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Minimax properties of some density measures in graphs and digraphs

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ABSTRACT

For a graph G, let $f(G)$ denote the connectivity $\kappa(G)$, or the edge-connectivity $\kappa'(G)$, or the minimum degree $\delta(G)$ of G, and define $\overline{f}(G) = \max\{f(H) : H$ is a subgraph of $G\}$. Matula in [*K-components, clusters, and slicings in graphs*, SIAM J. Appl. Math. 22 (1972), pp. 459–480] proved two minimax theorems related to $\overline{\delta}(G)$ and $\overline{\kappa}'(G)$, and obtained polynomial algorithms to determine $\overline{\delta}(G)$, $\overline{\kappa}'(G)$ and $\overline{\kappa}(G)$. The *restricted edge-connectivity* of G, denoted by $\lambda_2(G)$, is the minimum size of a restricted edge-cut of G. We define $\overline{\lambda_2}(G) = \max\{\lambda_2(H) : H \subseteq G\}$. For a digraph D, let $\kappa(D)$, $\lambda(D)$, $\delta^-(D)$ and $\delta^+(D)$ denote the strong connectivity, arc-strong connectivity, minimum in-degree and out-degree of D, respectively. For each $f \in \{\kappa, \lambda, \delta^-, \delta^+\}$, define $\overline{f}(D) = \max\{f(H) : H$ is a subgraph of $D\}$. In this paper, we obtain analogous minmax duality results, which are applied to yield polynomial algorithms to determine $\overline{\delta}^+(D)$, $\overline{\delta}^-(D)$, $\overline{\lambda}(D)$ and $\overline{\kappa}(D)$ for a digraph D and $\overline{\lambda_2}(G)$ for a graph G.

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1. Introduction

We consider finite simple graphs and simple digraphs. Usually, we use G to denote a graph and D a digraph. Undefined terms and notations will follow [5] for graphs and [3] for digraphs. For graphs H and G, we denote $H \subseteq G$ when H is a subgraph of G. Similarly, for digraphs H and D, $H \subseteq D$ when H is a subdigraph of D. In particular, $\kappa(G)$, $\kappa'(G)$ and $\delta(G)$ denote the connectivity, the edge-connectivity and the minimum degree of a graph G, respectively; $\kappa(D)$ and $\lambda(D)$ denote the strong connectivity and the arc-strong connectivity of a digraph D, respectively. A digraph D is strong if D is strongly connected. A *strong component* of a digraph D is a maximal strong subdigraph of D. A strong component H of D is *nontrivial* if $|A(H)| > 0$. Following [5], a digraph D is *strict* if D has no loops nor parallel arcs. Throughout this paper, we use the notation (u, v) to denote an arc oriented from u to v in a digraph, and $[u, v]$ to denote an arc which is in $\{(u, v), (v, u)\}$. A digraph D is *complete* if D is strict and for every pair u,v of distinct vertices of D, both (u, v) and $(v, u) \in A(D)$. The complete digraph on n vertices will be denoted by K_n^*. It is known that (see [3, p. 16], for example) for any integer $n > 1$, $\kappa(K_n^*) = n - 1$.

Using the notation in [3,5], for any disjoint subsets $X, Y \subseteq V(G)$, define

$$(X, Y)_G = \{xy \in E(G) : x \in X, y \in Y\} \quad \text{and} \quad \partial_G(X) = (X, V(G) - X)_G.$$

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When $X = \{v\}$, we often use $\partial_G(v)$ for $\partial_G(\{v\})$. Likewise, for any disjoint subsets $X, Y \subseteq V(D)$, define

$$(X,Y)_D = \{(x,y) \in A(D) : x \in X, y \in Y\}, \quad \partial_D^+(X) = (X, V(D) - X)_D$$
$$\text{and} \quad \partial_D^-(X) = \partial_D^+(V(D) - X).$$

For each $v \in V(D)$, we use $\partial_D^+(v)$ for $\partial_D^+(\{v\})$ and $\partial_D^-(v)$ for $\partial_D^-(\{v\})$. The *out-degree* (*in-degree*, respectively) of v in D is $d_D^+(v) = |\partial_D^+(v)|$ ($d_D^-(v) = |\partial_D^-(v)|$, respectively). We also define

$$N_D^+(v) = \{u \in V(D) : (v,u) \in A(D)\} \quad \text{and} \quad N_D^-(v) = \{u \in V(D) : (u,v) \in A(D)\}.$$

For a graph G, let $f(G)$ denote the edge-connectivity $\kappa'(G)$ or the minimum degree $\delta(G)$ of G, and define $\overline{f}(G) = \max\{f(H) : H \text{ is a subgraph of } G\}$. As indicated in [15], networks modelled as a graph G with $f(G) = \overline{f}(G)$ are of particular interest of investigations. Matula first studied the quantities

$$\overline{\kappa}'(G) = \max\{\kappa'(H) : H \subseteq G\} \quad \text{and} \quad \overline{\delta}(G) = \max\{\delta(H) : H \subseteq G\}.$$

These graph invariants have drawn the attention of researchers as early as in the 1960s. Graphs G with $\overline{\delta}(G) \le k$ are called k-degenerate graphs and were first investigated in [18]. For any fixed integer $k > 0$, the k-core of a graph G is the unique maximal subgraph H of G with $\delta(H) \ge k$, and can be obtained from G by repeatedly deleting vertices of degree less than k. The k-cores are considered as fundamental structures in graph theory, as seen in [6,7,20,25,28], among others. A weighted version of k-cores is introduced in [14] to study the communities cooperation level in social science. Other social network applications can be found in [27]. As commented in [16,20,24], both $\overline{\delta}(G)$ and $\overline{\kappa}'(G)$ are related to graph colouring problems.

In order to compute $\overline{\kappa}'(G)$ and $\overline{\delta}(G)$, Matula defined slicings.

Definition 1.1: *Let G be a graph with $E(G) \neq \emptyset$.*

(i) *A sequence of disjoint nonempty edge subsets $Z = (J_1, J_2, \ldots, J_m)$ is a* slicing *of G if J_1 is an edge-cut of G, and for each i with $2 \le i \le m$, J_i is an edge-cut of $G - \bigcup_{j=1}^{i-1} E(J_j)$.*
(ii) *If there exists a sequence $v_1, v_2, \ldots, v_m$ of vertices of G such that $J_1 = \partial_G(v_1)$ and for $i \ge 2$, $J_i = \partial_{G - \bigcup_{j=1}^{i-1} E(J_j)}(v_i)$, then the slicing $Z = (J_1, J_2, \ldots, J_m)$ is a* δ-slicing *of G.*
(iii) *If $Z = (J_1, J_2, \ldots, J_m)$ is a slicing of G, then the* width *of Z is*

$$w(Z) = \max\{|J_i| : 1 \le i \le m\}.$$

In [22], Matula discovered some minimax results involving $\overline{\kappa}'(G)$ and $\overline{\delta}(G)$.

Theorem 1.2 ([22]): *For any graph G with $|E(G)| \ge 1$, each of the following holds.*

(i) $\overline{\kappa}'(G) = \max\{\kappa'(H) : H \subseteq G\} = \min\{w(Z) : Z \text{ is a slicing of } G\}$.
(ii) $\overline{\delta}(G) = \max\{\delta(H) : H \subseteq G\} = \min\{w(Z) : Z \text{ is a } \delta\text{-slicing of } G\}$.

While the parameters $\overline{\delta}(G)$, $\overline{\kappa}'(G)$ and $\overline{\kappa}(G)$ have been intensively studied, to the best of our knowledge, the related problem on the other network reliability measures and the corresponding measures of digraphs have rarely been investigated. The purpose of this paper is to investigate whether digraphs will have similar behaviours, and to seek if Theorem 1.2 can be extended to other graph reliability measures. As in [3], the *minimum out-degree* and the *minimum in-degree* of a digraph D are

$\delta^+(D) = \min\{d_D^+(v) : v \in V(D)\}$ and $\delta^-(D) = \min\{d_D^-(v) : v \in V(D)\}$, respectively. Naturally, for a digraph D, we define

$$\overline{\lambda}(D) = \max\{\lambda(H) : H \subseteq D\}, \quad \overline{\delta}^+(D) = \max\{\delta^+(H) : H \subseteq D\}$$
$$\text{and} \quad \overline{\delta}^-(D) = \max\{\delta^-(H) : H \subseteq D\}.$$

Some of the recent studies on $\lambda(D)$ and $\overline{\lambda}(D)$ focused on extremal properties and the relationship with arc disjoint spanning arborescences, as seen in [1,2,17,19], among others. By the definition of $\overline{\lambda}(D)$, we observe that $\overline{\lambda}(D) = 0$ if and only if D does not contain a directed cycle. That is, D is acyclic. Therefore, throughout this paper, when discussing $\overline{\lambda}(D)$, we always assume that $\lambda(D) > 0$.

A natural model for digraph slicing will be a sequence of disjoint nonempty arc subsets in the form $\partial_{D_i}^+(X)$ for some subdigraph D_i of D. Similarly, δ^+-slicings (δ^--slicings, respectively) will be sequences of disjoint nonempty arc subsets in the form $\partial_{D_i}^+(v)$ ($\partial_{D_i}^-(v)$, respectively). We observe that in a nontrivial graph G, every edge lies in an edge-cut of the connected component of G containing the edge. But in a nontrivial digraph D, not every arc is lying in a directed cut of a strong component of D. Therefore, we would need to modify the definition of a graph slicing to define a digraph slicing to accommodate this difference, and the proving arguments would also be altered accordingly. In their studies of fault tolerance networks, Esfahanian [8] and Esfahanian and Hakimi [9] introduced restricted edge-connectivity of a graph. An edge-cut X of a graph G is *restricted* if for any $v \in V(G)$, $\partial_G(v) - X \neq \emptyset$. The restricted edge-connectivity of a graph G, denoted by $\lambda_2(G)$, is the minimum size of a restricted edge-cut of G. The concept of different slicing will be formally defined in the next section. Our main result on digraphs is stated below.

Theorem 1.3: *Let D be a digraph with $A(D) \neq \emptyset$. Let $\mathcal{S}(D)$ be the collection of all slicings of D and let $\mathcal{S}^+(D)$, $\mathcal{S}^-(D)$ be the collection of all δ^+-slicings of D and all δ^--slicings of D, respectively. Each of the following holds.*

(i) *Assume that $\overline{\lambda}(D) > 0$. Then* $\max\{\min\{|\partial_H^+(X)| : \emptyset \neq X \subset V(H)\} : H \subseteq D\} = \min\{\max\{|J_i| : 1 \leq i \leq m\} : S = (J_1, J_2, \ldots, J_m) \in \mathcal{S}(D)\}$.
(ii) $\max\{\min\{d_H^+(v) : v \in V(H)\} : H \subseteq D\} = \min\{\max\{|J_i| : 1 \leq i \leq m\} : S = (J_1, J_2, \ldots, J_m) \in \mathcal{S}^+(D)\}$.
(iii) $\max\{\min\{d_H^-(v) : v \in V(H)\} : H \subseteq D\} = \min\{\max\{|J_i| : 1 \leq i \leq m\} : S = (J_1, J_2, \ldots, J_m) \in \mathcal{S}^-(D)\}$.

Likewise, we define $\overline{\kappa}(D) = \max\{\kappa(H) : H \subseteq D\}$. Related properties on $\overline{\kappa}(D)$ are also discussed. In the next section, we present the proofs for these minimax relations stated in Theorem 1.3, as well as discussions of other related properties. In Section 3, we explain that these quantities $\overline{\delta}^+(D)$, $\overline{\delta}^-(D)$, $\overline{\lambda}(D)$ and $\overline{\kappa}(D)$ can be computationally determined in polynomial time. In the last section, we will develop the concept of λ_2-slicing of G and prove an analogous minimax duality result that determines the value of $\overline{\lambda_2}(G) = \max\{\lambda_2(H) : H \subseteq G\}$.

Our approaches to the digraph generalization of Theorem 1.2 are motivated by and similar to the work of Matula [20–23]. The minimax theorem on the restricted edge-connectivity of graph is also motivated by these results. We believe that there might be a more general theorem that can cover all these as special cases, and we have not yet found this general result yet.

2. Minimax theorems in some subdigraph density measures

Let $k \geq 0$ be an integer. A digraph D is *k-arc-strong* if $\lambda(D) \geq k$, or equivalently, for any proper nonempty subset $\emptyset \neq X \subset V(D)$, we always have $|\partial_D^+(X)| \geq k$. Thus in this sense, every digraph D is 0-arc-strong, and $\lambda(D) = 0$ if and only if D is not 1-arc-strong. Let D be a digraph and let D_1 and D_2 be two subdigraphs of D. Define $D_1 \cup D_2$ to be the subdigraph of D with $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$ and $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$. We start with some elementary properties. Proposition 2.1 follows by an argument similar to that by Matula [20,21]

Proposition 2.1 ([1]): *Let $D_1, D_2, \ldots, D_n$ be subdigraphs of a digraph D such that $\bigcup_{i=1}^{n} D_i$ is strongly connected. Then $\lambda(\bigcup_{i=1}^{n} D_i) \geq \min_{1 \leq i \leq n} \lambda(D_i)$.*

It follows from the definitions that for any strong digraph D,

$$\kappa(D) \leq \lambda(D) \leq \min\{\delta^+(D), \delta^-(D)\}. \tag{1}$$

Proposition 2.2: *Let D be a strong digraph. Then $\overline{\kappa}(D) \leq \overline{\lambda}(D) \leq \min\{\overline{\delta}^+(D), \overline{\delta}^-(D)\}$.*

Proof: Let $L \subseteq D$ such that $\overline{\kappa}(D) = \kappa(L)$. By Equation (1), $\overline{\kappa}(D) = \kappa(L) \leq \lambda(L) \leq \overline{\lambda}(D)$. To show that $\overline{\lambda}(D) \leq \min\{\overline{\delta}^+(D), \overline{\delta}^-(D)\}$, we now take a subdigraph $H \subseteq D$ such that $\overline{\lambda}(D) = \lambda(H)$. Let $v \in H$ such that $d_H^+(v) = \delta^+(H)$. As $D - \partial_H^+(v)$ is not strong, we have $\overline{\lambda}(D) = \lambda(H) \leq \delta^+(H) \leq \overline{\delta}^+(D)$. Similarly, we also have $\overline{\lambda}(D) \leq \overline{\delta}^-(D)$. ■

2.1. Slicing and proof of Theorem 1.3(i)

Throughout this subsection, we assume that D is a digraph with $\overline{\lambda}(D) > 0$. An arc subset W of D is a *direct cut* of D if there exists a nonempty proper vertex subset X such that $W = (X, V(D) - X)_D$ with $W \neq \emptyset$. We present a formal definition of digraph slicing below.

Definition 2.3: *Let D be a digraph with $\lambda(D) > 0$. Set $D_1 = D$.*

(i) *A* slicing *of D is a sequence $S = (J_1, J_2, \ldots, J_s)$ of arcs subsets of D with $s \geq 2$ such that each of the following holds.*
  (i-1) *J_1 is a direct cut of D_1.*
  (i-2) *Define $D_2 = D - J_1$. For $i = 2, 3, \ldots, s-1$, D_i is not acyclic, J_i is a nonempty direct cut of D_i and set $D_{i+1} = D_i - J_i$.*
  (i-3) *$D_s = D - \bigcup_{i=1}^{s-1} J_i$ is acyclic.*
(ii) *If for each i with $1 \leq i \leq s-1$, J_i is a minimum direct cut of a nontrivial strong component of D_i, then the slicing $S = (J_1, J_2, \ldots, J_s)$ is a* narrow slicing.
(iii) *The* width *of a slicing $S = (J_1, J_2, \ldots, J_s)$ is $w(S) = \max\{|J_i|, 1 \leq i \leq s-1\}$.*
(iv) *The collection of all slicings of D is denoted by $\mathcal{S}(D)$.*

Proof of Theorem 1.3(i): Let $k = \overline{\lambda}(D)$ and $k' = \min\{w(S) : S \in \mathcal{S}(D)\}$. By the assumption of Theorem 1.3(i), we have $k > 0$.

Suppose first that H is a subdigraph of D with $k = \lambda(H)$. Let $S = (J_1, J_2, \ldots, J_m)$ be a slicing of D. By the definition of a slicing, $D_m = D - \bigcup_{i=1}^{m-1} J_i$ is acyclic. Since $\lambda(H) = k \geq 1$, H is not a subdigraph of D_m. Hence there must be a smallest index ℓ with $1 \leq \ell < m$ such that the arc subset $J_\ell \cap A(H) \neq \emptyset$. It follows that $J_\ell \cap A(H)$ is a direct cut of H, and so $w(S) \geq |J_\ell| \geq |J_\ell \cap A(H)| \geq \lambda(H) = k$. Since S was arbitrary, we have $k' = \min\{w(S) : S \in \mathcal{S}(D)\} \geq k = \overline{\lambda}(D)$.

Conversely, let $k'' = \min\{w(S) : S \text{ is a narrow slicing of } D\}$. Then as the collection of all narrow slicings of D is a subset of $\mathcal{S}(D)$, it follows by definition that $k'' \geq k'$. We are to show that $k \geq k''$, which implies the desired $k \geq k'$. Arguing by contradiction, we assume that $k'' > k$. Thus there exists a narrow slicing $S = (J_1, J_2, \ldots, J_m)$ such that $k'' = w(S) \geq k + 1$. Hence there exists a smallest i with $1 \leq i \leq m$ such that $|J_i| = w(S) \geq k + 1$. Since S is a narrow slicing, by Definition 2.3(ii), J_i is a minimum direct cut of a strong component L of D_i. It follows that $\lambda(D_i) = |J_i| = w(S) \geq k + 1 > \overline{\lambda}(D) \geq \lambda(D_i)$. This contradiction implies that we must have $k \geq k'' \geq k'$. This establishes Theorem 1.3(i). ■

The argument deployed in the proof of Theorem 1.3(i) suggests some computational useful ways of determining $\overline{\lambda}(D)$, as stated in the following result.

Lemma 2.4: *Let D be a digraph with $A(D) \neq \emptyset$. If $S = (J_1, J_2, \ldots, J_s)$ is a narrow slicing of D, then $\overline{\lambda}(D) = \max_{1 \le i \le s}\{|J_i|\}$.*

Proof: Let $S = (J_1, J_2, \ldots, J_s)$ be a narrow slicing of D. Since a narrow slicing is a slicing, it follows by Theorem 1.3(i) that $\overline{\lambda}(D) \le \max_{1 \le i \le s}\{|J_i|\}$. Conversely, let ℓ be an integer with $1 \le \ell \le s$ satisfying $|J_\ell| = \max_{1 \le i \le s}\{|J_i|\}$. By the definition of a narrow slicing, there exists a subdigraph D_ℓ of D such that J_ℓ is a direct cut of D_ℓ with $|J_\ell| = \lambda(D_\ell)$. It follows that $\overline{\lambda}(D) = \max\{\min\{|\partial_H^+(X)| : \emptyset \neq X \subset V(H)\} : H \subseteq D\} \ge \lambda(D_\ell) = |J_\ell| = \max_{1 \le i \le s}\{|J_i|\}$. ∎

2.2. δ^+-slicing, δ^--slicing and proof of Theorem 1.3(ii) and (iii)

Throughout this subsection, we assume that D is a digraph with $A(D) \neq \emptyset$. For a digraph D, let $G(D)$, called the *underlying graph* of D, be the graph obtained from D by erasing all the orientation of the arcs of D. A digraph D is *weakly connected* if $G(D)$ is connected. A subdigraph H of D is a *weak component* of D if $G(H)$ is a component of $G(D)$ with $|A(H)| > 0$. (Thus, an isolated vertex of D is not a weak component.) Like in Section 2.1, we start with a formal definition of a δ^+-slicing, as well as one of a δ^--slicing.

Definition 2.5: *Let D be a digraph with $A(D) \neq \emptyset$.*

(i) *A sequence of disjoint arc subsets $S = (J_1, J_2, \ldots, J_s)$ of D is a* δ^+-slicing *(or* δ^--slicing, *respectively) of D if each $J_i \neq \emptyset$, $1 \le i \le s$, and if each of the following holds:*
 (i-1) *Let $D_1 = D$. There exists a vertex $v_1 \in V(D_1)$ such that $J_1 = \partial_{D_1}^+(v_1)$ ($J_1 = \partial_{D_1}^-(v_1)$, respectively).*
 (i-2) *For $i = 2, \ldots, s$, set $D_i = D_{i-1} - J_{i-1}$, and there exists a vertex $v_i \in V(D_i)$ such that $J_i = \partial_{D_i}^+(v_i)$ ($J_i = \partial_{D_i}^-(v_i)$, respectively).*
 (i-3) *$A(D_s) - J_s = \emptyset$.*

(ii) *A δ^+-slicing (or a δ^--slicing, respectively) $S = (J_1, J_2, \ldots, J_s)$ is* minimal *if for each i with $1 \le i \le s$, there exists a weak component L_i of D_i such that $|J_i| = \delta^+(L_i)$ (or $|J_i| = \delta^-(L_i)$, respectively).*

(iii) *Let $\mathcal{S}^+(D)$ and $\mathcal{S}^-(D)$ denote the collections of all δ^+-slicings and all δ^--slicings of D, respectively.*

By Definition 2.5, if $S = (J_1, J_2, \ldots, J_m)$ is a δ^+-slicing of D, then for each $i = 1, 2, \ldots, m$,

$$\text{there exists a weak component } D_i' \text{ of } D_i \text{ and a vertex } v_i \in V(D_i') \text{ such that } J_i = \partial_{D_i'}^+(v_i). \tag{2}$$

Proof of Theorem 1.3(ii) and (iii): By symmetry, it suffices to prove Theorem 1.3(ii). By Definition 2.5, there exists a subdigraph $H \subseteq D$ such that $\overline{\delta}^+(D) = \delta^+(H)$.

Let $S = (J_1, J_2, \ldots, J_s) \in \mathcal{S}(D)$ be an arbitrary δ^+-slicing of D. By Equation (2), there exist a vertex $z \in V(H)$ and an index j with $1 \le j \le s$ such that $J_j \cap A(H) = \partial_H^+(z)$. It follows that $\max_{1 \le i \le m}\{|J_i|\} \ge |J_j| \ge |J_j \cap A(H)| \ge \delta^+(H) = \overline{\delta}^+(D)$. Since $S = (J_1, J_2, \ldots, J_s) \in \mathcal{S}(D)$ is arbitrary, we have

$$h = \min\{\max\{|J_i| \, : \, 1 \le i \le s\} : S = (J_1, J_2, \ldots, J_s) \in \mathcal{S}(D)\} \ge \overline{\delta}^+(D).$$

Conversely, let $h' = \min\{\max\{|J_i| : 1 \le i \le s\} : S = (J_1, J_2, \ldots, J_s) \in \mathcal{S}(D)$ be a minimal δ^+-slicing of $D\}$. Thus by definition, we have $h \ge h'$. We are to show that $h' \le \overline{\delta}^+(D)$, which would imply the needed $h \le h' \le \overline{\delta}^+(D)$ to complete the proof. Let $S = (J_1, J_2, \ldots, J_s)$ be an arbitrary minimal δ^+-slicing of D. By Definition 2.5(i-3), $A(D) - \bigcup_{i=1}^s J_i = A(D_s) - J_s = \emptyset$. As $H \neq \emptyset$, we observe that

$A(H) \subseteq \bigcup_{i=1}^{s} J_i$, and so there exists a smallest integer j such that $J_j \cap A(H) \neq \emptyset$. By Equation (2), for each i with $1 \leq i \leq m$, there exists a weak component L_i of D_i and a vertex $v_i \in V(L_i)$ such that $|J_i| = \delta^+_{L_i}(v_i) \leq \overline{\delta}^+(D)$. It follows that $\max\{|J_i| : 1 \leq i \leq m\} \leq \overline{\delta}^+(D)$, and so

$$h' = \min\{\max\{|J_i| \, : \, 1 \leq i \leq s\} : S = (J_1, J_2, \ldots, J_s) \in \mathcal{S}^+(D) \text{ is a minimal slicing of } D\} \leq \overline{\delta}^+(D).$$

This proves that Theorem 1.3(ii) must hold. The proof for Theorem 1.3(iii) is similar and will be omitted. ■

The arguments deployed in the proof of Theorem 1.3(ii) and (iii) also suggest some computational useful ways of determining $\overline{\delta}^+(D)$ and $\overline{\delta}^-(D)$, as stated in the following results.

Lemma 2.6: *Let D be a digraph with $A(D) \neq \emptyset$.*

(i) *If $S = (J_1, J_2, \ldots, J_s)$ is a minimal δ^+-slicing of D, then*

$$\overline{\delta}^+(D) = \max_{1 \leq i \leq s}\{|J_i|\}.$$

(ii) If $S = (J_1, J_2, \ldots, J_s)$ is a minimal δ^--slicing of D, then

$$\overline{\delta}^-(D) = \max_{1 \leq i \leq s}\{|J_i|\}.$$

Proof: By symmetry, it suffices to prove Lemma 2.6(i). Let $S = (J_1, J_2, \ldots, J_s)$ be a minimal δ^+-slicing of D. Since a minimal δ^+-slicing is also a δ^+-slicing, it follows from Theorem 1.3(ii) that

$$\overline{\delta}^+(D) = \max\{\min\{d^+_H(v) : v \in V(H)\} : H \subseteq D\} \leq \max_{1 \leq i \leq s}\{|J_i|\}.$$

On the other hand, there exists an ℓ with $1 \leq \ell \leq s$ such that $|J_\ell| = \max_{1 \leq i \leq s}\{|J_i|\}$. By Definition 2.5(ii), there exists a subdigraph L_ℓ of D such that $|J_\ell| = \delta^+(L_\ell)$. It follows that

$$\overline{\delta}^+(D) = \max\{\min\{d^+_H(v) : v \in V(H)\} : H \subseteq D\} \geq \delta^+(L_\ell) = |J_\ell| = \max_{1 \leq i \leq s}\{|J_i|\}.$$

This justifies Lemma 2.6(i). The proof for Lemma 2.6(ii) is similar and will be omitted. ■

2.3. Maximum subdigraph strong connectivity

Throughout this subsection, we assume that D is a digraph which is not spanned by a complete digraph. If $X \subseteq V(D)$ is a subset, then $D[X]$ denotes the subdigraph of D induced by X. Following [3], for a pair of distinct vertices $u, v \in V(D)$, a vertex subset $S \subset V(D) - \{u, v\}$ is an (u, v)-*separator* if $D - S$ contains no directed (u, v)-paths. A subset $S \subset V(D)$ is an (u, v)-*separator* of D if for some $u, v \in V(D)$, S is an (u, v)-separator. A separator of a strong digraph D is *minimum* if $|S|$ is the smallest among all separators of D. Thus by definition, for a strong digraph D, $\kappa(D) = |S|$ for any minimum separator of D.

Lemma 2.7: *Let D be a strongly connected digraph which is not spanned by a complete digraph, and $S \subset V(D)$ be a minimum separator of D. Let $H_1, H_2, \ldots, H_c$ be the strong components of $D - S$. Then $c \geq 2$ and*

$$\overline{\kappa}(D) = \max\left\{|S|, \max_{1 \leq i \leq c}\{\overline{\kappa}(D[V(H_i) \cup S])\}\right\}. \tag{3}$$

Proof: Since D is a strong connected digraph and S is a separator, we have $c \geq 2$. Since S is a minimum separator of D, we have $|S| = \kappa(D)$, and so by the definition of $\overline{\kappa}(D)$, and as for each i with $1 \leq i \leq c$, $D[V(H_i) \cup S] \subseteq D$, we have

$$\overline{\kappa}(D) \geq \max\left\{|S|, \max_{1\leq i\leq c}\{\overline{\kappa}(D[V(H_i) \cup S])\}\right\}. \tag{4}$$

Conversely, let H be a subdigraph of D such that $\overline{\kappa}(D) = \kappa(H)$. If for some i with $1 \leq i \leq c$, we have $V(H) \subseteq V(H_i) \cup S$, then $H \subseteq D[V(H_i) \cup S]$. In this case, $\overline{\kappa}(D) = \overline{\kappa}(D[V(H_i) \cup S]) \leq \max_{1\leq i\leq c}\{\overline{\kappa}(D[V(H_i) \cup S])\}$. Now assume that for any i with $1 \leq i \leq c$, $V(H) \subseteq V(H_i) \cup S$ does not hold. This implies that there exists some distinct i and j with $1 \leq i < j \leq c$ such that $V(H) \cap V(H_i) \neq \emptyset$ and $V(H) \cap V(H_j) \neq \emptyset$. Thus S contains a separator of H, and so in this case, $|S| \geq \kappa(H) = \overline{\kappa}(D)$. Hence we have proved that, in any case,

$$\overline{\kappa}(D) \leq \max\left\{|S|, \max_{1\leq i\leq c}\{\overline{\kappa}(D[V(H_i) \cup S])\}\right\}.$$

Thus, together with Equation (4), Lemma 2.7 follows. ■

3. Applications

In [23], Matula indicated that contrasting with the situation that computing the maximum clique of a graph is an NP-complete problem, the values $\overline{\delta}(G)$, $\overline{\kappa}'(G)$ and $\overline{\kappa}(G)$ of a graph G are polynomially determinable. In this section, we shall show that the corresponding computational problems in digraph also have polynomial time solutions. Throughout this section, we always assume that D is a digraph on n vertices and m edges, for some positive integers m and n. The main results stated in Section 2 can be applied to computationally determine the parameters $\overline{\delta}^+(D)$, $\overline{\delta}^-(D)$, $\overline{\lambda}(D)$ and $\overline{\kappa}(D)$.

To generate an arc subset of the form $\partial_D^+(v)$ for a vertex v satisfying $d_D^+(v) = \delta^+(D)$, it takes n steps of vertex scanning, and such a procedure is referred to as a minimum out-degree search. It takes at most $n-1$ minimum out-degree search to generate a minimal $\overline{\delta}^+$-slicing. Thus by Lemma 2.6, it takes $O(n^2)$ time to determine $\overline{\delta}^+(D)$. Similarly, determining $\overline{\delta}^-(D)$ also takes $O(n^2)$ time.

In [4], an $(O(|E(G)|)$-time algorithm is presented that, for any input graph G, and any integer k, the vertices in G that are contained in a subgraph $H \subseteq G$ with $\delta(H) \geq k$. Therefore, this also effectively determines the value $\overline{\delta}(G)$. This algorithm can be adopted and modified in a straightforward way to determine, for an input digraph D, both $\overline{\delta}^+(D)$ and $\overline{\delta}^-(D)$, which is better than the algorithm we presented above for sparse digraphs.

The efficiency of computing $\overline{\kappa}'(G)$ is not so straightforward. In [22,23], Matula showed that utilizing the Ford–Fulkerson network flow algorithm, $\overline{\kappa}'(G)$ can be determined in $O(|V(G)|^{5/3}|E(G)|)$-time. For sparse graphs, this can be improved to $O(|V(G)| \cdot |E(G)|^2)$-time. For digraphs, Schnorr [26] showed that $\lambda(D)$ can be computed in by $O(|V(D)|)$ maximum flow calculations. It is known, with the shortest augmentation path algorithm, each maximum flow calculation runs $O(|V(D)|^2|A(D)|)$ time and outputs a minimum direct cut $J = \partial_D^+(X)$ for some nonempty proper subset X with $|J| = \lambda(D)$. Thus a narrow slicing of D can be found in $O(|V(D)|^4|A(D)|)$ time, and so by Lemma 2.4, $\overline{\lambda}(D)$ can be computationally determined in $O(|V(D)|^4|A(D)|)$ time.

In the rest, we explain how Lemma 2.7 can be applied to obtain a polynomial algorithm to compute $\overline{\kappa}(D)$.

Gabow [12] found an algorithm to determine a separator S of D with $|S| = \kappa(D)$ (this algorithm will be referred to as Gabow's algorithm below). Gabow's algorithm runs in $O(|V(D)|^{5/2}|A(D)|)$-time. Tarjan [29] presented an $O(|V(D)| + |A(D)|)$-algorithm (referred to as Tarjan's algorithm below) to determine the strong components of a digraph D on n vertices and m arcs. Thus by Lemma 2.7, a polynomial algorithm to compute $\overline{\kappa}(D)$ can be found by utilizing Gabow's algorithm and Tarjan's algorithm.

An algorithm computing $\overline{\kappa}$
Input: A digraph D with $n = |V(D)| > 0$ and $m = |A(D)| > 0$.
Output: $\overline{\kappa}(D)$
(Step 1) Set $k := 0$; apply Tarjan's algorithm to determine the strong components of D; set $\mathcal{L} := \{L$ is a strong component of D with $|V(L)| \geq k + 2\}$.
(Step 2) While $\mathcal{L} \neq \emptyset$,
Find $H \in \mathcal{L}$ so that $|V(H)| = \max\{|V(H_i)| : H_i \in \mathcal{L}\}$.
(Step 2.1) If H is spanned by a complete digraph, then
(Step 2.1.1) **updating *k*:** set $k := \max\{k, |V(H)| - 1\}$, and
(Step 2.1.2) **updating $\mathcal{L}$:** set $\mathcal{L} := \mathcal{L} - \{H\}$.
(Step 2.2) If H is not spanned by a complete digraph, then run Gabow's algorithm to determine a minimum separator $S \subset V(H)$ of H.
(Step 2.2.1) **updating *k*:** Set $k := \max\{k, |S|\}$.
(Step 2.2.2) **updating $\mathcal{L}$:** Apply Tarjan's algorithm to determine the strong components H_1, $H_2, \ldots, H_{c'}$ of $H-S$. Set $\mathcal{L} := \mathcal{L} - \{H\} \cup \{D[V(H_j) \cup S]: |V(H_j) \cup S| \geq k + 2$, and $1 \leq j \leq c'\}$.

By the rule we update the value of k at Step 2.1.1 or Step 2.2.1, at any time, $k \leq \overline{\kappa}(D)$. By Lemma 2.7, when the algorithm stops, it will output $k = \overline{\kappa}(D)$ for any digraph D. It suffices to show that the algorithm will stop for any inputting digraph D. Define

$$h(\mathcal{L}) = \sum_{L \in \mathcal{L}} (|V(L)| - (\overline{\kappa}(D) + 2)).$$

Thus $h(\mathcal{L}) \leq |V(D)| - \kappa(D)$. By the rule that we update the value of $\mathcal{L}$ at Step 2.1.2 or Step 2.2.2, after each iteration of Step 2, the value of $h(\mathcal{L})$ is reduced by at least 1, and so it takes at most $h(\mathcal{L}) \leq n - \kappa(D)$ iterations executing Step 2. This implies that the algorithm must stop.

At each Step 2 iteration, Gabow's algorithm runs in $O(n^{5/2}m)$-time, and Tarjan's algorithm runs in $O(n + m)$-time. As there will be $O(n)$-time Step 2 iterations, it follows that this algorithm will run in $O(n^{7/2}m)$-time.

The main purpose of this section is to indicate that there exist polynomial algorithms to computationally determine $\overline{\delta}^+(D)$, $\overline{\delta}^-(D)$, $\overline{\lambda}(D)$ and $\overline{\kappa}(D)$. Efforts have not been spent on finding the fastest algorithms to compute these invariants. Improvement on computational complexity can be made with further discussions. As examples, it is known [10,13] that given a digraph D with n vertices and m edges, and an integer k, there exists an algorithm to determine if $\lambda(D) \geq k$ in $O(knm)$ time. Using matroid intersection and based on Edmonds branching theorem, Gabow [11] determines $\lambda(D)$ in $O(\lambda(D)m\log(n^2/m))$ time. These could be applied to improve the complexity of finding a narrow slicing of a digraph D, thereby determining $\overline{\lambda}(D)$ by Lemma 2.4.

4. A minimax theorem in restricted edge-connectivity

In their studies of fault tolerance networks, Esfahanian[8] and Esfahanian and Hakimi [8,9] introduced restricted edge-connectivity of a graph. There has been intensive researches on restricted edge-connectivity, as seen in the recent survey of Xu [30]. An edge-cut X of a graph G is *restricted* if for any $v \in V(G)$, $\partial_G(v) - X \neq \emptyset$. With this definition, not every connected graph may have a restricted edge-cut. Let $\mathcal{F}$ be a family of connected graph such that a graph G is in $\mathcal{F}$ if and only if either G is spanned by a K_3, or G has a vertex $v \in V(G)$ such that $E(G - v) = \emptyset$.

Lemma 4.1: *Let G be a connected graph with $|E(G)| > 0$. Then G does not have a restricted edge-cut if and only if $G \in \mathcal{F}$.*

Proof: Let G be a connected graph with $|E(G)| > 0$ which does not have a restricted edge-cut. Since every graph on two vertices must be in $\mathcal{F}$, we assume that $|V(G)| \geq 3$. Assume that $|V(G)| = 3$ and

G is not spanned by a K_3, then G has a cut vertex v, and so $E(G-v)=\emptyset$, whence $G \in \mathcal{F}$. Thus we assume that $|V(G)| \geq 4$. If G has a path of length at least 3, then G has a restricted edge-cut. Hence every longest path of G has length 2. Since $|V(G)| \geq 4$, G cannot have a cycle of length at least 3. It follows that G must be spanned by a $K_{1,n-1}$, where $n=|V(G)|$. Since G contains no cycles of length at least 3, if $v \in V(G)$ has maximum degree in G, then $E(G-v)=\emptyset$, and so $G \in \mathcal{F}$. Conversely, it follows by definition that every member in $\mathcal{F}$ does not have a restricted edge-cut. ■

Lemma 4.1 indicates that in order to define restricted edge-connectivity of a graph, we need to define restricted edge-cuts of graphs in $\mathcal{F}$. To facilitate the study of restricted edge-connectivity of a graph, we further define that for any $G \in \mathcal{F}$, we define an edge subset $X \subseteq E(G)$ such that it is a restricted edge-cut of G if and only if $|X|=|E(G)|-1$. The restricted edge-connectivity of a nontrivial connected graph G, denoted by $\lambda_2(G)$, is the minimum size of a restricted edge-cut of G. Note that $\lambda_2(K_2)=0$. If $G=K_1$ or if G is not connected, it is natural to define that $\lambda_2(G)=0$. In this section, we will develop the concept of λ_2-slicing of G and prove an analogous minimax duality result that determines the value of $\overline{\lambda_2}(G)=\max\{\lambda_2(H) : H \subseteq G\}$.

4.1. Restricted slicing of a graph

Let G be a connected graph such that $G \notin \mathcal{F}$. A restricted edge-cut S of G is *minimal* if it contains no other restricted edge-cut of G. Thus if S is a minimal restricted edge-cut of G, then $G{-}S$ has exactly two nontrivial connected components G', G''. If $G \in \mathcal{F}$, then for any restricted edge-cut S of G, $G{-}S$ has exactly one nontrivial component isomorphic to K_2. We start with a lemma below.

Lemma 4.2: *Let G be a nontrivial connected graph such that $G \notin \mathcal{F}$. If S is a minimal restricted edge-cut of G such that $G{-}S$ has components G', G'', then*

$$\overline{\lambda_2}(G)=\max\{|S|, \overline{\lambda_2}(G[E(G') \cup S]), \overline{\lambda_2}(G[E(G'') \cup S])\}. \tag{5}$$

Proof: By definition, there exists a connected subgraph H of G such that $\lambda_2(H)=\overline{\lambda_2}(G)$. Since S is a minimal, by definition, we have $H=G$ if and only if $\overline{\lambda_2}(G)=|S|$.

Assume first that $H=G$, or equivalently, $\overline{\lambda_2}(G)=|S|$. Then by definition of $\overline{\lambda_2}(G)$, we have $\overline{\lambda_2}(G) \geq \max\{\overline{\lambda_2}(G[E(G') \cup S]), \overline{\lambda_2}(G[E(G'') \cup S])\}$. Hence Equation (5) holds. Now assume that $H \neq G$. Thus $\overline{\lambda_2}(G) > |S|$. If H is a subgraph of $G[E(G') \cup S]$, then $\overline{\lambda_2}(G)=\lambda_2(H)=\overline{\lambda_2}(G[E(G') \cup S]) \geq \max\{|S|, \overline{\lambda_2}(G[E(G'') \cup S])\}$, whence Equation (5) holds. Thus it suffices to show that either $H \subseteq G[E(G') \cup S])$ or $H \subseteq G[E(G'') \cup S])$.

By contradiction, we assume that H is not a subgraph of $G[E(G') \cup S])$ and H is a subgraph of $G[E(G'') \cup S])$. These imply that $E(H) \cap E(G') \neq \emptyset$ and $E(H) \cap E(G'') \neq \emptyset$. It follows that $S \cap E(H)$ is a restricted edge-cut of H, and so

$$\overline{\lambda_2}(G) > |S| \geq |S \cap E(H)| \geq \lambda_2(H)=\overline{\lambda_2}(G),$$

showing that a contradiction obtains. This contradiction justifies that either $H \subseteq G[E(G') \cup S])$ or $H \subseteq G[E(G'') \cup S])$, and so Equation (5) must hold. ■

We will define the λ_2-slicing of a connected graph G. To do that, we introduce a subroutine as follows.

Subroutine $\Phi(\Gamma, S, F)$.
Input. A graph Γ with nontrivial connected components $H_1, H_2, \ldots, H_t$. Initially set $S=\emptyset$ and $F=\{H_1, H_2, \ldots, H_t\}$.

(S1) Choose $H \in F$ such that $|E(H)|=\max\{E(H_j)| : 1 \leq j \leq t\}$. If $|E(H)| \leq 1$, then set $S=\emptyset$ and stop.

(S2) Assume that $|E(H)| > 1$.

(S2-1) If $H \in F$, then pick any $e_H \in E(H)$, set $S = E(H) - \{e_H\}$ and $F := F - \{H\}$.

(S2-2) If $H \notin F$, then find a restricted edge-cut S of G. Let $G^{(1)}, G^{(2)}, \ldots, G^{(s)}$ be the nontrivial components of $G{-}S$. Define, for $1 \le i \le s$, $H_i' = G[E(G^{(i)}) \cup S]$. Set $S{:=}S$, $F := (F - \{H\}) \cup \{H_1', H_2', \ldots, H_s'\}$.

Output. An edge subset S of Γ such that either $S = \emptyset$, or S is a restricted edge-cut of G, as well as a collection F of graphs, each of which is isomorphic to a subgraph of Γ. ■

With Subroutine $\Phi(\Gamma, S, F)$, we have the following algorithm that generate the λ_2-slicings of G. Given a connected graph G.

Algorithm Slicing. Let G be a connected graph with $G \notin \{K_1, K_2\}$. Initially, we first set $G_0 = G$, $F_0 = \{G_0\}$ and set σ to be the empty sequence.

Apply Subroutine $\Phi(G_0, S_1, F_1)$. If the output $S_1 = \emptyset$, then stop and we conclude that $G \in \{K_1, K_2\}$, and so $\lambda_2(G) = \overline{\lambda_2}(G) = 0$. If $S_1 \neq 0$, the Subroutine $\Phi(G_0, S_1, F_1)$ outputs a restricted edge-cut S_1 of G_0 and a collection F_1 of graphs such that each of which is isomorphic to a subgraph of G. Update $\sigma = (S_1)$ as a one term sequence, and define G_1 to be the graph whose connected components are precisely those graphs in F_1. Thus up to isomorphism, graphs in F_1 are subgraphs of G.

Inductively, assume that $\sigma = (S_1, S_2, \ldots, S_k)$ and the graph G_k are found. We then apply Subroutine $\Phi(G_k, S_{k+1}, F_{k+1})$. If the output $S_{k+1} = \emptyset$, then stop, and we define the current value σ is a λ_2-slicing of G. Otherwise, $S_{k+1} \neq \emptyset$, and we update $\sigma := (S_1, S_2, \ldots, S_k, S_{k+1})$, and define G_{k+1} to be the graph whose connected components are precisely those graphs in F_{k+1}.

We shall show that this algorithm terminates in finite time so that if a connected graph $G \notin \{K_1, K_2\}$, then the algorithm will generate a λ_2-slicing of G. For each current value $F' = \{H_1, H_2, \ldots, H_t\}$, we assume that $|E(H_1)| \ge |E(H_2)| \ge \cdots \ge E(H_t)$. Let $f(F') = \{i : 1 \le i \le t\}$ and $|E(H_i)| = \max\{|E(H_j)|, 1 \le j \le t\}$. After one application of $\Phi(G, S, F)$, without lose of generality, we assume that H_1 is picked by the subroutine. In the execution of (S2-1), H_1 will be removed from the output F; in the execution of (S2-2), as each of the new edge-induced subgraphs has number of edges less than $|E(H_1)|$, we conclude that $f(F') > f(F)$. As $f(F)$ is integral and as each time running the subroutine $\Phi(G, S, F)$, the output value $f(F)$ is strictly less than the input value. The algorithm must terminate in a finite time. For a connected graph G, let $\sigma(G)$ denote the collection of all λ_2-slicings of G.

In the execution of Subroutine $\Phi(G, S, F)$, we do not require, in Step (S2-2), that the restricted edge-cut S to be a minimum one. We now define a similar Subroutine $\Phi'(G, S, F)$ by additionally requiring that in the execution of (S2-2) of the Subroutine $\Phi'(G, S, F)$, the restricted edge-cut S must be minimized. With this new subroutine $\Phi'(G, S, F)$, we again run the algorithm described above to generate λ_2-slicing of G. These slicings will be called the *restricted narrow slicing* or narrow λ_2-slicing of G. Let $\sigma'(G)$ denote the set of all narrow λ_2-slicings of G.

Lemma 4.3: *Let G be a connected graph not in $\{K_1, K_2\}$ and let $\sigma = (S_1, S_2, \ldots, S_s) \in \sigma(G)$. If H is a subgraph of G satisfying $\lambda_2(H) = \overline{\lambda_2}(G)$, then for some j with $1 \le j \le s$, S_j is a restricted edge-cut of H.*

Proof: We argue by induction on $|V(G) - V(H)|$. If $V(G) = V(H)$, then $G{=}H$ and so as S_1 is a restricted edge-cut of G, S_1 is a restricted edge-cut of H. Let h be the smallest integer with $1 \le h \le s$ such that $S_h \cap E(H) \neq \emptyset$.

If S_h is a restricted edge-cut of H, then the lemma is proved. Assume that S_h is not a restricted edge-cut of H. Then by Lemma 4.2 and by Algorithm Slicing, there must be a graph $H' \in F_h$ such that H is a subgraph of H', with $|V(H')| < |V(G)|$. By induction, there must be an index j with $h \le j \le s$ such that S_j is a restricted edge-cut of H. ■

4.2. A minimax theorem of restricted edge-connectivity

Throughout out this subsection, G is assumed to be a connected graph not in $\{K_1, K_2\}$. The main result of this section is the following minimax result.

Theorem 4.4: *Let G be a connected graph not in* $\{K_1, K_2\}$. *Then*

$$\overline{\lambda_2}(G) = \max_{H\subseteq G} \min\{|X| : X \textit{ is a restricted edge-cut of } H\}$$
$$= \min_{\sigma\in\sigma(G)} \max\{|S_i| : 1 \le i \le s, \sigma = (S_1, S_2, \ldots, S_s)\}. \tag{6}$$

Proof: Let $\ell = \min_{\sigma\in\sigma(G)} \max\{|S_i| : 1 \le i \le s, \sigma = (S_1, S_2, \ldots, S_s)\}$. We shall show that both $\overline{\lambda_2}(G) \le \ell$ and $\overline{\lambda_2}(G) \ge \ell$. By definition, there exists a nontrivial subgraph H of G such that $\overline{\lambda_2}(G) = \lambda_2(H)$.

For any $\sigma = (S_1, S_2, \ldots, S_s) \in \sigma(G)$, by Lemma 4.3, there must be an index j with $1 \le j \le s$, S_j is a restricted edge-cut of H. It follows that $\max\{|S_i| : 1 \le i \le s, \sigma = (S_1, S_2, \ldots, S_s)\} \ge |S_j| \ge \lambda_2(H) = \overline{\lambda_2}(G)$. Since $\sigma \in \sigma(G)$ is arbitrary, we must have $\ell \ge \overline{\lambda_2}(G)$.

Conversely, let $\sigma = (S_1, S_2, \ldots, S_s) \in \sigma'(G)$ be a narrow λ_2-slicing. By Algorithm Slicing, in each iteration, each graph H' in the resulting collection of subgraphs F_i is isomorphic to a subgraph of G. Thus by the definition of a narrow λ_2-slicing, each S_i is a minimum restricted edge-cut of some subgraph of G, and so $\overline{\lambda_2}(G) \ge |S_i|$ for each i with $1 \le i \le s$. It follows that

$$\overline{\lambda_2}(G) \ge \min_{\sigma\in\sigma'(G)} \max\{|S_i| : 1 \le i \le s, \sigma = (S_1, S_2, \ldots, S_s)\} \ge \ell.$$

This completes the proof of the theorem. ∎

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References

[1] J. Anderson, *A study of arc strong connectivity of digraphs*, Ph.D. dissertation, West Virginia University, 2017.
[2] J. Anderson, H.-J. Lai, X. Lin, and M. Xu, *On k-maximal strength digraphs*, J. Graph Theory 84 (2017), pp. 17–25.
[3] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, 2nd ed., Springer-Verlag, London, 2009.
[4] V. Batagelj and M. Zaversnik, *An $O(m)$ algorithm for cores decomposition of networks*, preprint (2003). Available at arXiv preprint cs/0310049.
[5] J.A. Bondy and U.S.R. Murty, *Graph Theory*, Springer, New York, 2008.
[6] P. Erdös, *On the structure of linear graphs*, Israel J. Math. 1 (1963), pp. 156–160.
[7] P. Erdös and A. Rényi, *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kutató Int. K`ozl. 5 (1960), pp. 17–61.
[8] A.H. Esfahanian, *Generalized measures of fault tolerance with application to N-cube networks*, IEEE Trans. Comput. 38 (1989), pp. 1586–1591.

[9] A.H. Esfahanian and S.L. Hakimi, *On computing a conditional edge-connectivity of a graph*, Inf. Process. Lett. 27 (1988), pp. 195–199.

[10] S. Even, *Graph Algorithms*, Computer Science Press, Woodland Hills, CA, 1979.

[11] H.N. Gabow, *A matroid approach to finding edge-connectivity and packing arborescences*, J. Comput. Syst. Sci. 50(2) (1995), pp. 259–273.

[12] H.N. Gabow, *Using expander graphs to find vertex connectivity*, J. Assoc. Comput. Mach. 53 (2006), pp. 800–844.

[13] Z. Galil, *Finding the vertex connectivity of graphs*, SIAM J. Comput. 9(1) (1980), pp. 197–199.

[14] C. Giatsidis, D. Thilikos, and M. Vazirgiannis, *Evaluating cooperation in communities with the k-core structure*, in 2011 International Conference on Advances in Social Networks Analysis and Mining (ASONAM), IEEE, Kaohsiung, Taiwan, 2011, pp. 87–93.

[15] A.M. Hobbs, *Network survivability*, in *Applications of Discrete Mathematics*, J.G. Michaels and K.H. Rosen, eds., McGraw-Hill, Inc., New York, 1991, pp. 332–353.

[16] T.R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley and Sons, New York, 1995, pp. 210–211.

[17] P. Li, H.-J. Lai, and M. Xu, *Disjoint spanning arborescences in k-arc-strong digraphs*, Ars Combin., accepted.

[18] D.R. Lick and A.T. White, *k-degenerate subgraphs*, Canad. J. Math. 22 (1970), pp. 1082–1096.

[19] X. Lin, S. Fan, H.-J. Lai, and M. Xu, *On the lower bound of k-maximal digraphs*, Discrete Math. 339 (2016), pp. 2500–2510.

[20] D.W. Matula, *A min-max theorem for graphs with application to graph coloring*, SIAM Rev. 10(168) (1968), pp. 481–482.

[21] D.W. Matula, *The cohesive strength of graphs*, in *The Many Facets of Graph Theory*, G. Chartrand and S.F. Kapoor, eds., Lecture Notes in Mathematics, No. 110, Springer-Verlag, Berlin, 1969, pp. 215–221.

[22] D.W. Matula, *K-components, clusters, and slicings in graphs*, SIAM J. Appl. Math. 22 (1972), pp. 459–480.

[23] D.W. Matula, *Subgraph connectivity number of a graph*, in *Theory and Applications of Graphs*, Y. Alavi and D.R. Ricks, eds., Lecture Notes in Mathematics, No. 642, Springer-Verlag, Berlin, 1978, pp. 371–383.

[24] J.G. Oxley, *Colouring, packing and the critical problem*, Quart. J. Math. Oxford Ser. (2) 29 (1978), pp. 11–22.

[25] B. Pitte, J. Spencer, and N. Wormald, *Sudden emergence of a giant k-core in a random graph*, J. Comb. Theory Ser. B 67 (1996), pp. 111–151.

[26] C.-P. Schnorr, *Bottlenecks and edge connectivity in unsymmetrical networks*, SIAM J. Comput. 8(2) (1979), pp. 265–274.

[27] S.B. Seidman, *Network structure and minimum degree*, Soc. Networks 5(3) (1983), pp. 269–287.

[28] G. Szekeres and H.S. Wilf, *An inequality for the chromatic number of a graph*, J. Comb. Theory 4 (1968), pp. 1–3.

[29] R. Tarjan, *Depth-first search and linear graph algorithms*, SIAM J. Comput. 2 (1972), pp. 146–160.

[30] J.M. Xu, Super or restricted connectivity of graphs, a survey, preprint.