3. □

Theorem 2.3 immediately leads the following corollary, which reveals that there are finitely many $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced graph in the family of graphs with independence number at most t.

Corollary 2.4. *Every $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced graph G with $\alpha(G) \leq t$ has order at most $6pt - 2p$.*

2.3. Proof of Theorem 1.7

We need one more lemma before presenting the proof of Theorem 1.7. For a graph G, let $\xi(G)$ be the number of nontrivial maximal $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-subgraphs of G.

Lemma 2.5. *If G is a simple graph with $\alpha(G) \leq 2$, then $\xi(G) \leq 2$. Furthermore, $\xi(G) = 2$ if and only if $V(G)$ consists of vertex sets of exactly two maximal $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-subgraphs.*

Proof. Assume that $c = \xi(G) \geq 2$ and let $H_1, H_2, \ldots, H_c$ be the nontrivial maximal $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-subgraphs of G. By Lemma 2.1(iv), every strongly $\mathbb{Z}_{2p+1}$-connected simple graph other than K_1 has order at least $4p+1$, and so $|V(H_i)| \geq 4p+1$ for each $1 \leq i \leq c$.

By contradiction, we assume that $c \geq 3$, and so there exists a vertex $v \in V(G) \setminus (V(H_1) \cup V(H_2))$. By Lemma 2.1(ii)(iii), both $|[v, V(H_1)]_G| \leq 2p - 1$ and $|[V(H_2), V(H_1)]_G| \leq 2p - 1$. Since $|V(H_1)| \geq 4p + 1$, there exists $u_1 \in V(H_1)$ such that $u_1v \notin E(G)$ and $|[u_1, V(H_2)]_G| = 0$. Similarly, there exists $u_2 \in V(H_2)$ such that $u_2v \notin E(G)$ and $|[u_2, V(H_1)]_G| = 0$. Then it follows that $\{u_1, u_2, v\}$ is an independent set of size 3, contradicting to $\alpha(G) \leq 2$. This proves that we must have $\xi(G) \leq 2$, and when $\xi(G) = 2$, $V(G) = V(H_1) \cup V(H_2)$. □

Proof of Theorem 1.7. Since $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle \subseteq \mathcal{M}_{2p+1}$, we have $\mathcal{K}_1(2p+1) = \mathcal{K}_2(2p+1) \setminus \mathcal{M}_{2p+1}$ by (1). Thus by Lemma 2.1(v), Theorem 1.7(i) follows from Theorem 1.7(ii), and so it suffices to show Theorem 1.7(ii). Let G be a graph satisfying the hypotheses of Theorem 1.7, and let $H_1, H_2, \ldots, H_c$ denote the collection of all maximal strongly $\mathbb{Z}_{2p+1}$-connected subgraphs of G, where $|V(H_1)| \geq |V(H_2)| \geq \cdots \geq |V(H_c)|$ and $c \geq 2$, and $G' = G/(H_1 \cup \cdots \cup H_c)$ is the $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction of G.

Proof of (ii). We prove that if G is not strongly $\mathbb{Z}_{2p+1}$-connected, then G' is in $\mathcal{K}_2(2p+1)$.

If G is not connected, then as $\alpha(G) \leq 2$, G must be a disjoint union of two complete graphs, where the larger one has order at least $5p + 1$. By (1) and Lemma 2.1(iv), the $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction of G is a member in $\mathcal{K}_2(2p+1)$ with $s_1 = \cdots = s_{n-1} = 0$. Hence we assume that G is connected and not strongly $\mathbb{Z}_{2p+1}$-connected. By Lemma 2.1(iv) and Corollary 2.4, $|V(H_1)| \geq 4p + 1$. By Lemma 2.5, either $|V(H_2)| > 1$ and $V(G) = V(H_1) \cup V(H_2)$ or $|V(H_2)| = 1$. If $V(G) = V(H_1) \cup V(H_2)$, let $m = |[V(H_2), V(H_1)]_G|$. If $m \geq 2p$, then as $G/(H_1 \cup H_2)$ is an $mK_2 \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$, it follows by Lemma 2.1(ii) that $G \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$, contrary to the assumption that G is not strongly $\mathbb{Z}_{2p+1}$-connected. Hence $m \leq 2p - 1$, and so $G' = mK_2 \in \mathcal{K}_2(2p+1)$.

Assume that $|V(H_2)| = 1$. Then H_1 is the only non-trivial maximal strongly $\mathbb{Z}_{2p+1}$-connected subgraphs of G. Let $V' = V(G) \setminus V(H_1)$. We claim that $G[V']$ is a complete graph. Suppose to the contrary that there exist vertices $v_1, v_2 \in V'$

such that $v_1v_2 \notin E(G[V'])$. By Lemma 2.1(ii)(iii), $|[v_1, V(H_1)]_G| \le 2p-1$ and $|[v_2, V(H_1)]_G| \le 2p-1$. Thus there exists $u \in V(H_1)$ such that $uv_1 \notin E(G)$ and $uv_2 \notin E(G)$ by $|V(H_1)| \ge 4p+1$. It follows that $\{u, v_1, v_2\}$ is an independent set, contrary to the assumption of $\alpha(G) \le 2$. Therefore, $G[V']$ is a complete graph. By Lemma 2.1(iv), we have $|V'| \le 4p$. Thus the $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction of G is in $\mathcal{K}_2(2p+1)$. This proves (ii).

Proof of (iii). If $\kappa'(G) \ge 2p$ and $\delta(G) \ge 4p$, we show that the $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduction G' is not in $\mathcal{K}_2(2p+1)$, and so $G \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$ follows from (ii). By Lemma 2.5, if G has two nontrivial maximal strongly $\mathbb{Z}_{2p+1}$-connected subgraphs H_1 and H_2, then $V(G) = V(H_1) \cup V(H_2)$, and so $G/(H_1 \cup H_2)$ is a mK_2, where $m = |[V(H_2), V(H_1)]_G|$. If $m \le 2p-1$, then $G' = mK_2 \in \mathcal{K}_2(2p+1)$, contrary to the assumption that $\kappa'(G') \ge \kappa'(G) \ge 2p$. Thus $m \ge 2p$ and so by Lemma 2.1(ii) that $G \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$. Hence we assume that G does not have two nontrivial maximal strongly $\mathbb{Z}_{2p+1}$-connected subgraphs. By Corollary 2.4 and Lemma 2.5, G has exactly one nontrivial maximal strongly $\mathbb{Z}_{2p+1}$-connected subgraph H_1. Moreover, $G - V(H_1)$ is a complete graph as showed above in the proof of (ii). Let u^* be the vertex in G' onto which H_1 is contracted. Since $\delta(G) \ge 4p$, for any vertex $v \in V(G' - u^*)$, we have $|[u^*, v]_{G'}| \ge 4p+1-|V'|$, and so G' contains a spanning subgraph isomorphic to $K_{4p+1}/K_{4p+1-|V'|}$. By Lemma 2.1(i)(iv), $K_{4p+1}/K_{4p+1-|V'|} \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$, and so $G' \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$. This contradicts that G' is $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced, unless $|V(G')| = 1$. Therefore, $G \in \langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$ by Lemma 2.1(vi). □

3. On mod 5-orientations

The odd-edge-connectivity of a graph is defined as the size of a smallest edge-cut of odd size. A $6p$-edge-connected graph must be odd-$(6p+1)$-edge-connected, but not vice versa. Tutte's 3-Flow Conjecture was originally proposed for odd-5-edge-connected graphs (see [1]). Lovász, Thomassen, Wu and Zhang [14] proved the following result for mod $(2p+1)$-orientations concerning odd-edge-connectivity, which strengths their theorem on modulo orientations.

Theorem 3.1 (*Lovász et al. [14]*). *Every odd-$(6p+1)$-edge-connected graph admits a mod $(2p+1)$-orientation.*

The main result of this section is Theorem 3.2. For the class of graphs with independence number at most 2, Theorem 3.2 improves Theorem 3.1 for $p = 2$ and verifies Conjecture 1.2 for those values.

Theorem 3.2. *Every odd-9-edge-connected graph G of order at least 21 and with $\alpha(G) \le 2$ has a mod 5-orientation.*

We need a few more tools for the proof of Theorem 3.2.

Theorem 3.3 (*Hakimi [2]*). *Let G be a graph and $\ell : V(G) \mapsto \mathbb{Z}$ be a function such that $\sum_{v \in V(G)} \ell(v) = 0$ and $\ell(v) \equiv d_G(v) \pmod 2$, $\forall v \in V(G)$. Then the following are equivalent.*
(i) G has an orientation D such that $d_D^+(v) - d_D^-(v) = \ell(v)$, $\forall v \in V(G)$.
(ii) $|\sum_{v \in S} \ell(v)| \le |\partial_G(S)|$, $\forall S \subset V(G)$.

Let u_1v and u_2v be two distinct edges in G. We define $G_{[v,u_1u_2]}$ to be the graph obtained from G by deleting the edges u_1v, u_2v and adding a new edge u_1u_2, which is called the *lifting operation* (see [16,14]). The following lemma of Zhang [22] shows that the odd-edge-connectivity is preserved under certain lifting operation.

Lemma 3.4 (*Zhang [22]*). *Let G be a graph with odd edge-connectivity k. Assume there is a vertex $v \in V(G)$ with $d(v) \ne k$ and $d(v) \ne 2$. Then there exists a pair of edges u_1v, u_2v in $E(v)$ such that $G_{[v,u_1u_2]}$, the graph obtained from G by lifting u_1v, u_2v, remains odd edge-connectivity k.*

Lemma 3.5. *Let J_0, J_1 and J_2 be the graphs depicted in Fig. 1. Each of the following holds.*
(i) J_0 is strongly $\mathbb{Z}_5$-connected.
(ii) If G' is a $\langle \mathcal{S}\mathbb{Z}_5\rangle$-reduced graph on 3 vertices, then $|E(G')| \le 7$, where $|E(G')| = 7$ if and only if G' is isomorphic to either J_1 or J_2.

Proof. *Proof of (i).* Let $b \in Z(J_0, \mathbb{Z}_5)$. If $b(v_1) \ne 0$, lift two edges v_1v_2, v_1v_3 to obtain the graph $G_{[v_1,v_2v_3]}$. Since $|[v_1, \{v_2, v_3\}]_{G_{[v_1,v_2v_3]}}| = 3$ and $b(v_1) \ne 0$, we can modify the boundary $b(v_1)$ with the three edges in $[v_1, \{v_2, v_3\}]_{G_{[v_1,v_2v_3]}}$. Specifically, orient 1, 3, 0, 2 edges towards v_1 when $b(v_1) = 1, 2, 3, 4$, respectively. As $|[v_2, v_3]_{G_{[v_1,v_2v_3]}}| = 4$ and by Lemma 2.1(iii), we can also modify the boundaries $b(v_2)$, $b(v_3)$ with those four edges. By symmetry, we assume $b(v_1) = b(v_2) = 0$, then $b(v_3) = 0$ since $b \in Z(J_0, \mathbb{Z}_5)$. Orient all the edges in $E(v_1)$ towards v_1 and orient all the edges in $E(v_2)$ from v_2 to obtain an orientation D of J_0. Then D is a mod 5-orientation of G, which agrees with the boundary $b(v_1) = b(v_2) = b(v_3) = 0$. Therefore, (i) must hold.

Proof of (ii). Set $b_1(v_1) = b_1(v_2) = 3$ and $b_1(v_3) = 4$. Then $b_1 \in Z(J_1, \mathbb{Z}_5)$. It is routine to check that there is no orientation agreeing with the boundary b_1 in J_1. Set $b_2(v_1) = b_2(v_2) = 4$ and $b_2(v_3) = 2$. Then $b_2 \in Z(J_2, \mathbb{Z}_5)$. It is easy to see that there is no orientation agreeing with the boundary b_2 in J_2. Notice that J_1 and J_2 are the only two nonisomorphic graphs on 3 vertices and 7 edges with edge multiplicity at most 3. Now, Lemma 3.5 follows by Lemma 2.1(ii) and the fact that $J_0 \in \langle \mathcal{S}\mathbb{Z}_5\rangle$, $J_1, J_2 \notin \langle \mathcal{S}\mathbb{Z}_5\rangle$. □

Lemma 3.6. *Let G be an odd-9-edge-connected graph of order $n \ge 2$. If G contains a subgraph isomorphic to K_{n-1}, then G admits a mod 5-orientation.*

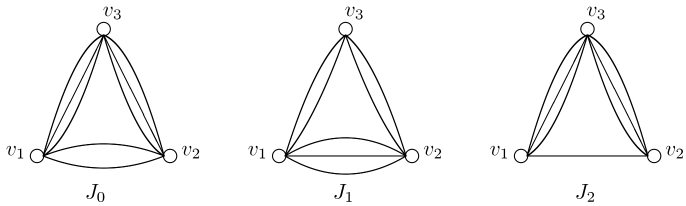

Fig. 1. Graphs in Lemma 3.5, where $J_0 \in \langle \mathcal{S}\mathbb{Z}_5\rangle$ and $J_1, J_2 \notin \langle \mathcal{S}\mathbb{Z}_5\rangle$.

Proof. It is straightforward to verify the statement when $n = 2$ and $n \geq 10$ by Lemma 2.1(iii)(iv). Let G be a counterexample with $|V(G)| + |E(G)|$ minimized. The minimality of G implies that G is $\langle \mathcal{S}\mathbb{Z}_5\rangle$-reduced. Let x be a vertex of G such that $G - x$ contains a subgraph isomorphic to K_{n-1} whose vertex set is denoted by $\{y_1, \dots, y_{n-1}\}$. We may further assume $|[x, y_i]_G| \geq |[x, y_{i+1}]_G|$, $\forall i \in [n-2]$. If G contains an even degree vertex, say v, then, by Lemma 3.4, there exist $\frac{d_G(v)}{2}$ pairs of edges incident with v such that lifting them results a graph, which contains a subgraph isomorphic to K_{n-2}, is still odd-9-edge-connected and has a mod 5-orientation, a contradiction. This implies every vertex has an odd degree, $\delta(G) \geq 9$ and n is even. Moreover, again by Lemma 3.4 and the minimality of $|V(G)| + |E(G)|$, we have $d_G(x) = 9$.

If $n = 4$, then $|E(G)| \geq 18$. Since $|[u, v]_G| \leq 3$ for any $u, v \in V(G)$ by Lemma 2.1(iii), we have $|E(G)| = 18$, and this, in addition, implies that G is isomorphic to $3K_4$. By Lemma 3.5, $3K_3 \in \langle \mathcal{S}\mathbb{Z}_5\rangle$, and so $G \cong 3K_4$ is not $\langle \mathcal{S}\mathbb{Z}_5\rangle$-reduced, contrary to the assumption that G is $\langle \mathcal{S}\mathbb{Z}_5\rangle$-reduced. Hence we assume that $n > 4$.

As every vertex of G has an odd degree, we must have $n \geq 6$. The following observations, stated as Claims 1 and 2, follow from Theorem 3.3 and Lemma 3.5.

Claim 1. *Let $\ell : V(G) \mapsto \{5, -5\}$ be a function such that $\sum_{v\in V(G)} \ell(v) = 0$. Then*

$$\text{there exists } S \subset V(G) \text{ such that } \Big|\sum_{v\in S} \ell(v)\Big| > |\partial_G(S)|. \tag{6}$$

In fact, if (6) fails, then by Theorem 3.3, G has a mod 5-orientation, contrary to the assumption that G is a counterexample. As $n \leq 9$, by the symmetry between S and $V(G) - S$, we may assume that there exists $S \subset V(G)$ satisfying (6) with $|S| \leq 4$ for any given ℓ.

Claim 2. *Let S be a vertex subset of G. Each of the following holds.*
(i) $|\partial_G(S)| \geq \begin{cases} 9 & \text{if } |S| = 1, \\ 12 & \text{if } |S| = 2. \end{cases}$
(ii) If $|S| = 3$, then $|\partial_G(S)| \geq 13$. Moreover, if $|\partial_G(S)| = 13$, then $d_G(s) = 9$, $\forall s \in S$, and $G[S] \in \{J_1, J_2\}$ (see Fig. 1).
(iii) If $n = 8$ and $|S| = 4$, then $|\partial_G(S)| \geq 12$ since G contains K_{n-1}.

When $n = 6$, denote $X = \{x, y_4, y_5\}$ and $Y = \{y_1, y_2, y_3\}$. As $d_G(x) = 9$, we have $|[x, y_5]_G| \leq 1$ and $|[x, y_4]_G| \leq 2$. These, together with $|[y_4, y_5]_G| \leq 3$, imply that

$$\begin{aligned} |[X, Y]_G| &= d_G(x) + d_G(y_4) + d_G(y_5) - 2(|[x, y_4]| + |[x, y_5]| + |[y_4, y_5]|) \\ &\geq 21 - 2(2 + 1 + 3) = 15. \end{aligned} \tag{7}$$

Set $\ell(x) = \ell(y_4) = \ell(y_5) = 5$ and $\ell(y_1) = \ell(y_2) = \ell(y_3) = -5$. We will obtain a contradiction by showing that ℓ violates Claim 1. Choose an $S \subset V(G)$ satisfying (6) with $|S|$ minimized. Then $|S| \leq 3$. By Claim 2(i), $|S| \neq 1, 2$, and so $|S| = 3$. Thus $|\sum_{v\in S} \ell(v)| \in \{5, 15\}$. By Claim 2, $|\sum_{v\in S} \ell(v)| = 15$ implying $S \in \{X, Y\}$, contrary to (7).

Therefore, we assume $n = 8$ in the following. Since $d_G(x) = 9$ and $|[x, y_i]_G| \geq |[x, y_{i+1}]_G|$, $\forall i \in [7]$, we have

$$|[x, y_7]_G| \leq |[x, y_6]_G| \leq |[x, y_5]_G| \leq 1, \tag{8}$$

and

$$|[x, \{y_5, y_6, y_7\}]| \leq |[x, \{y_4, y_6, y_7\}]| \leq 3. \tag{9}$$

Let $X_1 = \{x, y_5, y_6, y_7\}$, $Y_1 = \{y_1, y_2, y_3, y_4\}$, $X_2 = \{x, y_4, y_6, y_7\}$, and $Y_2 = \{y_1, y_2, y_3, y_5\}$. Define two functions ℓ_1 and ℓ_2 to be as follows.

$$\ell_1(v) = \begin{cases} 5, & \text{if } v \in X_1; \\ -5, & \text{if } v \in Y_1. \end{cases} \quad \text{and} \quad \ell_2(v) = \begin{cases} 5, & \text{if } v \in X_2; \\ -5, & \text{if } v \in Y_2. \end{cases}$$

We are to show that either ℓ_1 or ℓ_2 violates Claim 1, leading to a contradiction.

For $i = 1, 2$, choose $S_i \subset V(G)$ satisfying (6) with $|S_i|$ minimized. By Claim 2(i), we have $3 \leq |S_i| \leq 4$.

Claim 3. *If $|S_i| = 3$, then $|\partial_G(S_i)| = 13$ and $S_i = X_i \setminus \{x\}$.*

As $|S_i| = 3$, $|\sum_{v\in S_i} \ell_i(v)| \in \{5, 15\}$. By (6) and Claim 2(ii), we must have $15 = |\sum_{v\in S_i} \ell_i(v)| > |\partial_G(S_i)| = 13$. Thus $S_i \subset X_i$ or $S_i \subset Y_i$. Moreover, $G[S_i]$ is isomorphic to J_1 or J_2 as $|\partial_G(S_i)| = 13$ and by Claim 2(ii).

If $x \in S_i$, then by Claim 2(ii), $S_i \subset X_i$ and $|[x, S_i \setminus \{x\}]| \geq 4$ as $G[S_i]$ is isomorphic to J_1 or J_2, contradicting to (8). If $S_i \subset Y_i$, then we have $13 = |\partial_G(S_i)| = |[x, S_i]_G| + |[S_i, V(G) \setminus (S_i \cup \{x\})]_G| \geq |[x, S_i]_G| + 12$. Thus $|[x, S_i]_G| \leq 1$, and so $|[x, \{y_4, y_5, y_6, y_7\}]_G| = 0$. Denote $\{y_t\} = Y \setminus S_i$. Then $|[x, y_t]_G| \geq 9 - |[x, S_i]_G| - |[x, \{y_4, y_5, y_6, y_7\}]_G| \geq 8$. So, by Lemma 2.1(iii), G is not $\langle \mathcal{S}\mathbb{Z}_5\rangle$-reduced, a contradiction to the assumption on G. Therefore, we conclude that $S_i = X_i \setminus \{x\}$ if $|S_i| = 3$.

Claim 4. *If $|S_i| = 3$, then $|S_{3-i}| \notin \{3, 4\}$.*

Assume $|S_1| = |S_2| = 3$ first. We claim that there exists $s \in S_1 \cup S_2 = \{y_4, y_5, y_6, y_7\}$ such that $d_{G[S_1\cup S_2]}(s) \geq 7$. If one of $G[S_1]$, $G[S_2]$ is isomorphic to J_2, it is routine to verify that the vertex s corresponding to v_3 in J_2 has degree at least 7 in $G[S_1 \cup S_2]$. Otherwise, we have $G[S_1] \cong G[S_2] \cong J_1$ by Claim 2(ii), and so one of the vertices y_6, y_7 has degree at least 7 in $G[S_1 \cup S_2]$. Since $d_{G[S_1\cup S_2]}(s) \geq 7$, it follows by $|[s, \{y_1, y_2, y_3\}]| \geq 3$ that $d_G(s) \geq 10$, contradicting to $d_G(s) = 9$ by Claim 2(ii).

We assume $|S_i| = 3$ and $|S_{3-i}| = 4$. By Claim 3, we have $y_{6-i} \in S_i \subset X_i$, and it follows by Claim 2(ii) and Claim 3 that

$$|[y_{6-i}, \{y_6, y_7\}]| \geq 4. \tag{10}$$

Since $|S_{3-i}| = 4$ and by Claim 2(iii), we have $20 > |\partial_G(S_{3-i})| = |[X_{3-i}, Y_{3-i}]|$ from (6). However, it follows from (9), (10) and $y_{6-i} \in X_i$ that

$$\begin{aligned} |[X_{3-i}, Y_{3-i}]_G| &= d_G(x) - |[x, \{y_{3+i}, y_6, y_7\}]_G| + |[\{y_{3+i}, y_6, y_7\}, Y_{3-i}]_G| \\ &\geq 9 - 3 + 10 + |[y_{6-i}, \{y_6, y_7\}]_G| \\ &\geq 20 = |\sum_{v\in S_{3-i}} \ell_{3-i}(v)|, \end{aligned}$$

a contradiction to (6). Hence Claim 4 holds.

The final step. By Claim 4, we may assume that $|S_1| = |S_2| = 4$. Thus, for $i \in \{1, 2\}$, $20 = |\sum_{v\in S_i} \ell_i(v)| > |\partial_G(S_i)| = |[X_i, Y_i]|$ by (6) and Claim 2(iii). Then $|\partial_G(S_i)| = |[X_i, Y_i]| \leq 18$, since $|X_i|$ is even. However, it follows from (8) and (9) that

$$\begin{aligned} 36 &\geq |[X_1, Y_1]_G| + |[X_2, Y_2]_G| \\ &= 2d_G(x) - |[x, \{y_4, y_6, y_7\}]_G| - |[x, \{y_5, y_6, y_7\}]_G| \\ &\quad + 2|[\{y_6, y_7\}, \{y_1, y_2, y_3\}]_G| + (d_G(y_4) - |[x, y_4]_G|) + (d_G(y_5) - |[x, y_5]_G|) \\ &\geq 18 - 3 - 3 + 12 + 6 + 8 = 38, \end{aligned}$$

a contradiction. The proof is completed. □

Proof of Theorem 3.2. Let G be an odd-9-edge-connected graph with $\alpha(G) \leq 2$ and G' be the $\langle \mathcal{S}\mathbb{Z}_5\rangle$-reduction of G. We shall show that $|V(G')| \leq 9$ *and G' contains a subgraph isomorphic to $K_{|V(G')|-1}$*. Then G' admits a mod 5-orientation by Lemma 3.6, and so Theorem 3.2 follows from Lemma 2.1(v).

Denote G_1 to be the underline simple graph of G. Since $|V(G_1)| \geq 21$, G_1 is not $\langle \mathcal{S}\mathbb{Z}_5\rangle$-reduced by Corollary 2.4, and hence $\xi(G_1) \neq 0$. By Lemma 2.5, we have $1 \leq \xi(G_1) \leq 2$. If $\xi(G_1) = 2$, again by Lemma 2.5, G_1', the $\langle \mathcal{S}\mathbb{Z}_5\rangle$-reduction of G_1, is a graph with at most two vertices, so does G'. Notice that $|V(G')| \leq |V(G_1')|$. Assume $\xi(G_1) = 1$ and let H_1 be the corresponding nontrivial maximal $\langle \mathcal{S}\mathbb{Z}_5\rangle$-subgraphs of G_1. Clearly, $|V(H_1)| \geq 9$ by Lemma 2.1(iv). Let H be a nontrivial maximal $\langle \mathcal{S}\mathbb{Z}_5\rangle$-subgraphs of G with $|V(H)|$ maximized. As $G[V(H_1)] \in \langle \mathcal{S}\mathbb{Z}_5\rangle$, we have $|V(H)| \geq |V(H_1)| \geq 9$. We claim that $\alpha(G - V(H)) = 1$. In fact, suppose that u, v are two non-adjacent vertices in $G - V(H)$. Then, by Lemma 2.1(ii)(iii), we have $|[u, V(H)]| \leq 3$ and $|[v, V(H)]| \leq 3$. Since $|V(H)| \geq 9$, there exists $w \in V(H)$ such that $\{w, u, v\}$ forms an independent set of size 3, a contradiction to $\alpha(G) \leq 2$. Hence $\alpha(G - V(H)) = 1$. Now, by Lemma 2.1(iv), the $\langle \mathcal{S}\mathbb{Z}_5\rangle$-reduction of $G - V(H)$ has size at most 8 and independence number 1. Hence G' has order at most 9 and contains a subgraph isomorphic to $K_{|V(G')|-1}$. Therefore, Theorem 3.2 follows from Lemma 2.1(v) and Lemma 3.6. □

Note that Theorem 1.8 follows from Theorems 3.2 and 1.6.

4. Concluding remarks

As already mentioned in Section 1, we have proved that there are finitely many $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced graphs, which are contraction obstacles for admitting a mod $(2p+1)$-orientation, in the family of graphs with bounded independence number. However, there are infinitely many $(4p+1)$-edge-connected $\langle \mathcal{S}\mathbb{Z}_{2p+1}\rangle$-reduced graphs without mod $(2p+1)$-orientation for every $p \geq 5$ as proved in [3]. We ask a meta question that what kind of graph family may have only finitely many contraction obstacles for admitting a mod $(2p+1)$-orientation. Some dense conditions or degree conditions may work, and certain edge connectivity condition may not work well. The corresponding question on planar graphs is of particular interest, which is open for every $p \geq 2$.

Problem 4.1. For each integer $p \geq 2$, are there finite many $(4p+1)$-edge-connected $\langle \mathcal{S}\mathbb{Z}_{2p+1} \rangle$-reduced planar graphs?

Problem 4.1 can be viewed as a relaxed version of Jaeger's conjecture on planar graphs, and it can be also generalized to graphs embedded on surface.

Problem 4.2. For each positive integer p, are there finite many $(4p+1)$-edge-connected $\langle \mathcal{S}\mathbb{Z}_{2p+1} \rangle$-reduced graphs for the family of graphs embedded on a fixed surface?

Acknowledgments

The authors are very grateful to two anonymous referees for their careful reading of the manuscript and helpful comments which led to the improvement of the presentation of this paper.

Funding

The research of Hong-Jian Lai is partially supported by Chinese National Natural Science Foundation grants CNNSF 11771039 and CNNSF 11771443.

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