



## Degree sum and hamiltonian-connected line graphs

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### ARTICLE INFO

#### Article history:

Received 27 July 2016

Received in revised form 6 January 2018

Accepted 5 February 2018

Available online 9 March 2018

#### Keywords:

Hamiltonian-connected

Line graphs

Spanning trailable graphs

Collapsible graphs

Reduction

### ABSTRACT

In 1984, Bauer proposed the problems of determining best possible sufficient conditions on the vertex degrees of a simple graph (or a simple bipartite graph, or a simple triangle-free graph, respectively)  $G$  to ensure that its line graph  $L(G)$  is hamiltonian. We investigate the problems of determining best possible sufficient conditions on the vertex degrees of a simple graph  $G$  to ensure that its line graph  $L(G)$  is hamiltonian-connected, and prove the following.

(i) For any real numbers  $a, b$  with  $0 < a < 1$ , there exists a finite family  $\mathcal{F}(a, b)$  such that for any connected simple graph  $G$  on  $n$  vertices, if  $d_G(u) + d_G(v) \geq an + b$  for any  $u, v \in V(G)$  with  $uv \notin E(G)$ , then either  $L(G)$  is hamiltonian-connected, or  $\kappa(L(G)) \leq 2$ , or  $L(G)$  is not hamiltonian-connected,  $\kappa(L(G)) \geq 3$  and  $G$  is contractible to a member in  $\mathcal{F}(a, b)$ .

(ii) Let  $G$  be a connected simple graph on  $n$  vertices. If  $d_G(u) + d_G(v) \geq \frac{n}{4} - 2$  for any  $u, v \in V(G)$  with  $uv \notin E(G)$ , then for sufficiently large  $n$ , either  $L(G)$  is hamiltonian-connected, or  $\kappa(L(G)) \leq 2$ , or  $L(G)$  is not hamiltonian-connected,  $\kappa(L(G)) \geq 3$  and  $G$  is contractible to  $W_8$ , the Wagner graph.

(iii) Let  $G$  be a connected simple triangle-free (or bipartite) graph on  $n$  vertices. If  $d_G(u) + d_G(v) \geq \frac{n}{8}$  for any  $u, v \in V(G)$  with  $uv \notin E(G)$ , then for sufficiently large  $n$ , either  $L(G)$  is hamiltonian-connected, or  $\kappa(L(G)) \leq 2$ , or  $L(G)$  is not hamiltonian-connected,  $\kappa(L(G)) \geq 3$  and  $G$  is contractible to  $W_8$ , the Wagner graph.

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### 1. The problem

We consider finite loopless graphs but multiple edges are permitted and follow [4] for undefined terms and notation. As in [4],  $\kappa(G)$  and  $\kappa'(G)$  denote the connectivity and edge-connectivity of a graph  $G$ , respectively. We define  $\kappa'(K_1) = \infty$ . An edge cut with size  $k$  is called a  $k$ -edge-cut. For an integer  $i \geq 0$ , we define  $V_i(G) = \{v \in V(G) \mid d_G(v) = i\}$  and  $d_i(G) = |V_i(G)|$ . For vertices  $u, v \in V(G)$ , a  $(u, v)$ -path (a  $(u, v)$ -trail, respectively) is a path (a trail, respectively) from  $u$  to  $v$ . A graph is hamiltonian if it has a spanning cycle, and is **hamiltonian-connected** if for any distinct vertices  $u$  and  $v$ ,  $G$  contains a spanning  $(u, v)$ -path. It is well known that every hamiltonian-connected graph must be 3-connected. The **line graph** of a graph  $G$ , denoted by  $L(G)$ , has  $E(G)$  as its vertex set, where two vertices in  $L(G)$  are adjacent if and only if the corresponding edges in  $G$  have at least one vertex in common.

If  $X \subseteq E(G)$ , the **contraction**  $G/X$  is the graph obtained from  $G$  by identifying the two ends of each edge in  $X$  and then deleting the resulting loops. We define  $G/\emptyset = G$ . If  $H$  is a subgraph of  $G$ , we write  $G/H$  for  $G/E(H)$ . If  $H$  is a connected subgraph of  $G$  and  $v_H$  is the vertex in  $G/H$  onto which  $H$  is contracted, then  $H$  is the **preimage** of  $v_H$  and is denoted by  $PI_G(v_H)$ .

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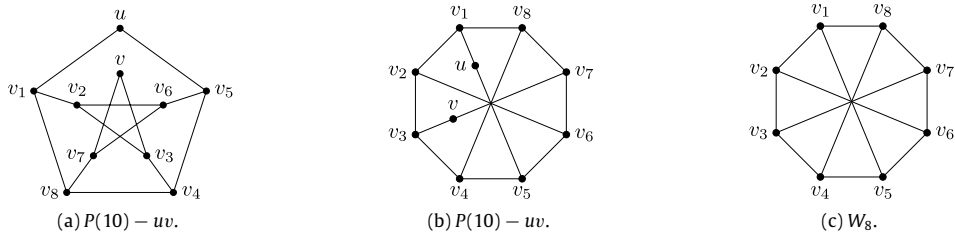


Fig. 1. Wagner graph and related graphs.

In [1,2], Bauer proposed the problems of determining best possible sufficient conditions on the vertex degrees of a simple graph (or a simple bipartite graph, or a simple triangle-free graph, respectively)  $G$  to ensure that its line graph  $L(G)$  is hamiltonian. These problems have been settled by Catlin [6] and Lai [15]. Similar problems are considered in this paper. We seek best possible sufficient degree conditions of a simple graph  $G$  to assure that  $L(G)$  is hamiltonian-connected. The graph  $W_8$  depicted in Fig. 1(c) is called the **Wagner graph**. Our main results in this paper are the following.

**Theorem 1.1.** *Let  $n \geq 3$  be an integer. For any real numbers  $a, b$  with  $0 < a < 1$ , there exists a family  $\mathcal{F}(a, b)$  of finitely many graphs each of which has a non-hamiltonian-connected line graph, such that for any connected simple graph  $G$  on  $n$  vertices, if*

$$d_G(u) + d_G(v) \geq an + b \text{ for any } u, v \in V(G) \text{ with } uv \notin E(G), \tag{1.1}$$

then exactly one of the following must hold:

- (i)  $L(G)$  is hamiltonian-connected;
- (ii)  $\kappa(L(G)) \leq 2$ ;
- (iii)  $L(G)$  is not hamiltonian-connected,  $\kappa(L(G)) \geq 3$  and  $G$  is contractible to a member in  $\mathcal{F}(a, b)$ .

**Theorem 1.2.** *Let  $n \geq 3$  be an integer, and  $G$  be a connected simple graph on  $n$  vertices. If*

$$d_G(u) + d_G(v) \geq \frac{n}{4} - 2 \text{ for any } u, v \in V(G) \text{ with } uv \notin E(G), \tag{1.2}$$

then for sufficiently large  $n$ , exactly one of the following must hold:

- (i)  $L(G)$  is hamiltonian-connected;
- (ii)  $\kappa(L(G)) \leq 2$ ;
- (iii)  $L(G)$  is not hamiltonian-connected,  $\kappa(L(G)) \geq 3$  and  $G$  is contractible to  $W_8$ .

**Theorem 1.3.** *Let  $G$  be a connected simple triangle-free (or bipartite) graph on  $n$  vertices. If*

$$d_G(u) + d_G(v) \geq \frac{n}{8} \text{ for any } u, v \in V(G) \text{ with } uv \notin E(G), \tag{1.3}$$

then for sufficiently large  $n$ , exactly one of the following must hold:

- (i)  $L(G)$  is hamiltonian-connected;
- (ii)  $\kappa(L(G)) \leq 2$ ;
- (iii)  $L(G)$  is not hamiltonian-connected,  $\kappa(L(G)) \geq 3$  and  $G$  is contractible to  $W_8$ .

In the next section, we present our associate results and develop some needed tools. In Section 3, we assume the truth of the associate results to prove our main results on hamiltonian-connected line graphs. The proofs for our associate results will be given in the last section.

## 2. Strongly spanning trailable graphs

For a graph  $G$ , let  $O(G)$  denote the set of odd degree vertices in  $G$ . A graph  $G$  is **eulerian** if  $G$  is connected with  $O(G) = \emptyset$ , and is **supereulerian** if  $G$  has a spanning eulerian subgraph. Supereulerian graphs are first introduced by Boesch, Suffel, and Tindell in [3], and are closely related to the study of hamiltonian line graphs. Catlin [7] presented the first survey on supereulerian graphs. Supplemented or updated surveys on supereulerian graphs can be found in [11,16].

A graph  $G$  is **collapsible** if for any subset  $R \subseteq V(G)$  with  $|R| \equiv 0 \pmod{2}$ ,  $G$  has a spanning connected subgraph  $H$  such that  $O(H) = R$ . If  $G$  is collapsible, then by definition with  $R = \emptyset$ ,  $G$  is supereulerian and so  $\kappa'(G) \geq 2$ . In [6], Catlin showed that for any graph  $G$ , every vertex of  $G$  lies in a unique maximal collapsible subgraph of  $G$ . The **reduction** of  $G$ , denoted by  $G'$ , is obtained from  $G$  by contracting all nontrivial maximal collapsible subgraphs of  $G$ . A graph is **reduced** if it is the reduction of some graph. As shown in [6], a reduced graph is simple.

For  $u, v \in V(G)$ , a  $(u, v)$ -**trail** is a trail of  $G$  from  $u$  to  $v$ . Thus a  $(u, u)$ -trail is an eulerian subgraph of  $G$ . For  $e, e' \in E(G)$ , an  $(e, e')$ -trail is a trail of  $G$  having end-edges  $e$  and  $e'$ . An  $(e, e')$ -trail  $T$  is **dominating** if each edge of  $G$  is incident with at least one internal vertex of  $T$ ; and  $T$  is **spanning** if  $T$  is a dominating trail with  $V(T) = V(G)$ . A graph  $G$  is **spanning trailable** if for each pair of edges  $e_1$  and  $e_2$ ,  $G$  has a spanning  $(e_1, e_2)$ -trail.

Suppose that  $e = u_1v_1$  and  $e' = u_2v_2$  are two edges of  $G$ . If  $e \neq e'$ , then the graph  $G(e, e')$  is obtained from  $G$  by replacing  $e = u_1v_1$  with a path  $u_1v_e v_1$  and by replacing  $e' = u_2v_2$  with a path  $u_2v_{e'} v_2$ , where  $v_e, v_{e'}$  are two new vertices not in  $V(G)$ . If  $e = e'$ , then  $G(e, e')$ , also denoted by  $G(e)$ , is obtained from  $G$  by replacing  $e = u_1v_1$  with a path  $u_1v_e v_1$ . For the recovering operation, we let  $c_e(G(e, e'))$  be the graph obtained from  $G(e, e')$  by replacing the path  $u_1v_e v_1$  with the edge  $e = u_1v_1$ . Thus,  $c_e(c_e(G(e, e'))) = G$ .

For a graph  $G$  and an integer  $k > 0$ , a  $k$ -edge-cut  $X$  of  $G$  is an **essential  $k$ -edge-cut** of  $G$  if each side of  $G - X$  has at least one edge. If a connected graph  $G$  does not have an essential  $k'$ -edge-cut for any  $k' < k$ , then  $G$  is **essentially  $k$ -edge-connected**. Thus when  $L(G)$  is not a complete graph,  $L(G)$  is  $k$ -connected if and only if  $G$  is essentially  $k$ -edge-connected. The largest integer  $k$  such that  $G$  is essentially  $k$ -edge-connected is denoted by  $ess'(G)$ .

As defined in [19], a graph  $G$  is **strongly spanning trailable** if for any  $e, e' \in E(G)$ ,  $G(e, e')$  has a  $(v_e, v_{e'})$ -trail  $T$  with  $V(G) = V(T) - \{v_e, v_{e'}\}$ . Since  $e = e'$  is possible, strongly spanning trailable graphs are both spanning trailable and supereulerian. The theorem below indicates that the study of strongly spanning trailable graphs should be focused on graphs with edge-connectivity less than 4.

**Theorem 2.1** (Luo et al. [20], see also Theorem 4 of [9]). *If  $\kappa'(G) \geq 4$ , then  $G$  is strongly spanning trailable.*

As the Wagner graph  $W_8$  is spanning trailable but not strongly spanning trailable [23], strongly spanning trailable graphs and spanning trailable graphs are not equivalent.

### 2.1. Associate results on strongly spanning trailable graphs

Harary and Nash-Williams showed that there is a close relationship between a graph and its line graph concerning Hamilton cycles.

**Theorem 2.2** (Harary and Nash-Williams, [13]). *Let  $G$  be a graph with  $|E(G)| \geq 3$ . Then  $L(G)$  is hamiltonian if and only if  $G$  has an eulerian subgraph  $H$  with  $E(G - V(H)) = \emptyset$ .*

Let  $G$  be a graph such that  $\kappa(L(G)) \geq 3$  and  $G \neq K_{1,n-1}$ . The **core** of this graph  $G$ , denoted by  $G_0$ , is obtained from  $G - V_1(G)$  by contracting exactly one edge  $xy$  or  $yz$  for each path  $xyz$  in  $G$  with  $d_G(y) = 2$ . Lemma 2.3(iii) is proved by using a similar argument in the proof of Theorem 2.2.

**Lemma 2.3** (Shao, [22]). *Let  $G$  be a connected nontrivial graph such that  $\kappa(L(G)) \geq 3$ , and let  $G_0$  denote the core of  $G$ .*

- (i)  $G_0$  is uniquely determined by  $G$  with  $\kappa'(G_0) \geq 3$ .
- (ii) (see also Lemma 2.9 of [17]) *If for any  $e, e' \in E(G_0)$ ,  $G_0(e, e')$  has a spanning  $(v_e, v_{e'})$ -trail, then  $L(G)$  is hamiltonian-connected.*
- (iii) (see also Proposition 2.2 of [17])  *$L(G)$  is hamiltonian-connected if and only if for any pair of edges  $e, e' \in E(G)$ ,  $G$  has a dominating  $(e, e')$ -trail.*

Lemma 2.3 indicates that the study of hamiltonian-connected line graphs is closely related to the study of strongly spanning trailable graphs. For an integer  $B > 0$ , define

$$S_0(B) = \{G : G \text{ is reduced with } |V(G)| \leq B \text{ and with vertices } u, v \in V(G) \text{ such that } G \text{ has no spanning } (u, v)\text{-trails}\}. \tag{2.1}$$

Since a reduced graph is simple, the number of reduced graphs with at most  $B$  vertices is finite, and so  $S_0(B)$  is a finite family.

Let  $v_e$  and  $v_{e'}$  be the two vertices in  $G(e, e')$  obtained by subdividing the edges  $e, e'$  in  $G$ , respectively. Let  $G(e, e')$  be the reduction of  $G(e, e')$ . Define

$$c(G(e, e')) = \begin{cases} G(e, e') & \text{if } v_e, v_{e'} \notin V(G(e, e')), \\ c_e(G(e, e')) & \text{if } v_e \in V(G(e, e')), v_{e'} \notin V(G(e, e')), \\ c_{e'}(G(e, e')) & \text{if } v_{e'} \in V(G(e, e')), v_e \notin V(G(e, e')), \\ c_e(c_{e'}(G(e, e'))) & \text{if } \{v_e, v_{e'}\} \subset V(G(e, e')). \end{cases}$$

Let  $\mathcal{N} = \{G : G \text{ is not a strongly spanning trailable graph}\}$ , and define

$$S(B) = \{\Gamma : \Gamma = c(G(e, e')), \text{ for some } G \in \mathcal{N} \text{ and } e, e' \in E(G), G(e, e') \in S_0(B)\}. \tag{2.2}$$

Since  $G(e, e') \in S_0(B)$  and since  $S_0(B)$  is a finite family,  $S(B)$  is also finite. In order to prove our main results in this paper, we shall prove the following associate results on strongly spanning trailable graphs.

**Theorem 2.4.** Let  $G$  be a connected simple graph on  $n$  vertices with  $\kappa'(G) \geq 2$ ,  $\text{ess}'(G) \geq 3$  and  $|V_2(G)| \leq 1$ . For any real numbers  $a$  and  $b$  with  $0 < a < 1$ , there exists an integer  $B = \max\{\lceil \frac{12-b}{a} \rceil, \lceil \frac{-(b+2)(a+2)}{a^2} \rceil, \lceil 19 + \frac{6}{a} \rceil, 5\}$  such that if (1.1) holds, then either  $G$  is strongly spanning trailable or  $G$  is contractible to a member in  $S(B)$ .

**Theorem 2.5.** Let  $G$  be a connected simple graph on  $n \geq 217$  vertices with  $\kappa'(G) \geq 2$ ,  $\text{ess}'(G) \geq 3$  and  $|V_2(G)| \leq 1$ . If (1.2) holds, then either  $G$  is strongly spanning trailable or  $G$  is contractible to  $W_8$  with  $n \equiv 0 \pmod{8}$  in such a way that the preimage of every vertex of  $W_8$  is the complete graph  $K_{\frac{n}{8}}$  or  $K_{\frac{n}{8}} - e$  for some  $e \in E(K_{\frac{n}{8}})$ .

**Theorem 2.6.** Let  $G$  be a connected simple triangle-free (or bipartite) graph on  $n \geq 577$  vertices with  $\kappa'(G) \geq 2$ ,  $\text{ess}'(G) \geq 3$  and  $|V_2(G)| \leq 1$ . If (1.3) holds, then either  $G$  is strongly spanning trailable or  $G$  is contractible to  $W_8$  with  $n \equiv 0 \pmod{16}$  in such a way that the preimage of every vertex of  $W_8$  is the complete bipartite graph  $K_{\frac{n}{16}, \frac{n}{16}}$  or  $K_{\frac{n}{16}, \frac{n}{16}} - e$  for some  $e \in E(K_{\frac{n}{16}, \frac{n}{16}})$ .

2.2. Some tools

Let  $F(G)$  be the minimum number of additional edges that must be added to  $G$  such that the resulting graph has two edge-disjoint spanning trees. We use the symbols  $P(10)$  and  $P(10)(e)$  to denote the Petersen graph and the graph obtained from  $P(10)$  by subdividing an edge, respectively. The following lemma summarizes some properties of collapsible graphs and reduced graphs.

**Lemma 2.7.** Let  $G$  be a connected graph and  $G'$  be the reduction of  $G$ . For integer  $k \geq 2$ , let  $C_k$  denote a cycle on  $k$  vertices. Then each of the following holds.

- (i) [6] The graph  $C_2$  is collapsible. Moreover, if  $n = 1$  or  $n \geq 3$ , then the complete graph  $K_n$  is also collapsible.
- (ii) (Lemma 1 of [5]) Every subdivision of  $K_4$  with at most 6 vertices is collapsible. Particularly,  $K_{3,3}^-$  is collapsible, where  $K_{3,3}^-$  is the graph obtained from  $K_{3,3}$  by deleting an edge.
- (iii) (Theorem 3 and 8 of [6]) Let  $H$  be a collapsible subgraph of  $G$ . Then  $G$  is collapsible if and only if  $G/H$  is collapsible. Particularly,  $G$  is collapsible if and only if  $G' = K_1$ .
- (iv) (Lemma 2.3 of [8]) If  $G$  is reduced, then  $F(G) = 2|V(G)| - |E(G)| - 2$ .
- (v) (Theorem 1.3 of [8]) If  $F(G) \leq 2$ , then  $G' \in \{K_1, K_2\} \cup \{K_{2,t} : t \geq 1\}$ .
- (vi) [23] Let  $G$  be a connected simple graph with  $n$  ( $n \leq 12$ ) vertices,  $d_1(G) = 0$  and  $d_2(G) \leq 1$ . Then either  $G' \in \{K_1, K_2, K_{1,2}, K_{2,3}, K_{2,3}^+, P(10), P(10)(e)\}$  or  $G$  is a super-eulerian graph on 12 vertices, where  $K_{2,3}^+$  is a graph obtained from  $K_{2,3}$  by adding a pendant edge to a vertex with degree 2 of  $K_{2,3}$ .
- (vii) [18] Let  $G$  be a connected simple graph with  $|V(G)| \leq 8$ ,  $V_1(G) = \emptyset$  and  $|V_2(G)| \leq 2$ . Then the reduction of  $G$  is  $K_1$  or  $K_2$ .

**Lemma 2.8** (Lemma 2.2 of [18]). If  $G$  is collapsible, then for any  $u, v \in V(G)$ ,  $G$  has a spanning  $(u, v)$ -trail.

**Lemma 2.9** (Lemma 2.5 of [19], see also [23]). Let  $e, e' \in E(G)$ ,  $H$  be a collapsible subgraph of  $G(e, e')$  and  $v_H$  denote the vertex in  $G(e, e')/H$  onto which  $H$  is contracted. Define

$$v'_e = \begin{cases} v_e & \text{if } v_e \notin V(H), \\ v_H & \text{if } v_e \in V(H), \end{cases} \quad \text{and} \quad v'_{e'} = \begin{cases} v_{e'} & \text{if } v_{e'} \notin V(H), \\ v_H & \text{if } v_{e'} \in V(H). \end{cases} \tag{2.3}$$

If  $G(e, e')/H$  has a spanning  $(v'_e, v'_{e'})$ -trail, then  $G(e, e')$  has a spanning  $(v_e, v_{e'})$ -trail.

**Lemma 2.10** ([23]). If  $\kappa'(G) \geq 3$  and if  $G$  is not strongly spanning trailable, then  $|V(G)| \geq 8$ , and  $|V(G)| = 8$  if and only if  $G \cong W_8$ .

**Lemma 2.11.** Let  $G$  be a connected simple graph with  $\kappa'(G) \geq 2$ ,  $\text{ess}'(G) \geq 3$  and  $|V_2(G)| \leq 1$ . Let  $e, e' \in E(G)$  and  $G(e, e')$  be the reduction of  $G(e, e')$ . Denote  $d_i = |V_i(G(e, e'))|$ . Then each of the following holds.

- (i)  $2F(G(e, e')) = 4|V(G(e, e'))| - 2|E(G(e, e'))| - 4 = \sum_{i \geq 2} (4 - i)d_i - 4$ .
- (ii) If  $G(e, e')$  has no spanning  $(v_e, v_{e'})$ -trails, then  $F(G(e, e')) \geq 3$  and  $2d_2 + d_3 \geq 10 + \sum_{i \geq 5} (i - 4)d_i$ .

**Proof.** As  $d_1 = 0$  follows from  $\kappa'(G) \geq 2$ , we have  $|V(G(e, e'))| = \sum_{i \geq 2} d_i$  and  $2|E(G(e, e'))| = \sum_{i \geq 2} id_i$ . By Lemma 2.7(iv), (i) holds immediately.

Assume that  $G(e, e')$  has no spanning  $(v_e, v_{e'})$ -trails. Now we show that  $F(G(e, e')) \geq 3$ . By Lemma 2.7(v), it suffices to show that  $G(e, e') \notin \{K_1, K_2\} \cup \{K_{2,t} : t \geq 1\}$ . If  $G(e, e') = K_1$ , then by Lemma 2.7(iii),  $G(e, e')$  is collapsible. Then by Lemma 2.8,  $G(e, e')$  has a spanning  $(v_e, v_{e'})$ -trail, contrary to the assumption. Hence  $G(e, e') \neq K_1$ .

Since  $\kappa'(G) \geq 2$ , we have  $\kappa'(G(e, e')) \geq 2$  and so  $G(e, e') \notin \{K_2, K_{2,1}\}$ . Since  $\text{ess}'(G) \geq 3$  and  $|V_2(G)| \leq 1$ ,  $G(e, e')$  has at most three vertices of degree 2. Hence  $G(e, e') \notin \{K_{2,t} : t \geq 1, t \neq 3\}$ . If  $G(e, e') = K_{2,3}$ , then for any  $u, v \in V_2(G(e, e'))$ ,  $G(e, e')$  has a spanning  $(u, v)$ -trail. It follows by Lemma 2.9 that  $G(e, e')$  has a spanning  $(v_e, v_{e'})$ -trail, a contradiction. Hence  $F(G(e, e')) \geq 3$ . This completes the proof of Lemma 2.11. ■

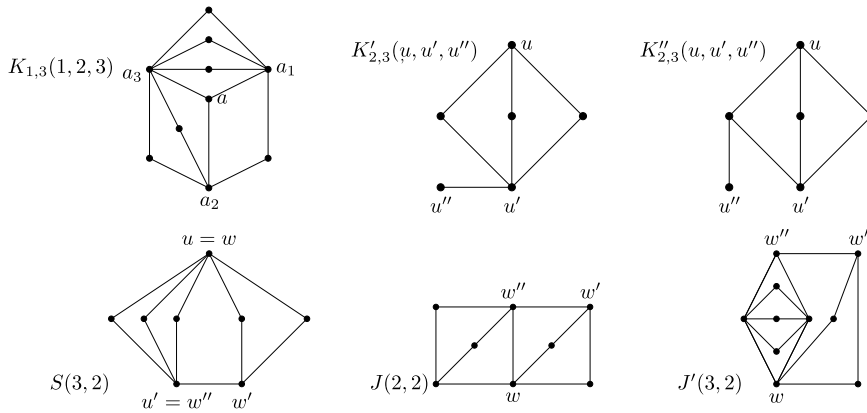


Fig. 2. Some graphs in Definition 2.13 with small parameters.

**Lemma 2.12.** Suppose that  $G$  is a simple graph with  $|V(G)| \leq 10$  and  $\kappa'(G) \geq 3$ . For any  $e_0 \in E(G)$ , if  $G(e_0)$  is not collapsible, then  $G(e_0) = P(10)(e_0)$ .

**Proof.** Suppose that  $G$  and  $G(e_0)$  are the graphs satisfying the conditions of Lemma 2.12. Then  $|V(G(e_0))| \leq 11$ ,  $d_1(G(e_0)) = 0$  and  $d_2(G(e_0)) = 1$ . Let  $H$  be the reduction of  $G(e_0)$ . By Lemma 2.7(vi),  $H \in \{K_1, K_2, K_{1,2}, K_{2,3}, K_{2,3}^+, P(10), P(10)(e_0)\}$ . Since  $G(e_0)$  is not collapsible,  $H \neq K_1$ . By  $\kappa'(G) \geq 3$ , we have  $\kappa'(G(e_0)) \geq 2$  and  $\kappa'(H) \geq 2$ . Hence  $H \notin \{K_2, K_{1,2}, K_{2,3}^+\}$ . Since  $\kappa'(G) \geq 3$ ,  $G(e_0)$  has a unique 2-edge-cut, and then  $H$  has at most one 2-edge-cut. Thus  $H \neq K_{2,3}$  and so  $H \in \{P(10), P(10)(e_0)\}$ . If  $G(e_0) \neq H$ , then as every collapsible simple graph has at least 3 vertices,  $|V(H)| \leq |V(G(e_0))| - 2 = 9$ . This forces that  $G(e_0) = H = P(10)(e_0)$ . ■

**Definition 2.13** ([12]). Let  $s_1, s_2, s_3, m, l, t$  be natural numbers with  $m, l, t \geq 1$ .

- (i) Let  $M \cong K_{1,3}$  with center  $a$  and ends  $a_1, a_2, a_3$ . Define  $K_{1,3}(s_1, s_2, s_3)$  to be the graph obtained from  $M$  by adding  $s_i$  vertices with neighbors  $\{a_i, a_{i+1}\}$ , where  $i \equiv 1, 2, 3 \pmod{3}$ ;
- (ii) Let  $K_{2,t}(u, u')$  be a  $K_{2,t}$  with  $u, u'$  being the nonadjacent vertices of degree  $t$ . Let  $K'_{2,t}(u, u', u'')$  be the graph obtained from a  $K_{2,t}(u, u')$  by adding a new vertex  $u''$  that joins to  $u'$  only. Hence  $u''$  has degree 1 and  $u$  has degree  $t$  in  $K'_{2,t}(u, u', u'')$ ;
- (iii) Let  $K''_{2,t}(u, u', u'')$  be the graph obtained from a  $K_{2,t}(u, u')$  by adding a new vertex  $u''$  that joins to a vertex of degree 2 of  $K_{2,t}$ . Hence  $u''$  has degree 1 and both  $u$  and  $u'$  have degree  $t$  in  $K''_{2,t}(u, u', u'')$ . We shall use  $K'_{2,t}$  and  $K''_{2,t}$  for a  $K'_{2,t}(u, u', u'')$  and a  $K''_{2,t}(u, u', u'')$ , respectively;
- (iv) Let  $S(m, l)$  be the graph obtained from a  $K_{2,m}(u, u')$  and a  $K'_{2,l}(w, w', w'')$  by identifying  $u$  with  $w$ , and  $w''$  with  $u'$ ;
- (v) Let  $J(m, l)$  denote the graph obtained from a  $K_{2,m+1}$  and a  $K'_{2,l}(w, w', w'')$  by identifying  $w$  with  $u$ , and  $w''$  with  $v$ , where  $uv \in E(K_{2,m+1})$ ,  $d_{K_{2,m+1}}(u) = 2$  and  $d_{K_{2,m+1}}(v) = m + 1$ ;
- (vi) Let  $J'(m, l)$  denote the graph obtained from a  $K_{2,m+2}$  and a  $K'_{2,l}(w, w', w'')$  by identifying  $w, w''$  with two vertices of degree 2 in  $K_{2,m+2}$ , respectively.

In Fig. 2, we depict some graphs in Definition 2.13 with small parameters. Let  $\mathcal{F}_0 = \{K_1, K_2, P(10)\} \cup \{K_{2,t}, K'_{2,t}, K''_{2,t} : t \geq 1\} \cup \{K_{1,3}(s_1, s_2, s_3) : s_1, s_2, s_3 \geq 0\} \cup \{S(m, l), J(m, l), J'(m, l) : m, l \geq 1\}$ .

**Lemma 2.14** ([12]). If  $G$  is a connected reduced graph with  $|V(G)| \leq 11$  and  $F(G) \leq 3$ , then  $G \in \mathcal{F}_0$ .

**Lemma 2.15.** Let  $G$  be a connected simple graph with  $\kappa'(G) \geq 2$  and  $|V_2(G)| = 1$ . Let  $G(e, e')$  be the reduction of  $G(e, e')$ . If  $7 \leq |V(G(e, e'))| \leq 9$ ,  $|V_2(G(e, e'))| = 3$  and  $F(G(e, e')) = 3$ , then  $G(e, e') \in \{J'(1, 2), J'(1, 1), K_{1,3}(1, 1, 1)\}$ .

**Proof.** Suppose that  $G$  is a graph satisfying the conditions of Lemma 2.15 and  $G(e, e')$  be the reduction of  $G(e, e')$ . Since  $7 \leq |V(G(e, e'))| \leq 9$  and  $F(G(e, e')) = 3$ , by Lemmas 2.14 and 2.7(v),  $G(e, e') \in \{K'_{2,t}, K''_{2,t} : t \geq 1\} \cup \{K_{1,3}(s_1, s_2, s_3) : s_1, s_2, s_3 \geq 0\} \cup \{S(m, l), J(m, l), J'(m, l) : m, l \geq 1\}$ . Since  $\kappa'(G) \geq 2$ , we have  $\kappa'(G(e, e')) \geq 2$ , which implies that  $G(e, e') \notin \{K'_{2,t}, K''_{2,t} : t \geq 1\}$ . Hence  $G(e, e') \in \{K_{1,3}(s_1, s_2, s_3) : s_1, s_2, s_3 \geq 0\} \cup \{S(m, l), J(m, l), J'(m, l) : m, l \geq 1\}$ . Since  $|V_2(G(e, e'))| = 3$ , it is routine to verify that  $G(e, e') \in \{J'(1, 2), J'(1, 1), K_{1,3}(1, 1, 1)\}$  (see Fig. 3 and the Appendix). ■

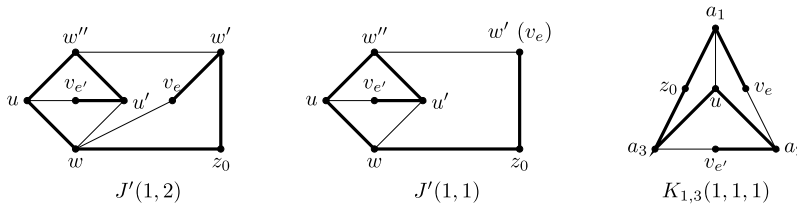


Fig. 3. The graphs in Lemma 2.15.

### 3. Applications of Theorems 2.4–2.6

In this section, we assume the validity of Theorems 2.4–2.6 to prove our main results stated in Theorems 1.1–1.3. Let  $a$  and  $b$  be real numbers with  $a > 0$ . Define a family of connected simple graphs as follows:

$$\mathcal{G}(a, b) = \{G : G \text{ satisfying (1.1) with } n = |V(G)| \geq \max\{\frac{5-b}{a}, 5\} \text{ and } \kappa(L(G)) \geq 3\}.$$

**Lemma 3.1.** *If  $G \in \mathcal{G}(a, b)$ , then  $ess'(G) \geq 3$  and  $|V_1(G) \cup V_2(G)| \leq 1$ .*

**Proof.** Since  $\kappa(L(G)) \geq 3$ , we have  $ess'(G) \geq 3$ . As  $n \geq 4$  and  $\kappa(L(G)) \geq 3$ ,  $V_1(G) \cup V_2(G)$  is an independent set of  $G$ . By contradiction, assume first that  $V_1(G) \cup V_2(G)$  contains two vertices  $u$  and  $v$ . By (1.1),  $4 \geq d_G(u) + d_G(v) \geq an + b$ , implying  $n \leq \frac{4-b}{a}$ , contrary to the assumption that  $n \geq \max\{\frac{5-b}{a}, 5\}$ . Hence we must have  $|V_1(G) \cup V_2(G)| \leq 1$ . ■

Lemma 3.1 motivates the following definition.

**Definition 3.2.** Let  $\mathcal{G} = \{G \in \mathcal{G}(a, b) : V_1(G) = \emptyset \text{ and } |V_2(G)| = 1\}$ . For each  $G \in \mathcal{G}$ , we assume that  $z \in V_2(G)$  with  $N_G(z) = \{z', z''\}$ . Define

- $\mathcal{G}_1 = \{G : G \in \mathcal{G} \text{ and } z'z'' \notin E(G)\},$
- $\mathcal{G}_2 = \{G : G \in \mathcal{G}, z'z'' \in E(G) \text{ and } ess'(G - z) \geq 3\},$
- $\mathcal{G}_3 = \{G : G \in \mathcal{G}, z'z'' \in E(G), ess'(G - z) \leq 2, \text{ and } z'z'' \text{ is not in any 3-cycle of } G - z\},$
- $\mathcal{G}_4 = \{G : G \in \mathcal{G}, z'z'' \in E(G), ess'(G - z) \leq 2, \text{ and } z'z'' \text{ is in a 3-cycle of } G - z\}.$

If  $G \in \mathcal{G}(a, b)$  with  $V_1(G) \neq \emptyset$  or if  $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ , then define

$$G_1 = G - V_1(G) \quad \text{if } V_1(G) \neq \emptyset;$$

$$G_2 = \begin{cases} G/z z'' & \text{if } G \in \mathcal{G}_1, \\ G - z & \text{if } G \in \mathcal{G}_2, \\ G/\{z z', z z'', z' z''\} & \text{if } G \in \mathcal{G}_3. \end{cases} \tag{3.1}$$

The following Lemma 3.3 can be proved by using a similar argument in the proof of Lemma 2.6 of [19] (see also Lemma 3.2.3 of [23]).

**Lemma 3.3.** *Let  $G$  be a graph with  $\kappa'(G) \geq 2$  and  $ess'(G) \geq 3$ , and let  $G_1, G_2, \dots, G_k$  be the blocks of  $G$ . Then the following are equivalent.*

- (i)  $G$  is strongly spanning trailable.
- (ii) For every  $i \in \{1, 2, \dots, k\}$ ,  $G_i$  is strongly spanning trailable.

**Lemma 3.4.** *Let  $G \in \mathcal{G}$ , and let  $z, z', z''$  be defined as in Definition 3.2. Each of the following holds.*

- (A) If  $z'z'' \in E(G)$  and  $ess'(G - z) \leq 2$ , then  $G$  must have the structure as depicted in Fig. 4(a).
- (B) If  $G \in \mathcal{G}_4$ , then there exist connected subgraphs  $H_1, H_2$  of  $G$  such that  $E(G) = E(H_1) \cup E(H_2)$  and  $V(H_1) \cap V(H_2) = \{z'\}$  (as depicted in Fig. 4(c)). Moreover, each of the following holds.
  - (i) For each  $1 \leq i \leq 2$ ,

$$d_{H_i}(u) + d_{H_i}(v) \geq a|V(H_i)| + b \text{ for any } u, v \in V(H_i) \text{ with } uv \notin E(H_i). \tag{3.2}$$

- (ii) If  $G$  is not strongly spanning trailable, then there exists  $H_i$  ( $1 \leq i \leq 2$ ) such that  $H_i$  is not strongly spanning trailable.
- (iii) Fix  $i \in \{1, 2\}$ . Then  $\kappa'(H_i) \geq 2$ ,  $ess'(H_i) \geq 3$  and  $|V_2(H_i)| \leq 1$ ; and if  $H_i$  is contractible to a graph  $\Gamma$ , then  $G$  is also contractible to  $\Gamma$ .

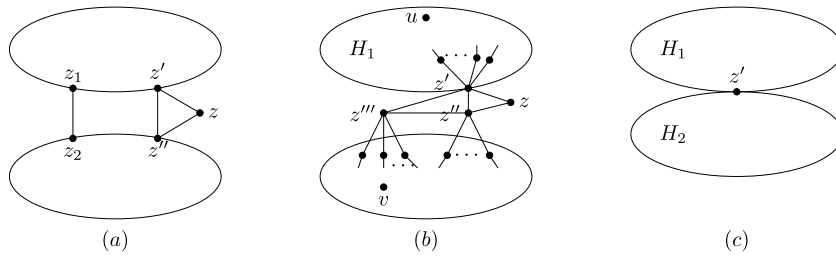


Fig. 4. Graphs in  $\mathcal{G}_3$  and  $\mathcal{G}_4$ .

**Proof.** (A) We use the notation in Definition 3.2. If  $z'z'' \in E(G)$  and  $ess'(G - z) \leq 2$ , then as  $ess'(G) \geq 3$ , there exists an essential edge cut  $X$  in  $G - z$  such that  $|X| = 2$  and  $z'z'' \in X$ , and so Lemma 3.4(A) holds.

(B) Assume that  $G \in \mathcal{G}_4$ , and so  $z'z''$  lies in a 3-cycle  $C'$  of  $G - z$ . As  $ess'(G) \geq 3$  and  $ess'(G - z) \leq 2$ ,  $G - z$  must have an essential 2-edge-cut containing  $z'z''$ , and so  $z'$  or  $z''$  (say  $z'$ ) must be a cut vertex of  $G$ , as depicted in Fig. 4(b). Hence there exist connected subgraphs  $H_1, H_2$  of  $G$  such that  $E(G) = E(H_1) \cup E(H_2)$  and  $V(H_1) \cap V(H_2) = \{z'\}$ . For  $1 \leq i \leq 2$ , to prove  $H_i$  satisfies (3.2), it is sufficient to show that for any

$$u \in V(H_i) \text{ with } uz' \notin E(H_i),$$

$$d_{H_i}(u) + d_{H_i}(z') \geq a|V(H_i)| + b.$$

As for any  $u \in V(H_i)$  with  $uz' \notin E(H_i)$ , we have  $uz \notin E(G)$ . It follows from (1.1) that  $d_G(u) + d_G(z) \geq a|V(G)| + b$ , whence we have  $d_{H_i}(u) = d_G(u) \geq a|V(G)| + b - 2$ . Since  $z'$  is a cut vertex of  $G$  and by  $ess'(G) \geq 3$ , we have  $d_{H_i}(z') \geq 3$ , and so (3.2) holds. This proves Lemma 3.4(B)(i).

As Lemma 3.4(B)(ii) follows from Lemma 3.3, and Lemma 3.4(B)(iii) follows from the definition of contractions and from the assumption that  $|V_2(G)| \leq 1$ , Lemma 3.4 is proved. ■

**Lemma 3.5.** Let  $G \in \mathcal{G}(a, b)$  with  $V_1(G) \neq \emptyset$  or  $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ , and let  $G_1$  and  $G_2$  be defined in (3.1). Each of the following holds.

(i)  $\kappa'(G_1) \geq 2, ess'(G_1) \geq 3, |V_2(G_1)| = 0$  and for any distinct nonadjacent vertices  $u, v \in V(G_1)$ ,

$$d_{G_1}(u) + d_{G_1}(v) \geq a|V(G_1)| + b.$$

(ii)  $\kappa'(G_2) \geq 2, ess'(G_2) \geq 3, |V_2(G_2)| \leq 1$  and for any distinct nonadjacent vertices  $u, v \in V(G_2)$ ,

$$d_{G_2}(u) + d_{G_2}(v) \geq a|V(G_2)| + b.$$

**Proof.** Let  $G \in \mathcal{G}(a, b)$ . By Lemma 3.1, we have  $ess'(G) \geq 3$  and  $|V_1(G) \cup V_2(G)| \leq 1$ .

(i) Suppose that  $V_1(G) \neq \emptyset$ . We may assume that  $V_1(G) \cup V_2(G) = V_1(G) = \{v_1\}$ , and then  $G_1 = G - V_1(G)$ . Since  $ess'(G) \geq 3$ , the degree of the unique vertex adjacent to  $v_1$  is no less than 4. Hence  $|V(G_1)| = n - 1, |V_1(G_1) \cup V_2(G_1)| = 0, \kappa'(G_1) \geq 2$  and  $ess'(G_1) \geq 3$ . Let  $u, v \in V(G_1)$  be two nonadjacent vertices. Then we may assume that  $uv_1 \notin E(G)$ . By (1.1),  $d_{G_1}(u) = d_G(u) \geq an + b - d_G(v_1) = an + b - 1$ . It follows that  $d_{G_1}(u) + d_{G_1}(v) \geq an + b - 1 + d_{G_1}(v) \geq an + b - 1 + 3 > a(n - 1) + b$ . As  $|V(G_1)| = n - 1$ , this proves (i).

(ii) Suppose that  $V_2(G) \neq \emptyset$ . We may assume that  $V_1(G) \cup V_2(G) = V_2(G) = \{z\}$ . By (3.1), we have  $\kappa'(G_2) \geq 2$ . We shall show that in any case,

$$\text{both } ess'(G_2) \geq 3 \text{ and } |V_2(G_2)| \leq 1. \tag{3.3}$$

If  $G \in \mathcal{G}_1$ , then  $G_2 = G/zz''$ , and so (3.3) must hold. Therefore, assume that  $G \in \mathcal{G}_2 \cup \mathcal{G}_3$ . We claim that  $|V_3(G) \cap \{z', z''\}| \leq 1$ . Assume, by contradiction, that  $|V_3(G) \cap \{z', z''\}| = 2$  and there exist vertices  $u', u''$  with  $N_G(z') = \{z, z'', u'\}$  and  $N_G(z'') = \{z, z', u''\}$ . Then  $\{z'u', z''u''\}$  is an essential 2-edge-cut of  $G$ , contrary to  $ess'(G) \geq 3$ . Hence  $|V_3(G) \cap \{z', z''\}| \leq 1$ . If  $G \in \mathcal{G}_2$ , then by (3.1),  $G_2 = G - z$ . Thus  $|V(G_2)| = n - 1, ess'(G_2) = ess'(G - z) \geq 3$  and  $|V_2(G_2)| = |V_3(G) \cap \{z', z''\}| \leq 1$ , and so (3.3) holds also. If  $G \in \mathcal{G}_3$ , then by (3.1),  $G_2 = G/\{zz', zz'', z'z''\}$ . Thus  $|V(G_2)| = n - 2, ess'(G_2) = ess'(G/\{zz', zz'', z'z''\}) \geq 3$  and  $|V_2(G_2)| = 0$ , and so (3.3) holds also.

To complete the proof of (ii), let  $u, v \in V(G_2)$  with  $uv \notin E(G_2)$ . By the definition of  $G_2$ , we may conclude that  $uz \notin E(G)$ . By (1.1),  $d_{G_2}(u) = d_G(u) \geq an + b - d_G(z) = an + b - 2$ . As  $d_{G_2}(v) \geq \kappa'(G_2) \geq 2$ , we have  $d_{G_2}(u) + d_{G_2}(v) \geq an + b - 2 + d_{G_2}(v) \geq an + b > a(n - 1) + b > a(n - 2) + b$ . This, together with (3.3), proves (ii). ■

**Lemma 3.6.** Let  $G_1$  and  $G_2$  be the graphs defined in (3.1) and  $G_0$  be the core of  $G$ . Then  $G_0$  is strongly spanning trailable if one of the following holds:

- (i)  $|V_1(G)| = 1$  and  $G_1$  is strongly spanning trailable, or
- (ii)  $|V_2(G)| = 1$  and  $G_2$  is strongly spanning trailable.

**Proof.** By Definition 3.2 and Lemma 3.1, we observe that if  $|V_1(G)| = 1$ , then  $G_0 = G_1$ , and so if  $G_1$  is strongly spanning trailable, then  $G_0$  is strongly spanning trailable. Assume that  $V_2(G) = \{z\}$  with  $N_G(z) = \{z', z''\}$ . If  $z'z'' \in E(G)$ , then let  $e', e''$  denote the two parallel edges of  $G_0$  with ends  $z'$  and  $z''$ . By (3.1),

$$G_2 = \begin{cases} G_0 & \text{if } G \in \mathcal{G}_1, \\ G_0 - e'' & \text{if } G \in \mathcal{G}_2, \\ G_0/\{e', e''\} & \text{if } G \in \mathcal{G}_3. \end{cases}$$

Hence when  $G \in \mathcal{G}_1$ , if  $G_2$  is strongly spanning trailable, then  $G_0$  is strongly spanning trailable.

Assume from now on that  $G \in \mathcal{G}_2 \cup \mathcal{G}_3$ . Then  $G_0[\{e', e''\}]$  is a 2-cycle. By contradiction, we assume further that

$$\text{there exist edges } e_1, e_2 \in E(G_0) \text{ such that } G_0(e_1, e_2) \text{ does not have a spanning } (v_{e_1}, v_{e_2})\text{-trail.} \tag{3.4}$$

**Case 1.**  $G \in \mathcal{G}_2$ .

If  $|\{e_1, e_2\} \cap \{e', e''\}| \leq 1$ , then any spanning  $(v_{e_1}, v_{e_2})$ -trail of  $G_2(e_1, e_2)$  is a spanning  $(v_{e_1}, v_{e_2})$ -trail of  $G_0(e_1, e_2)$ . Assume now that  $\{e_1, e_2\} = \{e', e''\}$ . Pick an edge  $f = uz' \in E(G_2)$  for some  $u \neq z''$ . As  $G_2$  is strongly spanning trailable,  $G_2(f, e')$  has a spanning  $(v_f, v_{e'})$ -trail  $T'_1$ . Thus exactly one of  $v_f u, v_f z'$  is in  $T'_1$ . Define

$$T_1 = \begin{cases} G_0[(E(T'_1) - \{v_f z'\}) \cup \{v_{e'} z'\}] & \text{if } v_f z' \in E(T'_1), \\ G_0[(E(T'_1) - \{v_f u\}) \cup \{v_{e'} z', f\}] & \text{if } v_f u \in E(T'_1). \end{cases}$$

Then  $T_1$  is a spanning  $(v_{e'}, v_{e''})$ -trail of  $G_0(e_1, e_2)$ , contrary to (3.4). This proves Case 1.

**Case 2.**  $G \in \mathcal{G}_3$ .

Then we use  $K_3$  to denote  $G[\{z, z', z''\}]$  and  $\{e_0, z'z''\}$  to denote an essential edge cut of  $G - z$ . If  $|\{e_1, e_2\} \cap \{e', e''\}| = 0$ ,  $G_2(e_1, e_2) = G_0(e_1, e_2)/C_2$ . As  $C_2$  is collapsible, it follows by Lemma 2.9 that if  $G_2(e_1, e_2)$  has a spanning  $(v_{e_1}, v_{e_2})$ -trail, then  $G_0(e_1, e_2)$  has a spanning  $(v_{e_1}, v_{e_2})$ -trail, contrary to (3.4). If  $|\{e_1, e_2\} \cap \{e', e''\}| = 1$ , then assume that  $e_1 = e'$  and, when  $e_1 \neq e_2$ , that  $e_2 \notin \{e', e''\}$ . Assume first that  $e_1 \neq e_2$ . Pick  $e_3 = uz' \in E(G_2) - \{e_2\}$ . By assumption,  $G_2(e_3, e_2)$  has a spanning  $(v_{e_3}, v_{e_2})$ -trail  $T'_2$ . Define

$$T_2 = \begin{cases} G_2(e_2)[E(T'_2 - uv_{e_3}) \cup \{e_3\}] & \text{if } uv_{e_3} \in E(T'_2), \\ G_2(e_2)[E(T'_2 - z'v_{e_3})] & \text{if } z'v_{e_3} \in E(T'_2). \end{cases}$$

Then  $T_2$  is a spanning  $(z', v_{e_2})$ -trail of  $G_2(e_2)$ . As  $G_2(e_2) = G_0(e_1, e_2)/K_3$  and  $K_3$  is collapsible, it follows by Lemma 2.9 that  $T_2$  can be lifted to a spanning  $(v_{e_1}, v_{e_2})$ -trail of  $G_0(e_1, e_2)$ , contrary to (3.4). If  $e_1 = e_2 = e'$ , then by assumption,  $G_2$  has a spanning eulerian subgraph  $L'_2$ . As  $G_2 = G_0/K_3$  and as  $e' \in E(K_3)$ , by Lemma 2.9,  $L'_2$  can be lifted to a spanning eulerian subgraph  $L_2$  of  $G_0(e_1)$ . This also leads to a contradiction of (3.4). Hence we only need to prove the case when  $\{e_1, e_2\} = \{e', e''\}$ . By assumption,  $G_2(e_0)$  has a spanning closed trail, implying that  $G_2$  has a spanning trail  $T'_3$  containing  $e_0$ . Since  $e_0 \in E(T'_3)$ , each of  $z'$  and  $z''$  is of odd degree in  $G[E(T'_3)]$ . It follows that  $T_3 = G_0(e_1, e_2)[E(T'_3) \cup \{v_{e'} z', v_{e''} z''\}]$  is a spanning  $(v_{e'}, v_{e''})$ -trail of  $G_0(e_1, e_2)$ , contrary to (3.4). This proves Case 2, as well as the lemma. ■

**Lemma 3.7.** Let  $G, G_1, G_2$  be the graphs defined in (3.1). Each of the following holds.

- (i) If  $G_1$  is contractible to a graph  $H$ , then  $G$  is also contractible to  $H$ .
- (ii) If  $G \in \mathcal{G}_1 \cup \mathcal{G}_3$  and  $G_2$  is contractible to a graph  $H$ , then  $G$  is also contractible to  $H$ .
- (iii) If  $G \in \mathcal{G}_2$  and  $G_2$  is contractible to a graph  $H$ , then there is a graph  $H^*$  with  $|V(H^*)| = |V(H)| + 1$  such that  $G$  is contractible to  $H^*$ .
- (iv) If  $G_1$  or  $G_2$  is contractible to a member in a finite family  $\mathcal{S}$ , then there is a finite family  $\mathcal{S}'$  such that  $G$  is contractible to a member in  $\mathcal{S}'$ .

**Proof.** By Definition 3.2, (i) and (ii) hold immediately, and (iv) follows from (i), (ii) and (iii). Suppose that  $G \in \mathcal{G}_2$ . By (3.1),  $G_2 = G - z$ . If  $G_2$  is contractible to a graph  $H$ , then there exists a subset  $X_G \subseteq E(G)$  such that  $(G - z)/X_G = H$  and so  $G/X_G$  is a graph with  $|V(H)| + 1$  vertices. Thus (iii) holds as well. ■

**Proof of Theorem 1.1.** Let  $G$  be a graph satisfying the hypotheses of Theorem 1.1 with  $\kappa(L(G)) \geq 3$  and  $n = |V(G)|$ . We assume that  $L(G)$  is not hamiltonian-connected to prove the existence of  $\mathcal{F}$ . By Lemma 2.3,  $G_0$  is 3-edge-connected and not strongly spanning trailable. By Lemma 2.3,  $G$  is not strongly spanning trailable.

Take  $B = \max\{\frac{5-b}{a}, 5\}$ . If  $n < B$ , clearly there exists a finite family  $\mathcal{S}$  such that  $G$  is contractible to a member in  $\mathcal{S}$ . If  $n \geq \max\{\frac{5-b}{a}, 5\}$ , then  $G \in \mathcal{G}(a, b)$ . We distinguish the following two cases.

Assume first that  $G \in \mathcal{G}_4$ . By Lemma 3.4,  $G$  has connected subgraphs  $H_1$  and  $H_2$  with  $E(G) = E(H_1) \cup E(H_2)$  and  $V(H_1) \cap V(H_2) = \{z'\}$ . By Lemma 3.3, we may assume that  $H_1$  satisfies the conditions of Theorem 2.4 and is not strongly spanning trailable. By Theorem 2.4, there exists a finite family  $\mathcal{S}'$  such that  $H_1$  is contractible to a member in  $\mathcal{S}'$ . Since  $H_1 = G/H_2$ ,  $G$  is also contractible to a member in  $\mathcal{S}'$ .



If  $G \notin \mathcal{G}_4$ , let  $G_1$  and  $G_2$  be the graphs defined in (3.1). By (3.1), if  $V_1(G) \neq \emptyset$ , then  $G_1 = G_0$ ; if  $V_2(G) = \{z\}$  with  $N_G(z) = \{z', z''\}$  and  $z'z'' \notin E(G)$ , then  $G_2 = G_0$ . For notational convenience, we redefine that  $G_1 = G_0$  if  $|V_2(G)| = 0$ . By Lemma 3.6, either  $|V_2(G)| = 0$  and  $G_1$  is not strongly spanning trailable, or  $|V_2(G)| = 1$  and  $G_2$  is not strongly spanning trailable. Fix  $i \in \{1, 2\}$ . By Lemma 3.5,  $G_i$  satisfies (1.1). By the assumption that  $G_i$  is not strongly spanning trailable, it follows from Theorem 2.4 that there exists a finite family  $\mathcal{S}_i = \mathcal{S}_i(a, b)$  such that  $G_i$  is contractible to a member in  $\mathcal{S}_i$ . By Lemma 3.7, if  $G \in \mathcal{G}_2$ , then  $G$  is contractible to a member in the finite family  $\mathcal{S}'_2$  whose existence is warranted by Lemma 3.7; otherwise,  $G$  is contractible to a member in the finite family  $\mathcal{S}_1 \cup \mathcal{S}_2$ . Hence  $G$  is contractible to a member in the finite family  $\mathcal{F} = \mathcal{S} \cup \mathcal{S}' \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}'_2$ . This proves Theorem 1.1. ■

To proceed the following arguments, we need one more lemma.

**Lemma 3.8.** *Let  $H$  be a graph with  $\kappa'(H) \geq 3$  which is not contractible to  $W_8$ . Suppose that for some edges  $e_1, e_2 \in E(H)$ ,  $H(e_1, e_2)$  does not have a spanning  $(v_{e_1}, v_{e_2})$ -trail. Let  $H(e_1, e_2)'$  be the reduction of  $H(e_1, e_2)$ ,  $n' = |V(H(e_1, e_2))'|$  and  $d_i = |V_i(H(e_1, e_2))'|$  for each  $i \geq 2$ . Then  $n' - d_2 \geq 9$ ,  $d_3 \geq 6$  and  $\sum_{i=3}^5 d_i \geq 8$ .*

**Proof.** As  $H(e_1, e_2)$  does not have a spanning  $(v_{e_1}, v_{e_2})$ -trail, it follows by  $\kappa'(H) \geq 3$  and by Lemma 2.11(ii),

$$2d_2 + d_3 \geq 10 + \sum_{i \geq 5} (i - 4)d_i. \tag{3.5}$$

If  $n' - d_2 \leq 8$ , then  $c(H(e_1, e_2)')$  is a 3-edge-connected graph with at most 8 vertices. By Lemma 2.10,  $c(H(e_1, e_2)')$  must be isomorphic to  $W_8$ . By definition,  $c(H(e_1, e_2)')$  is a contraction of  $H$ , and so  $H$  is contractible to  $W_8$  as well, contrary to the assumption of the lemma. Hence  $n' - d_2 \geq 9$ . As  $d_2 \leq 2$ , by (3.5),  $d_3 \geq 6$ . If  $\sum_{i=3}^5 d_i \leq 7$ , then as  $n' - d_2 \geq 9$ ,  $\sum_{i \geq 6} d_i \geq n' - d_2 - \sum_{i=3}^5 d_i \geq 2$ . By (3.5), we have  $2d_2 + d_3 \geq 10 + \sum_{i \geq 6} (i - 4)d_i \geq 14$ , and so  $d_3 \geq 14 - 2d_2 \geq 10$ , contrary to  $\sum_{i=3}^5 d_i \leq 7$ . Hence we always have  $d_3 \geq 6$  and  $\sum_{i=3}^5 d_i \geq 8$ . ■

**Proof of Theorem 1.2.** Let  $G$  be a graph satisfying the hypotheses of Theorem 1.2 with  $\kappa(L(G)) \geq 3$  and  $n = |V(G)| \geq 219$ . Thus  $G \in \mathcal{G}(a, b)$  with  $a = \frac{1}{4}$  and  $b = -2$ . We shall assume that  $L(G)$  is not hamiltonian-connected to show that Theorem 1.2(iii) must hold.

**Case 1.**  $G \in \mathcal{G}_4$ .

Let  $V_2(G) = \{z\}$  with  $N_G(z) = \{z', z''\}$  and  $z'z'' \in E(G)$ . By Lemma 3.4, we may assume that  $G$  has two connected subgraphs  $H_1$  and  $H_2$  such that  $E(G) = E(H_1) \cup E(H_2)$  and  $V(H_1) \cap V(H_2) = \{z\}$ . Let  $G_0 = G/zz''$ . As  $G_0$  is a contraction of  $G$ , for  $i \in \{1, 2\}$ ,  $G_0$  has a subgraph  $H_i^*$  which is the contraction image of  $H_i$ . Thus  $E(G_0) = E(H_1^*) \cup E(H_2^*)$  with  $V(H_1^*) \cap V(H_2^*) = \{z'\}$ . By Lemma 2.3,  $G_0$  is not strongly spanning trailable and  $\kappa'(G_0) \geq 3$ . By Lemma 3.3, we may assume that  $H_1^*$  is not strongly spanning trailable. If  $H_1^*$  is contractible to  $W_8$ , then as  $H_1^*$  is a contraction of  $G$ ,  $G$  is also contractible to  $W_8$ , and so Theorem 1.2(iii) must hold. Therefore, we assume that  $H_1^*$  is not contractible to  $W_8$ .

Choose a vertex  $u \in V(H_1) - \{z'\}$  such that  $d_{H_1}(u) = \min\{d_{H_1}(v) \mid v \in V(H_1) - \{z'\}\}$ , and set  $d = d_{H_1}(u)$ . Pick a vertex  $u' \in V(H_2) - \{z'\}$ . Then by (1.2), we have

$$|V(H_2)| \geq d_G(u') + 1 \geq \frac{n}{4} - 1 - d, \text{ and so } |V(H_1)| = n + 1 - |V(H_2)| \leq \frac{3n}{4} + d + 2. \tag{3.6}$$

Since  $H_1^*$  is not strongly spanning trailable, there exist edges  $e_1, e_2 \in E(H_1^*)$  such that  $H_1^*(e_1, e_2)$  has no spanning  $(v_{e_1}, v_{e_2})$ -trails. Let  $J$  be the reduction of  $H_1^*(e_1, e_2)$ ,  $n' = |V(J)|$  and  $d_i = |V_i(J)|$  for each  $i \geq 2$ . Hence by Lemma 3.8,  $n' - d_2 \geq 9$ ,  $d_3 \geq 6$  and  $\sum_{i=3}^5 d_i \geq 8$ . Since  $d_G(z) = 2$ , by (1.2), for any  $v \in V(G) - \{z, z', z''\}$ ,

$$d_G(v) \geq \frac{n}{4} - 4 > 5. \tag{3.7}$$

As  $J$  is a contraction of  $H_1(e_1, e_2)$ , there exist disjoint nontrivial connected subgraphs  $L_1^*, L_2^*, \dots, L_8^*$  of  $H_1(e_1, e_2)$  which are contracted to vertices  $w'_1, w'_2, \dots, w'_8 \in \cup_{i=3}^5 V_i(J)$ , respectively. Thus there exist disjoint connected subgraphs  $L_1, L_2, \dots, L_8$  of  $c_{e_1}(c_{e_2}(H_1(e_1, e_2))) = H_1$  such that  $L_i$  is mapped into  $L_i^*$  in the process of mapping  $H_1$  onto  $H_1(e_1, e_2)$ , where  $1 \leq i \leq 8$ .

If  $z \in V(H_1)$ , then  $d = 2$  and we may assume that  $z', z'' \notin V(L_i)$  for  $1 \leq i \leq 6$ . For each  $i$  with  $1 \leq i \leq 6$ , pick a  $w_i \in V(L_i)$  and so by (3.7),  $|V(L_i)| \geq 1 + d_G(w_i) - d_j(w'_i) \geq \frac{n}{4} - 8$ . By (3.6), we have  $\frac{3n}{4} + 4 \geq |V(H_1)| \geq \sum_{i=1}^6 |V(L_i)| \geq 6(\frac{n}{4} - 8) = \frac{6n}{4} - 48$ , and so  $n \leq 69$ , contrary to the assumption that  $n \geq 219$ .

If  $z \notin V(H_1)$ , then  $z'' \notin V(H_1)$  and so we may assume that  $z' \notin V(L_i)$  for  $1 \leq i \leq 7$ . For each  $i$  with  $1 \leq i \leq 7$ , pick a  $w_i \in V(L_i)$ . Then by (3.7) and by  $d_j(w'_i) \leq 5$ , we have  $|V(L_i)| \geq 1 + d_G(w_i) - d_j(w'_i) \geq \frac{n}{4} - 3 - d_j(w'_i) \geq \frac{n}{4} - 8$ . This, together with  $d_3 + d_4 + d_5 \geq 8$  and (3.6), implies that  $\frac{3n}{4} + d + 2 \geq |V(H_1)| \geq \sum_{i=1}^7 |V(L_i)| \geq 7(\frac{n}{4} - 8) = \frac{7n}{4} - 58$ , and so  $n \leq d + 60$ . On the other hand, as  $u \in V(H_1) - \{z'\}$ , we may assume that  $u \notin V(L_i)$  for  $1 \leq i \leq 6$ . For each  $i$  with  $1 \leq i \leq 6$ , since  $d_j(w'_i) \leq 5$ , we have  $|N_G(u) \cap V(L_i)| \leq 5$  and so  $d = |N_G(u)| \leq |V(G) - \bigcup_{i=1}^6 (V(L_i) - N_G(u))| \leq n - 6(\frac{n}{4} - 8 - 5) = 78 - \frac{n}{2}$ . This, together with  $n \leq d + 60$ , leading to  $n \leq 92$ , contradicts the assumption that  $n \geq 219$ .

**Case 2.**  $G \notin \mathcal{G}_4$ .

Let  $G_1$  and  $G_2$  be the graphs defined in (3.1). By (3.1), if  $V_1(G) \neq \emptyset$ , then  $G_1 = G_0$ ; if  $V_2(G) = \{z\}$  with  $N_G(z) = \{z', z''\}$  and  $z'z'' \notin E(G)$ , then  $G_2 = G_0$ . For notational convenience, we redefine that  $G_1 = G_0$  if  $|V_2(G)| = 0$ . By Lemma 2.3,  $G_0$  is not strongly spanning trailable. By Lemma 3.6, either  $|V_2(G)| = 0$  and  $G_1$  is not strongly spanning trailable, or  $|V_2(G)| = 1$  and  $G_2$  is not strongly spanning trailable. Fix  $i \in \{1, 2\}$ . By Lemma 3.5,  $G_i$  satisfies (1.2). By (3.1),  $|V(G_i)| \geq |V(G)| - 2 \geq 217$ . By Theorem 2.5,  $G_i$  is contractible to  $W_8$ .

We claim that  $V_1(G) \cup V_2(G) \neq \emptyset$  is impossible. Otherwise, there is a vertex  $z$  with  $d_G(z) \leq 2$ , and so by (1.2),  $d_G(v) \geq \frac{n}{4} - 4 > 3$  for any vertex  $v \notin \{z\} \cup N_G(z)$ . Then there are at least 6 vertices of  $W_8$  whose preimages are nontrivial graphs. Let  $L_1, L_2, \dots, L_6$  be the preimages of these 6 vertices. Pick  $u_i \in V(L_i)$ . Then  $|V(L_i)| \geq |N_G(u_i) \cap V(L_i)| + 1 \geq \frac{n}{4} - 4 - 3 + 1 = \frac{n}{4} - 6$ . It follows that  $n \geq \sum_{i=1}^6 |V(L_i)| \geq 6(\frac{n}{4} - 6)$ , leading to a contradiction that  $219 \leq n \leq 72$ . Hence  $V_1(G) \cup V_2(G) = \emptyset$ , and so  $G_1 = G$  is contractible to  $W_8$ . This completes the proof of Theorem 1.2. ■

**Proof of Theorem 1.3.** Let  $G$  be a graph satisfying the hypotheses of Theorem 1.3 with  $\kappa(L(G)) \geq 3$  and  $n = |V(G)| \geq 578$ . Thus  $G \in \mathcal{G}(a, b)$  with  $a = \frac{1}{8}$  and  $b = 0$ . We assume that  $L(G)$  is not hamiltonian-connected to show that Theorem 1.3(iii) must hold. As  $G$  is triangle-free,  $G \notin \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$ . Hence  $|V_2(G)| = 0$  or  $G \in \mathcal{G}_1$ .

Let  $G_1$  be the graph defined in (3.1). By (3.1), if  $V_1(G) \neq \emptyset$ , then  $G_1 = G_0$ . For notational convenience, we redefine that  $G_1 = G_0$  if  $|V_2(G)| = 0$ . By Lemma 2.3,  $G_0$  is not strongly spanning trailable. By Lemma 3.6,  $G_1$  is not strongly spanning trailable. By Lemma 3.5,  $G_1$  satisfies (1.2). As  $|V(G_1)| \geq |V(G)| - 1 \geq 577$ , by Theorem 2.6,  $G_1$  is contractible to  $W_8$ , and  $G$  is also contractible to  $W_8$ .

If  $G \in \mathcal{G}_1$ , then  $ess'(G) \geq 3$  and  $\kappa'(G) \geq 2$  and  $|V_2(G)| = 1$ . Let  $V_2(G) = \{z\}$  with  $e = zz', e' = zz'' \in E(G)$ . It is easy to see that  $G(e, e')$  has no spanning  $(v_e, v_{e'})$ -trails, and so  $G$  is not strongly spanning trailable. As  $G$  satisfies (1.3) and  $n \geq 578$ , it follows by Theorem 2.6 that  $G$  is contractible to  $W_8$ .

Now we show that it is impossible that  $|V_1(G)| = 1$  or  $G \in \mathcal{G}_1$ . Otherwise, there is a vertex  $z \in V(G)$  with  $d_G(z) \leq 2$ . By (1.3), for any vertex  $v \notin \{z\} \cup N_G(z)$ ,  $d_G(v) \geq \frac{n}{8} - 2 > 3$ . Hence there are at least 6 vertices of  $W_8$  whose preimages do not contain the vertices in  $N_G(z)$  and are nontrivial connected graphs. Let  $L_1, L_2, \dots, L_6$  be the preimages of these 6 vertices. Pick an edge  $u_i u'_i \in E(L_i)$ . Since  $G$  is triangle-free,  $N_G(u_i) \cap N_G(u'_i) = \emptyset$ , and so  $|V(L_i)| \geq |N_G(u_i) \cap V(L_i)| + |N_G(u'_i) \cap V(L_i)| \geq 2(\frac{n}{8} - 2) - 3 = \frac{n}{4} - 7$ . It follows that  $n \geq \sum_{i=1}^6 |V(L_i)| \geq 6(\frac{n}{4} - 7)$ , leading to a contradiction that  $578 \leq n \leq 84$ . This completes the proof of Theorem 1.3. ■

**4. Strongly spanning trailable graphs**

For any real numbers  $a, b$  with  $0 < a < 1$ , fix a graph  $G \in \mathcal{G}(a, b)$ . For  $e, e' \in E(G)$ , let  $G(e, e')$  be the reduction of  $G(e, e')$ . Define

$$W = W_{(a,b)} = \{v \in V(G) : d_G(v) < \frac{1}{2}(an + b)\} \tag{4.1}$$

and

$$\Gamma_{(a,b)} = \begin{cases} G[W] & \text{if } e, e' \notin E(G[W]), \\ G[W](x) & \text{if } \{|e, e'\} \cap E(G[W])\} = 1 \text{ and } \{x\} = \{e, e'\} \cap E(G[W]), \\ G[W](e, e') & \text{if } e, e' \in E(G[W]). \end{cases} \tag{4.2}$$

Let  $\Gamma'_{(a,b)}$  be the reduction of  $\Gamma_{(a,b)}$  and  $w = |W|$ . We denote

$$W^* = \{v' \in V(G(e, e')) : V(Pl_{G(e,e')}(v')) \cap V(\Gamma'_{(a,b)}) \neq \emptyset\} \text{ and } \Gamma^*_{(a,b)} = G(e, e')'(W^*). \tag{4.3}$$

For a vertex  $v \in V(G(e, e'))$ , if  $Pl_{G(e,e')}(v) = K_1$ , then  $v$  is called a trivial vertex of  $G(e, e')$ . Otherwise,  $v$  is called a nontrivial vertex. Thus if  $v$  is a trivial vertex of  $G(e, e')$ , then  $d_{G(e,e')}(v) = d_{G(e,e')}(v)$ . Then the following result holds.

**Lemma 4.1.** For any real numbers  $a, b$  with  $0 < a < 1$ , let  $\Gamma^*_{(a,b)}$  be the graph defined in (4.3). Then each of the following holds.

- (i) If  $w \geq 4$ ,  $\Gamma^*_{(a,b)} = K_1$ ; if  $1 \leq w \leq 3$ ,  $\Gamma^*_{(a,b)} \in \{K_1, K_2, K_{1,2}, C_4, C_5\}$ .
- (ii)  $\Gamma^*_{(a,b)}$  has at most three trivial vertices  $v \in V(G(e, e')) - \{v_e, v_{e'}\}$  with  $2 \leq d_{G(e,e')}(v) < \frac{1}{2}(an + b)$ .

**Proof.** Since  $G$  satisfies (1.1),  $G[W]$  is a complete graph. Hence if  $1 \leq w \leq 3$ , then  $\Gamma_{(a,b)} \in \{K_1, K_2, K_2(x), K_3, K_3(x), K_3(e, e')\}$ ; if  $w \geq 4$ , then  $\Gamma_{(a,b)} \in \{K_w, K_w(x), K_w(e, e')\}$ . Note that  $K_{1,2} = K_2(x)$ ,  $C_4 = K_3(x)$ , and  $C_5 = K_3(e, e')$ . By Lemma 2.7(i) and (ii),  $K_4, K_4(e, e')$  and  $K_4(x)$  are collapsible. It follows that if  $w \geq 4$ ,  $\Gamma'_{(a,b)} = K_1$ .

Assume that  $1 \leq w \leq 3$ . By definition, if  $\Gamma_{(a,b)} \in \{K_1, K_3\}$ , then  $\Gamma^*_{(a,b)} = K_1$ . If  $\Gamma_{(a,b)} = K_2$ , then  $\Gamma^*_{(a,b)} \in \{K_1, K_2\}$ . Assume that  $\Gamma_{(a,b)} = K_{1,2}$ , and let  $v$  denote the only vertex in  $V_2(\Gamma_{(a,b)})$  such that  $d_{G(e,e')}(v) = 2$ . Since collapsible graphs are 2-edge-connected, if there exists a nontrivial collapsible subgraph  $H$  of  $G(e, e')$  with  $v \in V(H)$ , then  $V(\Gamma_{(a,b)}) \subseteq V(H)$  and  $\Gamma^*_{(a,b)} = K_1$ . Thus if  $\Gamma_{(a,b)} = K_{1,2}$ , then  $\Gamma^*_{(a,b)} \in \{K_1, K_{1,2}\}$ . Similarly, if  $\Gamma_{(a,b)} = C_4$ , then  $\Gamma^*_{(a,b)} \in \{K_1, C_4\}$ ; if  $\Gamma_{(a,b)} = C_5$ , then  $\Gamma^*_{(a,b)} \in \{K_1, C_4, C_5\}$ . This completes the proof of (i), and (ii) follows immediately. ■

In the following proof, we always assume that  $G$  is a simple graph with  $n$  vertices. If  $G$  is not strongly spanning trailable, by definition, there exists  $e, e' \in E(G)$  such that  $G(e, e')$  has no spanning  $(v_e, v_{e'})$ -trails. Let  $G(e, e')$  be the reduction of  $G(e, e')$  and  $n' = |V(G(e, e'))|$ . Define  $v'_e$  and  $v'_{e'}$  as in (2.3). Thus  $G(e, e')$  has no spanning  $(v'_e, v'_{e'})$ -trails. For any  $v \in V(G(e, e'))$ , define  $H(v) = V(PI_{G(e, e')}(v)) - \{v_e, v_{e'}\}$  and  $h(v) = |H(v)|$ .

For each integer  $i > 0$ , we define  $V_i = V_i(G(e, e')) = \{v \in V(G(e, e')) \mid d_{G(e, e')}(v) = i\}$  and  $d_i = |V_i(G(e, e'))|$ . Let

$$T_i = \{v \in V_i - \{v_e, v_{e'}\} : PI_{G(e, e')}(v) = K_1\}, \bar{T}_i = V_i - T_i \cup \{v_e, v_{e'}\}, t_i = |T_i| \text{ and } \bar{t}_i = |\bar{T}_i|. \tag{4.4}$$

By  $ess'(G) \geq 3$  and  $d_2(G) \leq 1$ , we have  $t_2 \leq 1, \bar{t}_2 = 0$  and  $\bar{t}_i = d_i - t_i$  for  $i \geq 3$ . Let  $T = \bigcup_{i=2}^6 T_i, \bar{T} = \bigcup_{i=2}^6 \bar{T}_i, t = |T| = \sum_{i=2}^6 t_i$  and  $\bar{t} = |\bar{T}| = \sum_{i=2}^6 \bar{t}_i$ . Let  $v_1, v_2, \dots, v_{\bar{t}}$  be the nontrivial vertices in  $\bar{T}$  such that  $h(v_1) \leq h(v_2) \leq \dots \leq h(v_{\bar{t}})$ .

**Proof of Theorem 2.4.** Let  $G$  be a simple graph satisfying the hypotheses of Theorem 2.4 and  $B = \max\{\lceil \frac{12-b}{a} \rceil, \lceil \frac{-(b+2)(a+2)}{a^2} \rceil, \lceil 19 + \frac{6}{a} \rceil, 5\}$ . Since nonstrongly spanning trailable graphs with at most  $B$  vertices can always be contractible to a member in  $S(B)$ , we assume that  $n = |V(G)| > B$  and  $G$  is not strongly spanning trailable. We shall show that  $G$  can be contracted to  $S(B)$ . Define  $W, \Gamma_{(a,b)}$  and  $\Gamma_{(a,b)}^*$  as those in (4.1)–(4.3). Let

$$X_5 = \{v \in V(G(e, e')) : d_{G(e, e')}(v) \leq 5\} \text{ and } X'_5 = \{v \in X_5 - V(\Gamma_{(a,b)}^*) : PI_{G(e, e')}(v) \neq K_1\}. \tag{4.5}$$

**Claim 1.** For any  $v \in X'_5$ , then  $h(v) \geq \frac{1}{2}(an + b) + 1$ .

Let  $v \in X'_5$  and  $xy \in E(G)$  be an arbitrary edge such that  $x \in V(PI_{G(e, e')}(v))$  and  $y \notin V(PI_{G(e, e')}(v))$ . As  $d_{G(e, e')}(v_e) = d_{G(e, e')}(v_{e'}) = 2$ , if  $x \in \{v_e, v_{e'}\}$ , then  $k'(PI_{G(e, e')}(v)) = 1$ , contrary to the fact that  $PI_{G(e, e')}(v)$  is collapsible. Thus  $x \notin \{v_e, v_{e'}\}$ . Since  $n \geq \lceil \frac{12-b}{a} \rceil$ , we have  $\frac{1}{2}(an + b) \geq 6$ . By (4.5),  $d_{G(e, e')}(x) = d_G(x) \geq \frac{1}{2}(an + b) \geq 6$ . Let  $N'_x = N_G(x) - V(PI_{G(e, e')}(v))$  and  $N' = \{w \in V(PI_{G(e, e')}(v)) : w \text{ is incident with an edge not in } V(PI_{G(e, e')}(v))\} - N'_x$ . Since  $d_{G(e, e')}(v) \leq 5$ , we have  $|N'| + |N'_x| \leq 5$  and there are at least  $6 - |N' \cup \{x\}| \geq 1$  vertices  $z \in N_G(x) \cap V(PI_{G(e, e')}(v))$  with  $N_G(z) \subseteq H(v)$ . Assume that we have  $w \in \{v_e, v_{e'}\} \cap V(PI_{G(e, e')}(v))$  with  $w_1, w_2$  being the two neighbors of  $w$  in  $PI_{G(e, e')}(v)$ . If  $|N'| \geq 3$ , then we can choose  $x \notin \{w_1, w_2\}$ ; if  $|N'| \leq 2$ , then  $6 - |N' \cup \{x\}| \geq 3$ , we choose  $z \in N_G(x) \cap V(PI_{G(e, e')}(v)) - \{v_e, v_{e'}\}$ . Hence we can always assume that  $z \notin \{v_e, v_{e'}\}$  with  $N_G(z) \subseteq H(v)$ . It follows that  $h(v) \geq d_G(z) + 1 \geq \frac{1}{2}(an + b) + 1$ . This completes the proof of Claim 1.

By Claim 1,  $|X'_5| (\frac{1}{2}(an + b) + 1) \leq \sum_{v \in X'_5} h(v) \leq n$ . Since  $n \geq \lceil \frac{-(b+2)(a+2)}{a^2} \rceil$ , we have

$$|X'_5| \leq \frac{2n}{an + b + 2} \leq \frac{2}{a} + 1. \tag{4.6}$$

As  $n \geq \frac{10-b}{a}$ , we have  $\frac{an+b}{2} > 5$ , and so together with (4.5),  $X_5 - X'_5 = \{v \in X_5 : PI_{G(e, e')}(v) = K_1\} \cup \{v \in X_5 : PI_{G(e, e')}(v) \neq K_1 \text{ and } v \in V(\Gamma_{(a,b)}^*)\} \subseteq V(\Gamma_{(a,b)}^*) \cup \{v_e, v_{e'}\}$ . By Lemma 4.1(i),

$$|X_5 - X'_5| \leq |V(\Gamma_{(a,b)}^*)| + 2 \leq 7. \tag{4.7}$$

To show that  $n'$  is a finite number, pick any  $v \in V(G(e, e')) - X_5$ , we have  $d_{G(e, e')}(v) \geq 6$ . Hence

$$6 |V(G(e, e')) - X_5| \leq \sum_{v \in V(G(e, e')) - X_5} d_{G(e, e')}(v) \leq \sum_{v \in V(G(e, e'))} d_{G(e, e')}(v) = 2 |E(G(e, e'))|.$$

By Lemma 2.11, we have  $2 |E(G(e, e'))| \leq 4n' - 10$ , and so

$$|V(G(e, e')) - X_5| \leq \frac{2n' - 5}{3}. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8), we have

$$n' = |X'_5| + |X_5 - X'_5| + |V(G(e, e')) - X_5| \leq \frac{2}{a} + 8 + \frac{2n' - 5}{3}.$$

Solving the inequality, we have  $n' \leq 19 + \frac{6}{a} \leq B$ . Then  $G$  can be contracted to  $S(B)$ . This completes the proof of Theorem 2.4. ■

Since the proof of Theorem 2.6 (see Appendix) is similar to that of Theorem 2.5, we only prove Theorem 2.5 here.

**Proof of Theorem 2.5.** Suppose that  $G$  is a simple graph satisfying the hypotheses of Theorem 2.5, and  $G$  is not strongly spanning trailable. We shall show that  $G$  can be contracted to  $W_8$ . Let

$$W = W_{(\frac{1}{8}, -2)} = \{v \in V(G) : d_G(v) < \frac{1}{8}n - 1\},$$

$w = |W|$  and define  $\Gamma = \Gamma_{(\frac{1}{4}, -2)}$  and  $\Gamma^* = \Gamma_{(\frac{1}{4}, -2)}^*$  as those in (4.1)–(4.3). Then by Lemma 4.1(i),

$$\text{if } w \geq 4, \text{ then } \Gamma^* = K_1. \tag{4.9}$$

By Lemma 4.1(ii),

$$\Gamma^* \text{ has at most three trivial vertices } v \in V(G(e, e')) - \{v_e, v_{e'}\} \text{ with } 2 \leq d_{G(e, e')}(v) < \frac{n}{8} - 1. \tag{4.10}$$

We have the following claims.

**Claim A.** (i) If there is a vertex  $v_i \in \bar{T}$  such that  $h(v_i) \leq 6$ , then  $d_G(v) \leq 11$  for any  $v \in H(v_i)$ , and  $3 \leq h(v_i) \leq 6$ .

(ii) For  $2 \leq i \leq \bar{t}$ ,  $h(v_i) \geq 7$ .

(iii) For  $1 \leq i \leq \bar{t}$ , if  $h(v_i) \geq 7$ , then there is a vertex  $z \in H(v_i)$  such that  $N_G(z) \subseteq H(v_i)$ , and then

$$h(v_i) \geq d_G(z) + 1 = d_{G(e, e')}(z) + 1.$$

Suppose that  $h(v_i) \leq 6$ . As  $d_{G(e, e')}(v_i) \leq 6$ ,  $d_G(v) \leq 11$  for any  $v \in H(v_i)$ . Since  $PI_{G(e, e')}(v_i)$  is a simple collapsible graph, it follows by Lemma 2.7(i) that  $3 \leq h(v_i) \leq 6$ . Hence Claim A(i) holds.

Suppose that  $h(v_1) \leq h(v_2) < 7$ . As  $11 < \frac{n}{8} - 1$ , by Claim A(i),  $H(v_1) \cup H(v_2) \subseteq W$  and so by Claim A(i),  $w \geq h(v_1) + h(v_2) \geq 6$ . By (4.9),  $\Gamma^* = K_1$ , a contradiction. Hence  $h(v_i) \geq 7$  for  $2 \leq i \leq \bar{t}$ . Thus Claim A(ii) must hold.

Suppose that  $h(v_i) \geq 7$ . By the argument similar to that of Claim 1 in Theorem 2.4, there is a vertex  $z \in H(v_i)$  such that  $N_G(z) \subseteq H(v_i)$ , and then  $h(v_i) \geq d_G(z) + 1 = d_{G(e, e')}(z) + 1$ . Thus Claim A(iii) holds as well.

**Claim B.** Let  $v_i, v_j \in \bar{T}$ . Then each of the following holds.

(i) If there is a vertex  $u \in T$ , then for any  $v_i \in \bar{T}$ ,  $h(v_i) \geq \frac{n}{4} - d_{G(e, e')}(u) - 1 \geq \frac{n}{4} - 7$ .

(ii) If  $7 \leq h(v_i) \leq h(v_j)$ , then  $h(v_i) + h(v_j) \geq \frac{n}{4}$ .

(iii) If  $3 \leq h(v_1) \leq 6$  and  $v_i \neq v_1$ , then  $h(v_1) + h(v_i) \geq \frac{n}{4} - d_{G(e, e')}(v_1) \geq \frac{n}{4} - 6$ , and  $\bar{t} \leq 5$ .

Let  $v_i \in \bar{T}$  and  $u \in T$ . Then  $d_{G(e, e')}(v_i) \leq 6$  and  $d_{G(e, e')}(u) = d_{G(e, e')}(u) = d_G(u) \leq 6$ . We first assume that  $3 \leq h(v_i) \leq 6$ . If there is a vertex  $z \in H(v_i)$  such that  $uz \notin E(G)$ , by Claim A(i) and (1.2), we have

$$17 = 11 + 6 \geq d_{G(e, e')}(z) + d_{G(e, e')}(u) = d_G(z) + d_G(u) \geq \frac{n}{4} - 2,$$

contrary to  $n \geq 217$ . Hence for any  $z \in H(v_i)$ , we have  $uz \in E(G)$ . By  $3 \leq h(v_i) \leq 6$  and (1.2), there are three vertices  $z_1, z_2, z_3 \in H(v_i)$  such that  $G(e, e')[\{z_1, z_2, z_3, u\}]$  is  $K_4$ , or the graph obtained from  $K_4$  by subdivided one edge, or the graph obtained from  $K_4$  by subdivided two edges. By Lemma 2.7(ii),  $G(e, e')[\{z_1, z_2, z_3, u\}]$  is a collapsible graph, contrary to the assumption. Hence  $h(v_i) \geq 7$ . By Claim A(iii), there is a vertex  $z \in H(v_i)$  such that  $N_G(z) \subseteq H(v_i)$ , and then  $h(v_i) \geq d_G(z) + 1 = d_{G(e, e')}(z) + 1$ . Since  $N_G(z) \subseteq H(v_i)$ , we have  $zu \notin E(G)$ . By (1.2),  $h(v_i) - 1 + d_{G(e, e')}(u) \geq d_{G(e, e')}(z) + d_{G(e, e')}(u) = d_G(z) + d_G(u) \geq \frac{n}{4} - 2$ . As  $u \in T$ ,  $d_{G(e, e')}(u) \leq 6$ , and so  $h(v_i) \geq \lceil \frac{n}{4} - 1 - d_{G(e, e')}(u) \rceil \geq \frac{n}{4} - 7$ . This proves Claim B(i).

Let  $v_i, v_j \in \bar{T}$  and  $7 \leq h(v_i) \leq h(v_j)$ . By Claim A(iii), there is a vertex  $z_i \in H(v_i)$  such that  $N_G(z_i) \subseteq H(v_i)$  and  $h(v_i) \geq d_G(z_i) + 1$ . Similarly, there is a vertex  $z_j \in H(v_j)$  such that  $N_G(z_j) \subseteq H(v_j)$  and  $h(v_j) \geq d_G(z_j) + 1$ . Since  $z_i z_j \notin E(G)$ , by (1.2),  $h(v_i) + h(v_j) \geq d_G(z_i) + d_G(z_j) + 2 \geq \frac{n}{4}$ . Hence Claim B(ii) holds.

Suppose that  $3 \leq h(v_1) \leq 6$  and  $v_i \neq v_1$ . By Claim A(ii),  $h(v_i) \geq 7$ . By Claim A(iii), there is a vertex  $z_i \in H(v_i)$  such that  $N_G(z_i) \subseteq H(v_i)$  and  $h(v_i) \geq d_G(z_i) + 1$ . For any  $z_1 \in H(v_1)$ , we have

$$h(v_1) \geq |N_{H(v_1)}(z_1) \cup \{z_1\}| \geq d_G(z_1) - d_{G(e, e')}(v_1) + 1.$$

Since  $z_1 z_i \notin E(G)$ , by (1.2),

$$h(v_1) + h(v_i) \geq d_G(z_1) + d_G(z_i) - d_{G(e, e')}(v_1) + 2 \geq \frac{n}{4} - d_{G(e, e')}(v_1) \geq \frac{n}{4} - 6.$$

If  $\bar{t} \geq 6$ , by  $n \geq 217$ , we have  $n \geq \sum_{i=1}^{\bar{t}} h(v_i) \geq \sum_{i=2}^{\bar{t}} (h(v_1) + h(v_i)) - 4h(v_1) \geq 5(\frac{n}{4} - 6) - 24 = \frac{5n}{4} - 54 > n$ , a contradiction. Thus  $\bar{t} \leq 5$ . This proves Claim B(iii).

**Claim C.** (i)  $T \subseteq V(\Gamma^*)$  and  $t \leq 3$ .

(ii) If  $t \neq 0$ , then  $\bar{t} \leq 4$ ; if  $t = 0$ , then  $\bar{t} \leq 8$ .

Let  $v \in T$ . For  $n \geq 217$ , we have  $d_G(v) = d_{G(e, e')}(v) \leq 6 < \frac{n}{8} - 1$  and then  $v \in V(\Gamma^*)$ . So  $T \subseteq V(\Gamma^*)$  and by (4.10),  $t \leq 3$ .

Suppose that  $t \neq 0$  and  $\bar{t} \geq 5$ . By Claim B(i), (ii) and as  $n \geq 217$ , we have  $n \geq \sum_{i=1}^{\bar{t}} h(v_i) \geq 5(\frac{n}{4} - 7) > n$ , a contradiction. It follows that  $\bar{t} \leq 4$ .

Now we assume that  $t = 0$  and  $\bar{t} \geq 9$ . Since  $\bar{t} \geq 9$ , by Claim B(iii), for each  $1 \leq i \leq \bar{t}$ ,  $h(v_i) \geq 7$ . By Claim B(ii) and  $n \geq 217$ ,  $n \geq \sum_{i=1}^{\bar{t}} h(v_i) \geq 4 \cdot \frac{n}{4} + 7 > n$ , a contradiction. Therefore  $\bar{t} \leq 8$ . This completes the proof of Claim C.

**Table 1**  
The cases occurred in Case 3 of the proof of Theorem 2.5.

$d_2$	$t_3$	$\bar{t}_3$	$t_4$	$\bar{t}_4$	$t_5$	$\bar{t}_5$	$t_6$	$\bar{t}_6$	$n'$
2	2	4	$\leq 1$	0	0	0	0	0	$\leq 9$
2	3	3	0	$\leq 1$	0	0	0	0	$\leq 9$
2	0	6	0	$\leq 2$	0	0	0	0	$\leq 10$
2	0	7	0	0	0	1	0	0	10
2	0	8	0	0	0	0	0	0	10

By Claim C, we have

$$t_3 \leq 3; \tag{4.11}$$

$$\text{if } t_3 \neq 0, \text{ then } \bar{t}_3 = d_3 - t_3 \leq 4; \text{ if } t_3 = 0, \text{ then } \bar{t}_3 = d_3 \leq 8. \tag{4.12}$$

Since  $\kappa'(G) \geq 2$ ,  $\text{ess}'(G) \geq 3$  and  $|V_2(G)| \leq 1$ , we have  $d_1 = 0$ ,  $d_2 \in \{0, 1, 2, 3\}$  and  $n' = |V(G(e, e'))| = \sum_{i \geq 2} d_i$ . By Lemma 2.11(ii),

$$2d_2 + d_3 \geq \sum_{i \geq 5} (i - 4)d_i + 10. \tag{4.13}$$

In the following analysis, we will show that  $G$  is contractible to  $W_8$  in the desirable way in Case 3 below, and a contradiction is obtained in all other cases.

**Case 1**  $d_2 = 0$ . By (4.13),  $d_3 \geq 10$ . By (4.11), if  $t_3 \neq 0$ , then  $\bar{t}_3 = d_3 - t_3 \geq 7$ ; if  $t_3 = 0$ , then  $\bar{t}_3 = d_3 \geq 10$ , contrary to (4.12).

**Case 2**  $d_2 = 1$ . By (4.13),  $d_3 \geq 8$ . By (4.11), if  $t_3 \neq 0$ , then  $\bar{t}_3 \geq 5$ ; if  $t_3 = 0$ , then  $\bar{t}_3 = d_3 \geq 8$ . By (4.12),  $(t_3, \bar{t}_3) = (0, 8)$ . By Claim C(ii), we have  $\bar{t}_3 = \bar{t} = 8$  and so  $t = 0$ . Thus  $d_i = 0$  for any  $i \geq 4$ , forcing  $n' = d_2 + \bar{t}_3 = 9$ . Note that  $\text{ess}'(G(e, e')) \geq 3$ ,  $d_2 = 1$  and  $G(e, e')$  has no spanning  $(v_e, v_{e'})$ -trails. By Lemma 2.12, we have  $G(e, e') = P(10)(e)$ , which contradicts  $n' = 9$ .

**Case 3**  $d_2 = 2$ . By (4.13),  $t_3 + \bar{t}_3 = d_3 \leq 6$ . Then by (4.11) and (4.12),  $t_3 \neq 1$  and  $(t_3, \bar{t}_3) \in \{(2, 4), (3, 3), (3, 4), (0, 6), (0, 7), (0, 8)\}$ . By  $2d_2 + d_3 \leq 12$  and (4.13),  $d_i = 0$  for  $i \geq 7$ . If  $(t_3, \bar{t}_3) = (3, 4)$ , by Claim C,  $d_4 = d_5 = d_6 = 0$ . Then  $G(e, e')$  has 7 vertices with odd degrees, which is impossible. For other cases, by (4.13) and Claim C(ii), it is routine to verify that  $n' \leq 10$  (see Table 1).

Suppose that  $u$  and  $v$  are the two vertices with degree 2 in  $G(e, e')$ . We claim that  $\{u, v\} = \{v_e, v_{e'}\}$ . If  $\{u, v\} \neq \{v_e, v_{e'}\}$ , then  $t_2 = 1$ . By Claim C,  $t_3 \leq 2$  and  $\bar{t}_3 \leq 4$ , which implies  $(t_3, \bar{t}_3) \notin \{(3, 3), (3, 4), (0, 6), (0, 7), (0, 8)\}$ , forcing  $(t_3, \bar{t}_3) = (2, 4)$ ,  $d_4 = d_5 = d_6 = 0$  and  $n' = d_2 + t_3 + \bar{t}_3 = 8$ . As  $G(e, e')$  is reduced, by Lemma 2.7(vii),  $G(e, e')$  is  $K_1$  or  $K_2$ , contrary to  $n' = 8$ . Hence  $\{u, v\} = \{v_e, v_{e'}\}$ .

Adding a new vertex  $w$  and two new edges  $wu, wv$  in  $G(e, e')$ , we get a new graph  $H$  with  $|V(H)| \leq 11$ . As  $d_2(H) = 1$  and  $G(e, e')$  has no spanning  $(u, v)$ -trails,  $H$  is not super-eulerian. By Lemma 2.12,  $H = P(10)(e)$ , which implies  $d_3 = 8$ . Hence  $(t_3, \bar{t}_3) = (0, 8)$ . Since  $\bar{t}_3 = 8 > 5$ , by Claim B(iii), for each  $1 \leq i \leq \bar{t} = 8$ ,  $h(v_i) \geq 7$ . By Claim B(ii),  $n \geq \sum_{i=1}^8 h(v_i) \geq 4 \cdot \frac{n}{4} = n$ . Hence for any  $1 \leq i, j \leq 8$  and  $i \neq j$ ,  $h(v_i) + h(v_j) = \frac{n}{4}$ . It follows that for any  $1 \leq i \leq 8$ ,  $h(v_i) = \frac{n}{8}$ . Since  $d_{G(e, e')}(v_i) = 3$ , it implies that  $G$  contains 3 edges which have exactly one end in  $H(v_i)$ . By (1.2), there are at least  $h(v_i) - 2$  vertices in  $H(v_i)$  with degree  $\frac{n}{8} - 1$ . Thus for  $1 \leq i \leq 8$ ,  $P_{G(e, e')}(v_i)$  is  $K_{\frac{n}{8}}$  or a graph obtained from  $K_{\frac{n}{8}}$  by deleting an edge. Furthermore,  $G(e, e')$  is the graph obtained from  $H = P(10)(e)$  by deleting  $w$  and the edges  $wu$  and  $wv$ , and so the graph is obtained from  $W_8$  (see Fig. 1(c)) by subdividing two edges. That is,  $G(e, e') = P(10) - uv$  (see Fig. 1(a) and (b)). Hence in this case,  $G$  is contractible to  $W_8$  with the preimage of every vertex of  $W_8$  being a  $K_{\frac{n}{8}}$  or a graph obtained from  $K_{\frac{n}{8}}$  by deleting an edge.

**Case 4**  $d_2 = 3$ . In this case,  $|V_2(G)| = 1$ . Suppose that  $V_2(G) = \{z_0\}$  and  $N_G(z_0) = \{z', z''\}$ . Then  $z_0, v_e$  and  $v_{e'}$  are the three vertices with degree 2 in  $G(e, e')$ , which implies  $t_2 = 1$ . By Claim C(i),  $t - t_2 = \sum_{i=3}^6 t_i \leq 2$ . By Claim C(ii),  $\bar{t} \leq 4$ . By (4.13), we have  $d_3 \geq 4$ . It follows that  $(t_3, \bar{t}_3) \in \{(0, 4), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$ . Since  $2d_2 + d_3 \leq 12$ , by (4.13),  $d_i = 0$  for any  $i \geq 7$ . Hence  $n' = \sum_{i=2}^6 d_i = 3 + \sum_{i=3}^6 d_i$ .

If  $(t_3, \bar{t}_3) \in \{(1, 4), (2, 3)\}$ , by (4.13),  $d_6 = 0$  and  $d_5 \leq 1$ . Since the number of odd degree vertices is an even number and  $d_3 = t_3 + \bar{t}_3 = 5$ , we have  $d_5 = 1$ . If  $(t_3, \bar{t}_3) = (1, 4)$ , by  $\bar{t} \leq 4$ , we have  $t_5 = 1$  and  $\bar{t}_5 = 0$ . By  $\sum_{i=3}^6 t_i \leq 2$  and  $\bar{t} \leq 4$ ,  $d_4 = 0$ . Hence  $n' = d_2 + t_3 + \bar{t}_3 + t_5 = 3 + 1 + 4 + 1 = 9$ . If  $(t_3, \bar{t}_3) = (2, 3)$ , by  $\sum_{i=3}^6 t_i \leq 2$  and  $\bar{t} \leq 4$ , we have  $t_5 = 0, \bar{t}_5 = 1$  and  $d_4 = 0$ . Hence  $n' = d_2 + t_3 + \bar{t}_3 + \bar{t}_5 = 3 + 2 + 3 + 1 = 9$ . For other cases, it is routine to verify that  $7 \leq d_2 + d_3 \leq n' \leq 9$  (see Table 2).

By Lemma 2.11(i),  $2F(G(e, e')) = \sum_{i \geq 2} (4 - i)d_i - 4$ . Hence, if  $(t_3, \bar{t}_3) \in \{(0, 4), (1, 3), (1, 4), (2, 2), (2, 3)\}$ , then  $7 \leq (d_2 + d_3) \leq n' \leq 9$  and  $F(G(e, e')) = 3$ . By Lemma 2.15,  $G(e, e') \in \{J'(1, 2), J'(1, 1), K_{1,3}(1, 1, 1)\}$ . Then there exists a spanning  $(v_e, v_{e'})$ -trail in  $G(e, e')$  (see Fig. 3), contrary to the assumption. Hence  $(t_3, \bar{t}_3) = (2, 4)$ . By (1.2), for any vertex  $w \in V(G) - \{z_0, z', z''\}$ ,  $d_G(w) = d_{G(e, e')}(w) \geq \frac{n}{4} - 4 > 3$ . Hence  $d_G(z') = d_G(z'') = 3$  and  $z'z'' \in E(G)$ . It follows that  $G$  has an essential 2-edge-cut, contrary to the assumption  $\text{ess}'(G) \geq 3$ . This completes the proof of Theorem 2.5. ■

**Table 2**  
The cases occurred in Case 4 of the proof of Theorem 2.5.

$d_2$	$t_3$	$\bar{t}_3$	$t_4$	$\bar{t}_4$	$t_5$	$\bar{t}_5$	$t_6$	$\bar{t}_6$	$n'$
3	0	4	$\leq 2$	0	0	0	0	0	$\leq 9$
3	1	3	$\leq 1$	$\leq 1$	0	0	0	0	$\leq 9$
3	2	2	0	$\leq 2$	0	0	0	0	$\leq 9$
3	1	4	0	0	1	0	0	0	9
3	2	3	0	0	0	1	0	0	9
3	2	4	0	0	0	0	0	0	9

**Acknowledgments**

We would like to thank the referees for their carefully reading the manuscript and their many helpful comments and suggestions.

**Funding**

The research of Jianping Liu is partially supported by the NNSF of China (Nos. 11471077 and 11301440), the 2016 program for New Century Excellent Talents in Fujian Province University (No. 2016), and by the fund “Overseas Training Program of Excellent Talents in Universities of Fujian in 2013” (No. 2013). The research of Aimei Yu is partially supported by the National Natural Science Foundation of China (No. 11371193), the Beijing Higher Education Young Elite Teacher Project (No. YETP0573), Fundamental Research Funds for the Central Universities of China (No. 2015JBM107) and the 111 Project of China (B16002). The research of Hong-Jian Lai is partially supported by National Natural Science Foundation of China grants CNNSF 11771039 and CNNSF 11771443.

**Appendix**

**Lemma 2.15.** Let  $G$  be a simple graph with  $\kappa'(G) \geq 2$  and  $|V_2(G)| = 1$ . Let  $G(e, e')$  be the reduction of  $G(e, e')$ . If  $7 \leq |V(G(e, e'))| \leq 9$ ,  $|V_2(G(e, e'))| = 3$  and  $F(G(e, e')) = 3$ , then  $G(e, e') \in \{J'(1, 2), J'(1, 1), K_{1,3}(1, 1, 1)\}$ .

**Proof.** Suppose that  $G$  is a graph satisfying the conditions of Lemma 2.15 and  $G(e, e')$  be the reduction of  $G$ . Since  $7 \leq |V(G(e, e'))| \leq 9$  and  $F(G(e, e')) = 3$ , by Lemmas 2.14 and 2.7,  $G(e, e') \in \{K'_{2,t}, K''_{2,t} : 1 \leq t \leq 7\} \cup \{K_{1,3}(s_1, s_2, s_3) : s_1 + s_2 + s_3 \leq 5\} \cup \{S(m, l), J(m, l), J'(m, l) : m, l \geq 0 \text{ and } m + l \leq 5\}$ . Since  $\kappa'(G) \geq 2$ , we have  $\kappa'(G(e, e')) \geq 2$ , and then  $G(e, e') \in \{K_{1,3}(s_1, s_2, s_3) : s_1 + s_2 + s_3 \leq 5\} \cup \{S(m, l), J(m, l), J'(m, l) : m, l \geq 0 \text{ and } m + l \leq 5\}$ . Now we distinguish four cases.

**Case 1**  $G(e, e') \cong K_{1,3}(s_1, s_2, s_3)$ .

Without loss of generality, we suppose that  $s_1 \geq s_2 \geq s_3$ . Since  $\kappa'(G) \geq 2$ , we have  $\kappa'(G(e, e')) \geq 2$ , which implies  $s_1 \geq s_2 \geq 1$ . If  $(s_1, s_2, s_3) \in \{(1, 1, 0), (2, 1, 0)\}$ , then  $|V_2(K_{1,3}(s_1, s_2, s_3))| = 4$ ; otherwise,  $|V_2(K_{1,3}(s_1, s_2, s_3))| = s_1 + s_2 + s_3$ . By  $|V_2(G(e, e'))| = 3$ , we have  $s_1 = s_2 = s_3 = 1$ , and then  $G(e, e') \cong K_{1,3}(1, 1, 1)$ .

**Case 2**  $G(e, e') \cong S(m, l)$ .

Without loss of generality, we suppose that  $m \geq l$ . If  $m \geq l \geq 2$ , then  $|V_2(S(m, l))| = m + l \geq 4$ ; if  $l = 1$  and  $m \geq 2$ , then  $|V_2(S(m, l))| = m + l + 1 \geq 4$ ; if  $m = l = 1$ , then  $|V_2(S(m, l))| = 5$ . By  $|V_2(G(e, e'))| = 3$ ,  $G(e, e') \not\cong S(m, l)$ .

**Case 3**  $G(e, e') \cong J(m, l)$ .

Suppose that  $m \geq l$ . If  $m \geq l \geq 2$ , then  $|V_2(J(m, l))| = m + l \geq 4$ ; if  $l = 1$  and  $m \geq 2$ , then  $|V_2(J(m, l))| = m + l + 1 \geq 4$ ; if  $m = l = 1$ , then  $|V_2(J(1, 1))| = 4$ . By  $|V_2(G(e, e'))| = 3$ ,  $G(e, e') \not\cong J(m, l)$ .

**Case 4**  $G(e, e') \cong J'(m, l)$ .

If  $l = 1$ , then  $|V_2(J'(m, l))| = m + 2$ ; if  $l \geq 2$ , then  $|V_2(J'(m, l))| = m + l$ . By  $|V_2(G(e, e'))| = 3$ ,  $G(e, e') \cong J'(1, 2)$  or  $J'(1, 1)$ . This completes the proof of Lemma 3.7. ■

**Proof of Theorem 2.6.** Suppose that  $G$  is a simple graph satisfying the hypotheses of Theorem 2.6, and that  $G$  is not strongly spanning trailable. We will show that  $G$  can be contracted to  $W_8$ . Let

$$W = W_{(\frac{1}{8}, 0)} = \{v \in V(G) : d_G(v) < \frac{1}{16}n\},$$

$w = |W|$  and define  $\Gamma = \Gamma_{(\frac{1}{8}, 0)}$ ,  $\Gamma^* = \Gamma_{(\frac{1}{8}, 0)}^*$  as those in (4.1)–(4.3). As  $G[W] \subset G$  is a complete graph, and  $G$  is a triangle-free graph, it follows that  $0 \leq w \leq 2$ . Since  $G$  is triangle-free, by Lemma 4.1,

$$|V(\Gamma^*)| \leq 3. \text{ Furthermore, if } 1 \leq w \leq 2, \Gamma^* \in \{K_1, K_2, K_{1,2}\}. \tag{A.1}$$

Moreover,

$$\Gamma^* \text{ has at most 2 trivial vertices } v \notin \{v_e, v_{e'}\} \text{ with } 2 \leq d_{G(e, e')}(v) < \frac{1}{16}n. \tag{A.2}$$

We make the following claims.

**Claim D.** *Each of the following holds.*

- (i) For any  $1 \leq i \leq \bar{t}$ ,  $h(v_i) \geq 7$ .
- (ii) If there exists a trivial vertex  $u \in V(G(e, e')) - \{v_e, v_{e'}\}$ , then

$$h(v_i) \geq \frac{3n}{16} - 12.$$

- (iii)  $h(v_i) \geq \frac{n}{8} - 6$ .

If there exists some  $v_i$  with  $h(v_i) \leq 6$ , then as  $Pl_{G(e, e')}(v_i)$  is a simple collapsible graph, by Lemma 2.7(i), we have  $3 \leq h(v_i) \leq 6$ . As  $d_{G(e, e')}(v_i) \leq 6$ ,  $d_G(v) \leq 11$  for any  $v \in H(v_i)$ . By  $n \geq 577$ , we have  $11 < \frac{n}{16}$ . It follows that  $H(v_i) \subseteq W$ , contrary to  $w \leq 2$ . Hence Claim D(i) holds.

Let  $v_i \in \bar{T}$ . Then  $h(v_i) \geq 7$ . Since  $w \leq 2$ , there exists a vertex  $x \in H(v_i)$  such that  $d_G(x) \geq \frac{n}{16}$ . Since  $d_G(x) \geq \frac{n}{16} \geq 10$  and  $d_{G(e, e')}(v_i) \leq 6$ , by the argument similar to that of Claim 1 in Theorem 2.4, there is a vertex  $z \in H(v_i)$  such that  $x \in N_G(z) \subseteq H(v_i)$ . If  $u \notin \{v_e, v_{e'}\}$  is a trivial vertex of  $G(e, e')$ , then  $d_{G(e, e')}(u) = d_{G(e, e')}(u) = d_G(u)$  and  $zu \notin E(G)$ . By (1.3), it follows that  $d_G(z) + d_{G(e, e')}(u) = d_G(z) + d_G(u) \geq \frac{n}{8}$ . Thus  $d_G(z) \geq \frac{n}{8} - 6$ . Since  $G$  is triangle-free, we have  $N_G(z) \cap N_G(x) = \emptyset$ , which implies

$$h(v) \geq d_G(x) + d_G(z) - d_{G(e, e')}(v_i) \geq \frac{n}{16} + (\frac{n}{8} - 6) - 6 = \frac{3n}{16} - 12.$$

Hence Claim D(ii) holds as well.

For any  $v_i \in \bar{T}$ ,  $G_i = G[V(Pl_{G(e, e')}(v_i)) - \{v_e, v_{e'}\}]$  is a triangle-free graph. Since  $w \leq 2$  and  $h(v_i) \geq 7$ , there exist two vertices  $u_1, u_2 \in V(G_i)$  with  $d_G(u_1) \geq \frac{n}{16}$ ,  $d_G(u_2) \geq \frac{n}{16}$  and  $N_G(u_1) \cap N_G(u_2) = \emptyset$ . Since  $d_{G(e, e')}(v_i) \leq 6$ , it follows that  $h(v_i) \geq 2 \cdot \frac{n}{16} - 6 = \frac{n}{8} - 6$ , justifying Claim D(iii).

**Claim E.** *Each of the following holds.*

- (i)  $T \subseteq V(\Gamma^*)$  and  $t \leq 2$ .
- (ii) If  $t \neq 0$ , then  $\bar{t} \leq 5$ ; otherwise,  $\bar{t} \leq 8$ .

Let  $v \in T$ . Since  $n \geq 577$ , we have  $d_G(v) = d_{G(e, e')}(v) \leq 6 < \frac{n}{16} - 1$ . Hence  $v \in \Gamma^*$ , and so  $T \subseteq V(\Gamma^*)$ . By (A.2),  $t \leq 2$ , and so Claim E(i) holds.

Suppose that  $t \neq 0$  and  $\bar{t} \geq 6$ . By Claim D(ii) and  $n \geq 577$ ,  $n \geq \sum_{i=1}^{\bar{t}} h(v_i) \geq 6(\frac{3n}{16} - 12) > n$ , a contradiction. It follows that  $\bar{t} \leq 5$ . If  $t = 0$  and  $\bar{t} \geq 9$ , by Claim D(iii) and  $n \geq 577$ , we have  $n \geq \sum_{i=1}^{\bar{t}} h(v_i) \geq 9(\frac{n}{8} - 6) > n$ , a contradiction. Therefore  $\bar{t} \leq 8$ . Thus Claim E(ii) must hold.

By Claim E, we have

$$t_3 \leq 2; \tag{A.3}$$

$$\text{if } t_3 \neq 0, \text{ then } \bar{t}_3 = d_3 - t_3 \leq 5; \text{ if } t_3 = 0, \text{ then } \bar{t}_3 = d_3 \leq 8. \tag{A.4}$$

Since  $\kappa'(G) \geq 2$ ,  $ess'(G) \geq 3$ ,  $|V_2(G)| \leq 1$  and  $G(e, e')$  is the reduction of  $G(e, e')$ , we have  $d_1 = 0$ ,  $d_2 \in \{0, 1, 2, 3\}$  and  $n' = |V(G(e, e'))| = \sum_{i \geq 2} d_i$ . By Lemma 2.11(ii),

$$2d_2 + d_3 \geq \sum_{i \geq 5} (i - 4)d_i + 10. \tag{A.5}$$

In the following analysis, we will show that  $G$  is contractible to  $W_8$  in the desirable way in Case 3 below, and a contradiction is obtained in all other cases.

**Case 1**  $d_2 = 0$ .

By (A.5),  $d_3 \geq 10$ . By (A.3), if  $t_3 \neq 0$ , then  $\bar{t}_3 = d_3 - t_3 \geq 8$ ; if  $t_3 = 0$ , then  $\bar{t}_3 = d_3 \geq 10$ , contrary to (A.4).

**Table 3**  
The cases occurred in Case 3 of the proof of Theorem 2.6.

$d_2$	$t_3$	$\bar{t}_3$	$t_4$	$\bar{t}_4$	$t_5$	$\bar{t}_5$	$t_6$	$\bar{t}_6$	$n'$
2	1	5	$\leq 1$	0	0	0	0	0	$\leq 9$
2	2	4	0	$\leq 1$	0	0	0	0	$\leq 9$
2	0	6	0	$\leq 2$	0	0	0	0	$\leq 10$
2	2	5	0	0	0	0	0	0	9
2	0	7	0	0	0	1	0	0	10
2	0	8	0	0	0	0	0	0	10

**Case 2**  $d_2 = 1$ .

By (A.5),  $d_3 \geq 8$ . By (A.3), if  $t_3 \neq 0$ , then  $\bar{t}_3 \geq 6$ ; if  $t_3 = 0$ , then  $\bar{t}_3 = d_3 \geq 8$ . By (A.4),  $t_3 = 0$  and  $\bar{t}_3 = 8$ . By Claim E(ii), we have  $\bar{t}_3 = \bar{t} = 8$  and so  $t = 0$ . Thus  $d_i = 0$  for any  $i \geq 4$ , forcing  $n' = d_2 + \bar{t}_3 = 9$ . Note that  $ess'(G(e, e')) \geq 3$ ,  $d_2 = 1$  and  $G(e, e')$  has no spanning  $(v_e, v_{e'})$ -trails. By Lemma 2.12, we have  $G(e, e') = P(10)(e)$ , which contradicts  $n' = 9$ .

**Case 3**  $d_2 = 2$ .

By (A.5), we have  $d_3 \geq 6$ . By (A.3) and (A.4),  $(t_3, \bar{t}_3) \in \{(1, 5), (2, 4), (2, 5), (0, 6), (0, 7), (0, 8)\}$ . As  $2d_2 + d_3 \leq 12$  for each case, by (A.5),  $d_i = 0$  for any  $i \geq 7$ . Hence  $n' = \sum_{i=2}^6 d_i$ . If  $(t_3, \bar{t}_3) = (0, 7)$ , by (A.5),  $d_5 \leq 1$  and  $d_6 = 0$ . Since the number of odd degree vertices is an even number, we have  $d_5 = 1$ . By  $\bar{t}_3 = 7 > 5$  and Claim E(ii), we have  $t_4 = t_5 = 0, \bar{t}_4 = 0$  and  $\bar{t}_5 = 1$ . Then  $n' = d_2 + \bar{t}_3 + \bar{t}_5 = 2 + 7 + 1 = 10$ . For other cases, it is routine to verify that  $n' \leq 10$  (see Table 3).

Suppose that  $u$  and  $v$  are the two vertices with degree 2 in  $G(e, e')$ . We claim that  $\{u, v\} = \{v_e, v_{e'}\}$ . If  $|\{u, v\} \cap \{v_e, v_{e'}\}| = 1$ , then  $t_2 = 1$ . By Claim E,  $t_3 \leq 1$  and  $\bar{t}_3 \leq 5$ , which implies  $(t_3, \bar{t}_3) \notin \{(2, 4), (2, 5), (0, 6), (0, 7), (0, 8)\}$ . Hence  $(t_3, \bar{t}_3) = (1, 5), d_4 = d_5 = d_6 = 0$  and  $n' = d_2 + t_3 + \bar{t}_3 = 8$ . As  $G(e, e')$  is reduced, by Lemma 2.7(vii),  $G(e, e')$  is  $K_1$  or  $K_2$ , contrary to  $n' = 8$ . Hence  $\{u, v\} = \{v_e, v_{e'}\}$ .

Adding a new vertex  $w$  and two new edges  $wu, wv$  in  $G(e, e')$ , we get a new graph  $H$  with  $|V(H)| \leq 11$ . As  $d_2(H) = 1$  and  $G(e, e')$  has no spanning  $(u, v)$ -trails,  $H$  is not supereulerian. By Lemma 2.12,  $H = P(10)(e)$ , which implies  $d_3 = 8$ . Hence  $(t_3, \bar{t}_3) = (0, 8)$ . In this case,  $G(e, e')$  is the graph obtained from  $H = P(10)(e)$  by deleting  $w$  and the edges  $wu$  and  $wv$ , i.e., the graph obtained from  $W_8$  (see Fig. 1) by subdividing two edges. Thus  $G$  is contractible to  $W_8$ .

For any  $1 \leq i \leq 8$ , let  $H_i$  be the preimage of  $v_i$  in  $W_8$ . By Claim D(i),  $h(v_i) \geq 7$ . Let  $N'_i = \{u \in V(H_i) : u \text{ is incident with an edge in } E(G) - E(H_i)\}$ . By  $w \leq 2$  and  $d_{W_8}(v_i) = 3$ , we have  $|V(H_i) - (W \cap N'_i)| \geq 7 - (2 + 3) = 2$ . Thus there exists a vertex  $u_1 \in V(H_i) - (W \cap N'_i)$  such that  $N_G(u_1) \subseteq V(H_i)$ . Since  $u_1 \notin W$  and by the definition of  $W$ , we have  $|N_G(u_1)| = d_G(u_1) \geq \frac{n}{16} \geq \frac{577}{16} > 5$ . It follows that there exists a vertex  $u_2 \in N_G(u_1) - (W \cap N'_i)$  such that  $N_G(u_2) \subseteq V(H_i)$ . By the definition of  $W$  and as  $u_1, u_2 \notin W$ , we have  $d_G(u_1) \geq \frac{n}{16}$  and  $d_G(u_2) \geq \frac{n}{16}$ . Since  $G$  is a triangle-free graph and since  $u_1 u_2 \in E(G)$ , we have  $N_G(u_1) \cap N_G(u_2) = \emptyset$ . Hence  $h(v_i) \geq |N_G(u_1) \cup N_G(u_2)| = |N_G(u_1)| + |N_G(u_2)| \geq \frac{n}{16} + \frac{n}{16} = \frac{n}{8}$ . Then  $n = \sum_{i=1}^8 h(v_i) \geq 8 \cdot \frac{n}{8} = n$ , which implies that  $|V(H_i)| = \frac{n}{8}$ . It follows that  $H_i$  is a spanning subgraph of  $K_{\frac{n}{16}, \frac{n}{16}}$ . Remembering  $G[W] \in \{K_1, K_2\}$  or  $w = 0$ , there are at most two vertices in  $H_i$  with degree  $\frac{n}{16} - 1$ , and other vertices in  $H_i$  with degree  $\frac{n}{16}$ . Furthermore, those two vertices must be adjacent to vertices outside  $H_i$ . It implies that the preimage of every vertex of  $W_8$  is  $K_{\frac{n}{16}, \frac{n}{16}}$  or a graph obtained from  $K_{\frac{n}{16}, \frac{n}{16}}$  by deleting an edge.

**Case 4**  $d_2 = 3$ .

In this case, suppose that  $V_2(G) = \{z_0\}$  and  $N_G(z_0) = \{z', z''\}$ . Then  $z_0, v_e$  and  $v_{e'}$  are the three vertices with degree 2 in  $G(e, e')$ , and  $t_2 = 1$ . By Claim E(i),  $t - t_2 \leq 1$ .

By (1.3), for any vertex  $u$  nonadjacent to  $z_0$ , we have

$$d_G(u) \geq \frac{n}{8} - d_G(z_0) \geq \frac{n}{8} - 2. \tag{A.6}$$

Let  $v \in \bar{T}$ . By  $n \geq 577$  and Claim D(iii), we have  $h(v) \geq \frac{n}{8} - 6 \geq 66$ . As  $d_G(z_0) = 2$ , there are at least two vertices  $u_1, u_2 \in H(v)$  such that  $u_1 u_2 \in E(G), u_i \notin N_G(z_0)$  and  $N_G(u_i) \subseteq H(v)$  for any  $1 \leq i \leq 2$ . Since  $G$  is a triangle-free graph,  $N_G(u_1) \cap N_G(u_2) = \emptyset$ . Thus  $h(v) \geq d_G(u_1) + d_G(u_2) + 2$ . By (A.6),  $h(v) \geq \frac{n}{4} - 2$ . If  $\bar{t} \geq 5$ , we have  $n \geq \sum_{i=1}^{\bar{t}} h(v_i) \geq 5(\frac{n}{4} - 2) > n$ , a contradiction. Hence  $\bar{t} \leq 4$ .

By (A.5), we have  $d_3 \geq 4$ . As  $t - t_2 \leq 1$  and  $\bar{t} \leq 4, (t_3, \bar{t}_3) \in \{(0, 4), (1, 3), (1, 4)\}$ . If  $(t_3, \bar{t}_3) = (1, 4)$ , by (A.5),  $d_i = 0$  for any  $i \geq 6$  and  $d_5 \leq 1$ . By  $t - t_2 \leq 1$  and  $\bar{t} \leq 4$ , we have  $d_5 = 0$ . Then the number of odd degree vertices is an odd number, which is impossible. Hence  $(t_3, \bar{t}_3) \in \{(0, 4), (1, 3)\}$ . By (A.5),  $d_i = 0$  for any  $i \geq 5$ . Furthermore, if  $(t_3, \bar{t}_3) = (0, 4)$ , then  $t_4 \leq 1, \bar{t}_4 = 0$  and  $n' = d_2 + t_3 + \bar{t}_3 + d_4 \leq 3 + 0 + 4 + 1 = 8$ ; if  $(t_3, \bar{t}_3) = (1, 3)$ , then  $t_4 = 0, \bar{t}_4 \leq 1$  and  $n' = d_2 + t_3 + \bar{t}_3 + d_4 \leq 3 + 1 + 3 + 1 = 8$ . By Lemma 2.11(i),  $F(G(e, e')) = 3$ . Hence by Lemma 2.15,  $G(e, e') \in \{J'(1, 2), J'(1, 1), K_{1,3}(1, 1, 1)\}$ . Then  $G(e, e')$  must be one of the graphs depicted in Fig. 3. It is easy to see that for any case, there is a spanning  $(v_e, v_{e'})$ -trail in  $G(e, e')$ , contrary to the assumption. This completes the proof of Theorem 2.6. ■

**Lemma 3.3.** Let  $G$  be a graph with  $\kappa'(G) \geq 2$  and  $ess'(G) \geq 3$ , and let  $G_1, G_2, \dots, G_k$  be the blocks of  $G$ . Then the following are equivalent.

- (i)  $G$  is strongly spanning trailable.
- (ii) For every  $i \in \{1, 2, \dots, k\}$ ,  $G_i$  is strongly spanning trailable.



**Proof.** Since each block of  $G$  is also 2-edge-connected and essentially 3-edge-connected, (i) implies (ii). To prove (ii) implies (i), we argue by induction on  $k$ , the number of blocks of  $G$ . As (ii) trivially implies (i) when  $k = 1$ , we assume that  $k > 1$  and for any graph with fewer than  $k$  blocks, (ii) implies (i).

Since  $k \geq 2$ ,  $G$  has two connected subgraphs  $H$  and  $L$  and a vertex  $z_0$  such that  $E(G) = E(H) \cup E(L)$  and  $V(H) \cap V(L) = \{z_0\}$ . Let  $e, e' \in E(G)$ . If  $\{e, e'\} \cap E(L) = \emptyset$ , then by induction,  $H(e, e')$  has a spanning  $(v_e, v_{e'})$ -trail  $Q_1$ . By induction, for any edge  $e'' \in E(L)$ ,  $L(e'')$  has a spanning  $(v_{e''}, v_{e''})$ -trail, and so  $L$  has a spanning closed trail  $Q_2$ . It follows that  $Q = G[E(Q_1) \cup E(Q_2)]$  is a spanning  $(v_e, v_{e'})$ -trail of  $G$ . The proof for the case when  $\{e, e'\} \subseteq E(L)$  is similar, and will be omitted. Hence we may assume that  $e \in E(H)$  and  $e' \in E(L)$ .

Since  $G$  is essentially 3-edge-connected, we have  $d_H(z_0) \geq 3$ , and so  $H$  has an edge  $e'' \in E_H(z_0) - \{e\}$ . By induction,  $H$  has a spanning  $(v_e, v_{e''})$ -trail  $T'_1$ . Assume that  $e'' = z_0 w$ . Define

$$T_1 = \begin{cases} T'_1 - z_0 v_{e''} & \text{if } z_0 v_{e''} \text{ is the last edge in } T'_1, \\ H[E(T'_1 - v_{e''}) \cup \{e''\}] & \text{if } w v_{e''} \text{ is the last edge in } T'_1. \end{cases}$$

Thus  $T_1$  is a spanning  $(v_e, z_0)$ -trail of  $H$ . Similarly,  $L$  has a spanning  $(z_0, v_{e'})$ -trail  $T_2$ . It follows that  $T = T_1 \cup T_2$  is a spanning  $(v_e, v_{e'})$ -trail. ■

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