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List r-hued chromatic number of graphs with bounded maximum average degrees

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ABSTRACT

For integers $k, r > 0$, a (k, r)-coloring of a graph G is a proper coloring c with at most k colors such that for any vertex v with degree $d(v)$, there are at least $\min\{d(v), r\}$ different colors present at the neighborhood of v. The *r-hued chromatic number* of G, $\chi_r(G)$, is the least integer k such that a (k, r)-coloring of G exists. The *list r-hued chromatic number* $\chi_{L,r}(G)$ of G is similarly defined. Thus if $\Delta(G) \geq r$, then $\chi_{L,r}(G) \geq \chi_r(G) \geq r+1$. We present examples to show that, for any sufficiently large integer r, there exist graphs with maximum average degree less than 3 that cannot be $(r+1, r)$-colored. We prove that, for any fraction $q < \frac{14}{5}$, there exists an integer $R = R(q)$ such that for each $r \geq R$, every graph G with maximum average degree q is list $(r+1, r)$-colorable. We present examples to show that for some r there exist graphs with maximum average degree less than 4 that cannot be r-hued colored with less than $\frac{3r}{2}$ colors. We prove that, for any sufficiently small real number $\epsilon > 0$, there exists an integer $h = h(\epsilon)$ such that every graph G with maximum average degree $4 - \epsilon$ satisfies $\chi_{L,r}(G) \leq r + h(\epsilon)$. These results extend former results in Bonamy et al. (2014).

1. Introduction

Graphs in this paper are simple and finite. Undefined terms and notation will follow [4]. Thus for a graph G, $\Delta(G)$, $\delta(G)$, and $\chi(G)$ denote the maximum degree, the minimum degree, and chromatic number of G, respectively. For $v \in V(G)$, let $N_G(v)$ denote the set of vertices adjacent to v in G, $N_G[v] = N_G(v) \bigcup \{v\}$, and $d_G(v) = |N_G(v)|$. When G is understood from the context, the subscript G is often omitted. For a graph G which is not a forest, the girth of G, denoted $g(G)$, is the length of a shortest cycle in G.

Let k, r be positive integers, and define $\bar{k}$ ={1, 2, . . . , k}. If $c : V(G) \mapsto \bar{k}$, and if $V' \subseteq V(G)$, then define $c(V') = \{c(v)|v \in V'\}$. A (k, r)-**coloring** of a graph G is a mapping $c : V(G) \mapsto \bar{k}$ satisfying both the following:
(C1) $c(u) \neq c(v)$ for every edge $uv \in E(G)$;
(C2) $|c(N_G(v))| \geq \min\{d_G(v), r\}$ for any $v \in V(G)$.
Such a (k, r)-coloring is also called as an r-hued coloring using at most k colors. For a fixed integer $r > 0$, the *r-hued chromatic number* of G, denoted by $\chi_r(G)$, is the smallest k such that G has a (k, r)-coloring. It is easy to extend the concept to its list coloring version. The list r-hued chromatic number $\chi_{L,r}(G)$ of a graph G is similarly defined. The r-hued coloring was first introduced in [19] and [16], where $\chi_2(G)$ was called *the dynamic chromatic number* of G. Its research can be traced much earlier, as the square coloring is the special case when $r = \Delta$. Many have investigated r-hued colorings and list r-hued colorings, as seen in [1,2,7–9,12–17,19,20,22], among others.

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By definition, for any integer $h > 0$ and for any graph G with $\Delta(G) = \Delta$, we have $\chi_{\Delta+h}(G) = \chi_\Delta(G)$. If a graph G satisfies $\Delta(G) \geq r$, then by (C2), we must have $\chi_r(G) \geq r + 1$. It is natural to seek when a graph G would satisfy $\chi_r(G) \geq r + C$, for some given constant C. The case when $C = 1$ is of particular interest. In [23], Wang and Lih conjectured that for any integer $k \geq 5$, there exists an integer $N(k)$ such that every planar graph G with $g(G) \geq k$ and $\Delta(G) \geq N(k)$ satisfies $\chi_\Delta(G) = \Delta(G)+1$. It is shown in [5,6,10,11] that Wang and Lih's conjecture holds for $k \geq 7$ and fails for $k \in \{5, 6\}$. Wegner [24] conjectured the case when $r = \Delta(G)$ in Conjecture 1.1.

Conjecture 1.1 (*[20]*). *Let G be a planar graph. Then*

$$\chi_r(G) \leq \begin{cases} r+3, & \text{if } 1 \leq r \leq 2 \\ r+5, & \text{if } 3 \leq r \leq 7; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \geq 8. \end{cases}$$

Recently, it is proved in [22] that for $r \geq 3$, any planar graph G with girth at least 6 satisfies $\chi_r(G) \leq r+5$. In [18,22], this conjecture is verified for graphs without a minor isomorphic to K_4.

The **maximum average degree** of a graph is defined as

$$mad(G) = \max\left\{\frac{\sum_{v\in V(H)} d_H(v)}{|V(H)|} : H \text{ is a subgraph of } G\right\}.$$

By definition, any forest is of maximum average degree at most 2. It is proved in [15] that all forests are $(r + 1, r)$-colorable. Bonamy et al. [3] proved the following results:

Theorem 1.2 (*Bonamy et al. [3]*). *There exists a function f such that for a small enough $\epsilon > 0$, every graph with $mad(G) < 14/5-\epsilon$ and $\Delta(G) \geq f(\epsilon)$ satisfies $\chi_\Delta(G) \leq \Delta + 1$.*

Theorem 1.3 (*Bonamy et al. [3]*). *For any sufficiently small real number $\epsilon > 0$, there exists an integer $h(\epsilon)$ such that every graph G with $mad(G) < 4 - \epsilon$ satisfies $\chi_{L,\Delta}(G) \leq \Delta(G) + h(\epsilon)$.*

Motivated by Theorems 1.2, 1.3, Conjecture 1.1 and the results mentioned above, we consider the following problems.

Problem 1.4. For any real number $x > 0$, is there a smallest integer $f(x)$ such that, when $r \geq f(x)$, every graph G with $mad(G) < x$ satisfies $\chi_r(G) \leq r + 1$?

Problem 1.5. Determine the set $\mathcal{X}$ of positive real numbers such that $x \in \mathcal{X}$ if and only if there exists a smallest integer $h(x)$ such that every graph G with $mad(G) < x$ satisfies $\chi_r(G) \leq r + h(x)$, for all sufficiently large r.

In Section 2, we present examples to show that for certain values of x, $f(x)$ in Problem 1.4 may not exist, and that in Problem 1.5, $\sup\{x \in \mathcal{X}\} \leq 4$. The main purposes of this paper are, within reasonable ranges of the parameters, to extend Theorems 1.2 and 1.3 to r-hued colorings for arbitrary values of r. The main results of the paper are presented below. Theorem 1.6 shows the existence of $f(x)$ for any $x \in [0, \frac{14}{5})$ and Theorem 1.7 shows that $\sup\{x \in \mathcal{X}\} = 4$.

Theorem 1.6. *For any sufficiently small real number $\epsilon > 0$, there exists an integer $f(\epsilon)$ such that every graph G with $mad(G) < 14/5 - \epsilon$ and $r \geq f(\epsilon)$ satisfies $\chi_r(G) \leq r + 1$.*

Theorem 1.7. *For any sufficiently small real number $\epsilon > 0$, there exists an integer $h(\epsilon)$ such that every graph G with $mad(G) < 4 - \epsilon$ satisfies $\chi_r(G) \leq r + h(\epsilon)$.*

In Section 3, we introduce the necessary notations and present some basics that are useful in our arguments. The proofs for the main results are in the subsequent sections.

2. Examples

We in this section will present two families of examples that are related to Problems 1.4 and 1.5. In particular, Example 2.1 shows that in Problem 1.4, $f(x)$ does not exist for any $x \geq 3$. Example 2.2 suggests that $\sup\{x \in \mathcal{X}\} \leq 4$.

Example 2.1 (*[21]*). There exists an infinite fractional sequence q_r with $\frac{7}{3} \leq q_r < 3$ and $\lim_{r\to\infty} q_r = 3$, such that for any integer $r \geq 3$, there exists a graph G satisfying that $mad(G) \leq q_r$, $\Delta(G) \geq r$ and $\chi_r(G) \geq r+2$. Such graphs can be constructed as follows. Let $s \geq 1$ and $t \geq 1$ be integers. For $i = 1, \ldots, s$, let J_i be a graph with

$$V(J_i) = \{w_1^i, w_2^i, w_3^i, w_4^i, x_1^i, x_2^i, \ldots, x_t^i, y_1^i, y_2^i, \ldots, y_t^i\},$$

and

$$E(J_i) = \{w_1^i w_3^i, w_2^i w_3^i, w_1^i w_4^i, w_2^i w_4^i\} \cup \{w_1^i x_j^i, x_j^i y_j^i, y_j^i w_2^i : 1 \leq j \leq t\}.$$

Obtain a graph $G(s,t)$ from the disjoint union of $J_1, J_2, \ldots, J_s$ by identifying $w_1^1, w_1^2, \ldots, w_1^s$ into one vertex w_1. Then we have the following observations which justify the conclusions stated in this example.
(i) $\Delta(G(s,t)) = s(t+2)$;
(ii) $\frac{7}{3} \le mad(G(s,t)) = \frac{2s(3t+4)}{s(2t+3)+1} < 3$;
(iii) If $r = t+2$, then $\chi_r(G(s,t)) \ge r+2$.

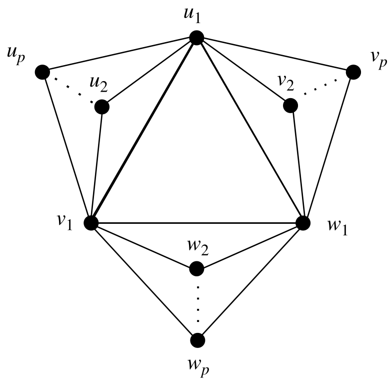

Fig. 1. Example 2.2, $G(p,1)$ with $r = 2p$, $mad(G(p,1)) = 4 - \frac{2}{p}$ and $\chi_r(G(p,1)) = \frac{3r}{2}$.

Example 2.2. There exists an infinite fractional sequence q'_r with $3 \le q'_r < 4$ and $\lim_{r\to\infty} q'_r = 4$, such that for any even integer $r > 0$, there exists a graph G satisfying that $mad(G) \le q'_r$, $\Delta(G) \ge r$ and $\chi_r(G) \ge \frac{3r}{2}$.

We will construct such graphs. Let $s \ge 1$ and $p \ge 2$ be integers. For $i = 1, \ldots, s$, let J_i be a graph with

$$V(J_i) = \{u_1^i, v_1^i, w_1^i, u_2^i, u_3^i, \ldots, u_p^i, v_2^i, v_3^i, \ldots, v_p^i, w_2^i, w_3^i, \ldots, w_p^i\},$$

and

$$E(J_i) = \{u_1^i v_1^i, v_1^i w_1^i, w_1^i u_1^i\} \cup \{u_1^i u_j^i, u_1^i v_j^i, v_1^i u_j^i, v_1^i w_j^i, w_1^i w_j^i, w_1^i v_j^i : 2 \le j \le p\}.$$

Obtain a graph $G(p,s)$ from the disjoint union of $J_1, J_2, \ldots, J_s$ by identifying $w_p^1, w_p^2, \ldots, w_p^s$ into one vertex w_p. See Fig. 1 for an example of $G(p,1)$. Then we have the following observations which justify the conclusions stated in this example.
(i) $\Delta(G(p,s)) = \max\{2p, 2s\}$.
(ii) $4 - \frac{2}{p} \le mad(G(p,s)) < 4$.
(iii) If $r = 2p$, then $\chi_r(G(p,s)) \ge \frac{3r}{2}$.

Proof. Direct computation yields Example 2.2(i) and that the average degree of $G(p,s)$ is

$$\frac{2|E(G(p,s))|}{|V(G(p,s))|} = \frac{2s(6p-3)}{s(3p-1)+1},$$

which is an increasing function in p as well as in s. As $p \ge 2$ and $s \ge 1$, with $q'_s = \frac{2s(6p-3)}{s(3p-1)+1}$, Example 2.2(ii) follows from the fact that

$$3 \le 4 - \frac{2}{p} \le \frac{2s(6p-3)}{s(3p-1)+1} \le \frac{2s(6p-3)}{s(3p-1)} = \frac{2(6p-3)}{(3p-1)} < \frac{2(6p-2)}{3p-1} = 4.$$

It remains to justify Example 2.2(iii). Let $r \ge 2p$. Suppose that $G(p,s)$ has a (k,r)-coloring $c : V(G(p,s)) \mapsto \bar{k} = \{1,2,\ldots,k\}$. Let $G = G(p,s)$. Since $N_G(u_1^1) = \{v_1^1, w_1^1, u_2^1, u_3^1, \ldots, u_p^1, v_2^1, v_3^1, \ldots, v_p^1\}$, it follows by $r \ge 2p$ that $|c(N_G(u_1^1))| = 2p$. Similarly, $|c(N_G(v_1^1))| = |c(N_G(w_1^1))| = 2p$. It follows that $|c(V(J_1))| = |V(J_1)| = 3p$, and so $k \ge |c(V(J_1))| = 3p = \frac{3r}{2}$. □

3. Preliminaries and reductions

For an integer $i \ge 0$ and a graph G, let $D_i(G) = \{v \in V(G) : d_G(v) = i\}$, and $D_{\ge i}(G) = \cup_{j\ge i} D_j(G)$. A vertex v is a k-**vertex**, (k^+-**vertex**, k^--**vertex**, respectively) of G if $v \in D_k(G)$ ($v \in D_{\ge k}(G)$, $v \in V(G) - D_{\ge k+1}(G)$, respectively). We define $n_i(v) = |D_i(G) \cap N_G(v)|$. For an integer $p \ge 1$ and $u, w \in V(G)$, a (u,w)-path $P = uv_1v_2\cdots v_pw$ of G is **internally divalent** in G if for every i with $1 \le i \le p$, $v_i \in D_2(G)$. An internally divalent (u,w)-path of length $p+1$ is also called as a p-**link**. When such a p-link P exists, the two end vertices u and w are said to be p-**linked**.

Let $V' \subseteq V(G)$ be a vertex subset of a graph G. As in [4], $G[V']$ is the subgraph of G induced by V'. A mapping $c : V' \mapsto \bar{k}$ is a **partial** (k,r)-**coloring** of G if c is a (k,r)-coloring of $G[V']$. The subset V', denoted by $S(c)$, is the **support** of c. If c_1, c_2 are two partial (k,r)-colorings of G such that $S(c_1) \subseteq S(c_2)$ and such that for any $v \in S(c_1)$, $c_1(v) = c_2(v)$, then we say that c_2

is an **extension** of c_1. Given a partial (k, r)-coloring c on $V' \subset V(G)$, for each $v \in V - V'$, define $\{c(v)\} = \emptyset$; and for every vertex $v \in V$, we extend the definition of $c(N_G(v))$ by setting $c(N_G(v)) = \cup_{z \in N_G(v)}\{c(z)\}$, and define

$$c[v] = \begin{cases} \{c(v)\}, & \text{if } |c(N_G(v))| \geq r; \\ \{c(v)\} \cup c(N_G(v)), & \text{otherwise.} \end{cases} \tag{1}$$

By (1), $|c[v]| \leq r$.

For any vertex $v \in V(G)$, to count the number of vertices in $N_G[v]$ which affects the color choices of its uncolored neighbors, we define the **modified degree** $d'(v)$ of v as follows.

$$d'(v) = \begin{cases} d(v), & \text{if } d(v) \leq r; \\ 1, & \text{if } d(v) \geq r + 1. \end{cases} \tag{2}$$

Observations 3.1 and 3.2 follow from (1) and (2) immediately.

Observation 3.1. *Let c be a partial (k, r)-coloring of G with support $S(c)$. For any $u \notin S(c)$, and for any $v \in N_G(u)$, by the definition of $c[v]$, we have $|c[v]| \leq \min\{d(v), r\}$ and $c[v]$ represents the colors that cannot be used as $c(u)$ if one wants to extend c to include u in the support. As the condition (C2) should hold for u under such an extension of c, the colors in $\overline{k} - \bigcup_{v \in N_G(u)} c[v]$ are available colors to define $c(u)$ in extending the support of c from $S(c)$ to $S(c) \cup \{u\}$ so that the extended c remains a partial (k, r)-coloring of G.*

Observation 3.2. *A partial (k, r)-coloring c of G is given. If v has only one uncolored neighbor, then $|c[v]| \leq d'(v)$.*

To build some tools to be applied in our arguments, we present a few lemmas in this section. Lemma 3.3 follows from the definition immediately.

Lemma 3.3. *Let G be a graph with components $G_1, G_2, \ldots, G_c$. Then $\chi_r(G) \leq k$ if and only if for every i, $\chi_r(G_i) \leq k$.*

Lemma 3.4 (*Lemma 3.2 of [22]*). *Let $v \in D_2(G)$ with $N_G(v) = \{u, w\}$, and c be a partial (k, r)-coloring of G with $v \notin S(c)$, $u, w \in S(c)$ such that $c(u) \neq c(w)$. If $|c[u] \bigcup c[w]| < k$, then G has a partial (k, r)-coloring c' such that $S(c) \cup \{v\} \subseteq S(c')$ and that for any $z \in S(c)$, $c(z) = c'(z)$.*

Lemma 3.5. *Let $\ell, r > 0$ be integers with $\ell > r$ and G be a graph. Each of the following holds.*
(i) Suppose that G has a vertex $v \in D_1(G)$. If $\chi_r(G - v) \leq \ell$, then $\chi_r(G) \leq \ell$.
(ii) Suppose that G has a vertex w_1 with $d'(w_1) \leq \ell - 2$ which is 2-linked to a vertex w_2 via an internally divalent path $P = w_1u_1u_2w_2$ with $d'(w_2) \leq \ell - 3$. If $\chi_r(G - \{u_1, u_2\}) \leq \ell$, then $\chi_r(G) \leq \ell$.
(iii) Suppose that G has a vertex u with $d_G(u) \leq 6$ which is 1-linked to a vertex w_1 with $d'(w_1) \leq 13$ via a divalent path uvw_1, and that $\sum_{x \in N_G(u) \setminus \{v\}} d'(x) \leq \ell - 2$. If $\ell \geq 20$ and $\chi_r(G - v) \leq \ell$, then $\chi_r(G) \leq \ell$.
(iv) Suppose for some integer $p \geq 2$, that G has a set of vertices $\{a_i\}$, where the subscripts are taken modulo p, such that for every i, $d(a_i) \leq r$ and a_i is 3-linked via an internally divalent path $a_ib_{2i}c_ib_{2i+1}a_{i+1}$ to a_{i+1}. Let $H = G - \{b_0, b_1, \ldots, b_{2p-1}, c_0, \ldots, c_{p-1}\}$. If $\ell \geq 5$ and $\chi_r(H) \leq \ell$, then $\chi_r(G) \leq \ell$.

Proof. (i) Suppose G has a vertex $v \in D_1(G)$ and $G - v$ has an (ℓ, r)-coloring c. Let u be the only neighbor of v. For $|c[u]| \leq r$, we extend c to an (ℓ, r)-coloring of G by letting $c(v) \in \overline{\ell} - c[u]$.
(ii) Suppose that $G - \{u_1, u_2\}$ has an (ℓ, r)-coloring c. As $|c[w_1] \bigcup \{c(w_2)\}| \leq d'(w_1) + 1 \leq \ell - 1$, c can be extended to c_1 by letting $c_1(u_1) \in \overline{\ell} - c[w_1] \bigcup \{c(w_2)\}$. Thus c_1 is a partial (ℓ, r)-coloring with $S(c_1) = V(G) - \{u_2\}$ and $c_1(u_1) \neq c_1(w_2)$. As $|c_1[u_1] \bigcup c_1[w_2]| \leq d(u_1) + d'(w_2) \leq 2 + d'(w_2) < \ell$, it follows by Lemma 3.4 that c_1 can be further extended to an (ℓ, r)-coloring c_2 of G. This proves (ii).
(iii) Suppose that $G - \{v\}$ has an (ℓ, r)-coloring c. Let c_0 be the restriction of c to $V(G) - \{v, u\}$. By Observation 3.2, $|\bigcup_{x \in N_G(u)} c_0[x]| \leq 1 + \sum_{x \in N_G(u), x \neq v} d'(x) \leq 1 + (\ell - 2) < \ell$, and so c_0 can be extended to c_1 by taking $c_1(u) \in \overline{\ell} - \bigcup_{x \in N_G(u)} c_0[x]$. Then c_1 is an (ℓ, r)-coloring of $V(G) - \{v\}$ with $c_1(u) \neq c_1(w_1)$. As $|c_1[w_1] \bigcup c_1[u]| \leq 13 + 6 < 20 \leq \ell$, it follows by Lemma 3.4 that c_1 can be extended to an (ℓ, r)-coloring c_2 of G. This proves (iii).
(iv) Let c be an (ℓ, r)-coloring of H. Since for every i with $0 \leq i \leq p - 1$, we have $|c[a_i]| \leq d(a_i) - 1 \leq r - 1 \leq \ell - 2$, it follows that for any i with $0 \leq i \leq p - 1$, there are at least two colors in $\overline{\ell} - c[a_i]$ available for coloring b_{2i-1} and b_{2i}. Hence coloring the set $\{b_0, b_1, \ldots, b_{2p-1}\}$ is equivalent to 2-list-coloring an even cycle. Thus we can extend c to an (ℓ, r)-coloring c_1 of $V(G) - \{c_0, \ldots, c_{p-1}\}$ satisfying $c_1(b_{2i}) \neq c_1(b_{2i+1})$ for any $0 \leq i \leq p - 1$, where the subscripts are taken modulo $2p$. For $\ell \geq 5$, by Lemma 3.4, c_1 can be extended to an (ℓ, r)-coloring of G. This proves (iv). □

4. Proof of Theorem 1.6

Throughout this section, let $\frac{1}{20} \geq \epsilon > 0$ and define $f(\epsilon) = \frac{16}{5\epsilon} + 2$. We will show that for any integer $r \geq f(\epsilon) \geq 66$, any graph with maximum average degree less than $14/5 - \epsilon$ has an $(r + 1, r)$-coloring. We shall argue by contradiction and assume that

$$G \text{ is a counterexample to Theorem 1.6 such that } |V(G)| + |E(G)| \text{ is minimized.} \tag{3}$$

By the assumption, we have $mad(G) < \frac{14}{5} - \epsilon$ and for some integer $r \geq f(\epsilon)$, G has no $(r+1, r)$-colorings, but for any non-empty proper subset $S \subset V(G) \cup E(G)$, $G - S$ has an $(r+1, r)$-coloring. In the following, we first investigate the structure of such a minimum counterexample G, and then use charge and discharge method to obtain a contradiction to complete the proof.

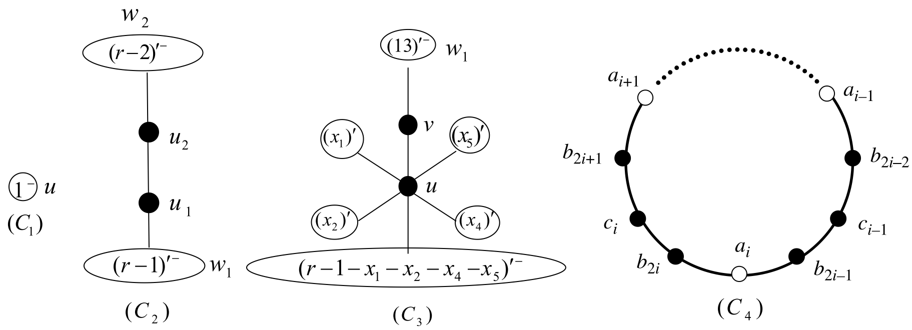

Fig. 2. Forbidden configurations for Theorem 1.6.

Lemma 4.1. *Each of the following holds for this graph G and the integer r.*
(i) G is connected and $\delta(G) \geq 2$.
(ii) G does not have a vertex w_1 with $d'(w_1) \leq r-1$ which is 2-linked to a vertex w_2 with $d'(w_2) \leq r-2$. (See Fig. 2 (C_2)).
(iii) G does not have a vertex u with $d_G(u) \leq 6$, that is, 1-linked to a vertex of modified degree at most 13, and such that the sum of the modified degrees of its neighbors is at most $r+1$. (See Fig. 2 (C_3)).
(iv) $p \geq 2$, and G does not have a set of vertices $\{a_i\}$, such that for any i, where i is taken modulo p, a_i is 3-linked to a_{i+1}. (See Fig. 2 (C_4)).
(v) G does not have a 4-link, and no link can be a cycle. Furthermore, if two vertices u, v are 3-linked in G, then $u, v \in D_r(G)$.

Proof. (i)–(iv) follow from Lemma 3.5(i)–(iv) by setting $\ell = r+1$ respectively.

To prove (v), we first observe that by (ii), there is no 4-link in G. Suppose G has a p-link $P = vv_1v_2 \ldots v_pv$ which is a cycle. Since G is simple, $2 \leq p \leq 3$. If $p = 2$ and $d_G(v) \geq r+1$, then by (3), $G - \{v_1\}$ has an $(r+1, r)$-coloring c, which can be extended to an $(r+1, r)$-coloring of G by letting $c(v_1) \in \overline{r+1} - \{c(v), c(v_2)\}$. If $p = 2$ and $d_G(v) \leq r$, then by (3), $G - v_1v_2$ has an $(r+1, r)$-coloring c which is also an $(r+1, r)$-coloring of G. In either case, a contradiction to (3) is obtained. Hence we assume that $p = 3$. By (3), $G - \{v_1, v_2\}$ has an $(r+1, r)$-coloring c. If $c[v] \neq \{c(v)\}$, then $c(v_3) \in c[v]$. As $|c[v] \bigcup \{c(v_3)\}| \leq r$, there exist an $\eta_1 \in \overline{r+1} - (c[v] \bigcup \{c(v_3)\})$ and $\eta_2 \in \overline{r+1} - \{c(v), c(v_3), \eta_1\}$. Let

$$c_1(z) = \begin{cases} c(z) & \text{if } z \in V(G) - \{v_1, v_2\} \\ \eta_1 & \text{if } z = v_1 \\ \eta_2 & \text{if } z = v_2. \end{cases}$$

Then c_1 is an $(r+1, r)$-coloring of G, contradicts (3). Suppose u is 3-linked to v via an internally divalent path $uv_1v_2v_3v$. u and v must be distinct. And by (ii) and (2), $d'_G(u) = d_G(u) = r = d'_G(v) = d_G(v)$. $\square$

We shall use discharge method to find a contradiction to complete the proof. For each vertex $x \in V(G)$, define the initial charge of x as $d_G(x)$. Let p_0 be a vertex not in $V(G)$, viewed as a common pot of the charges, define its initial charge equal to 0. The charge of a vertex will be renewed after every operation of charge transferring is done on it according to the following rules (R_1), (R_2), (R_3) and (R_g) (see Fig. 3 for an illustration). Any 2-vertex can only be the receiver during the operations of charge transferring. Now we list the rules from the point of view of the givers. Consider every vertex $x \in D_{\geq 3}(G) \cup \{p_0\}$.
(R_1) Suppose that $3 \leq d_G(x) \leq 13$. If no vertex in $N_G(x) \cap D_2(G)$ is adjacent to a vertex other than x of degree at most 13, x gives nothing away.

For any vertex $a \in N_G(x) \cap D_2(G)$, let $y \in N_G(a) - \{x\}$.

$(R_{1.1})$ When $d_G(y) = 2$, then x gives $\frac{3}{5}$ to a;

$(R_{1.2})$ When $3 \leq d_G(y) \leq 13$, then x gives $\frac{2}{5}$ to a.

(R_2) If $14 \leq d_G(x) \leq r-4$, for any vertex $a \in N_G(x)$, x gives $\frac{4}{5}$ to a.
(R_3) Suppose that $d_G(x) \geq r-3$. For any vertex $a \in N_G(x)$.

$(R_{3.1})$ When $d_G(a) = 2$ with $y \in N_G(a) - \{x\}$, then x gives $\frac{4}{5} - \epsilon$ to a and $\frac{1}{5}$ to y;

$(R_{3.2})$ When $d_G(a) \geq 3$, then x gives $1 - \epsilon$ to a.

(R_g) If $x \in D_{\geq r}(G)$, x gives additional $\frac{2}{5}$ to p_0; If $x = p_0$, for any vertex $a \in D_2(G)$ which is adjacent to two vertices of degree 2, x gives $\frac{2}{5}$ to a.

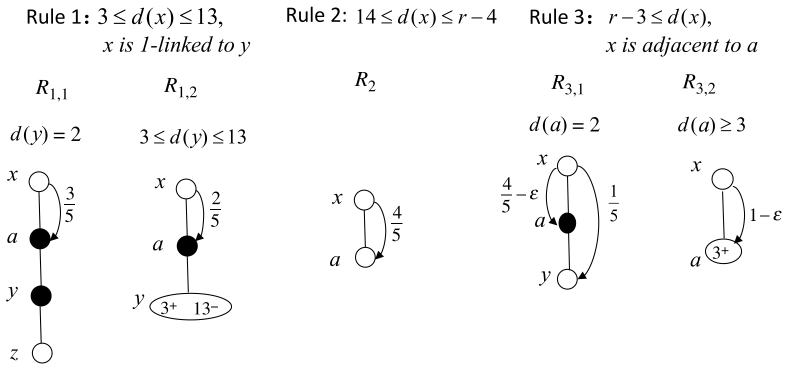

Fig. 3. Discharging rules R_1, R_2, R_3 for Theorem 1.6.

In the rest of the discussion, we let $w(x)$ denote the final charge of a vertex x, after all the recharging operations are complete on it. Let $\mathcal{P}$ be the set of all maximal divalent paths of length at least 4. By Lemma 4.1(v), every $P \in \mathcal{P}$ has length 4, and the two ends of P are in $D_r(G)$. Let $H = G[\cup_{P\in\mathcal{P}} V(P)] - E(G[D_r(G)])$. By Lemma 4.1(iv) and (v), the subgraph H of G is acyclic, and so $|D_r(G)|$ is not less than the number of vertices of degree 2 whose neighbors are of degree 2 in G. By R_g, we conclude that

$$w(p_0) \ge 0. \tag{4}$$

By Lemma 4.1(i), $\delta(G) \ge 2$. We will show that, for any $x \in V(G)$, $w(x) \ge \frac{14}{5} - \epsilon$ by justifying the following claims.

Claim 1. *Let $x \in D_2(G)$. Then $w(x) \ge \frac{14}{5} - \epsilon$.*

Proof of Claim 1. Since $x \in D_2(G)$, there exists a maximal internally divalent path P in G such that $x \in V(P)$. Let $p = |V(P)|-1$. By Lemma 4.1(v), P is not a cycle and $1 \le p \le 3$. Let u, u' be the two end vertices of the path P. Since P is maximal and by Lemma 4.1(i), $u, u' \in D_{\ge 3}(G)$. We assume that $d'(u) \ge d'(u')$.
Case 1 $p = 1$. Then $N_G(x) = \{u, u'\}$. If $\max\{d_G(u), d_G(u')\} \ge 14$, then by R_2 or $R_{3.1}$, $w(x) \ge d_G(x) + \frac{4}{5} - \epsilon = \frac{14}{5} - \epsilon$. Hence we assume that $\max\{d_G(u), d_G(u')\} \le 13$. Then by $R_{1.2}$, x receives $\frac{2}{5}$ from each of u and u', and so $w(x) = d_G(x) + 2 \times \frac{2}{5} = \frac{14}{5}$.
Case 2 $p = 2$. Let $P = uvv'u'$ with $x \in \{v, v'\} \subseteq D_2(G)$. By Lemma 4.1(ii), $d_G(u) \ge d'(u) \ge r-1$. By $R_{3.1}$, $w(v) \ge d_G(v) + \frac{4}{5} - \epsilon = \frac{14}{5} - \epsilon$. By $R_{3.1}$, $R_{1.1}$ or R_2, v' receives $\frac{1}{5}$ from u and at least $\frac{3}{5}$ from u'. Thus $w(v') \ge d_G(v') + \frac{1}{5} + \frac{3}{5} = \frac{14}{5}$.
Case 3 $p = 3$. Let $P = ua_2a_3a_4u'$ with $x \in \{a_2, a_3, a_4\} \subseteq D_2(G)$. By Lemma 4.1(v), $u, u' \in D_r(G)$. By $R_{3.1}$, $w(a_i) = d_G(a_i) + \frac{4}{5} - \epsilon = \frac{14}{5} - \epsilon$, for $i \in \{2, 4\}$. By $R_{3.1}$ and R_g, a_3 receives $\frac{1}{5}$ from u and u' and $\frac{2}{5}$ from p_0, and so $w(a_2) = d_G(a_2) + \frac{1}{5} + \frac{1}{5} + \frac{2}{5} = \frac{14}{5}$. □

Claim 2. *Let $x \in D_3(G)$. Then $w(x) \ge \frac{14}{5} - \epsilon$.*

Proof of Claim 2. Let $N_G(x) = \{x_1, x_2, x_3\}$. Since $r \ge f(\epsilon) \ge 66$ and by Lemma 4.1(iii), if $\{x_1, x_2, x_3\} \subseteq D_2(G)$, then every one x_i is adjacent to a vertex of degree at least 14.
Case 1 $N_G(x) \cap D_2(G)$ contains two vertices (x_1, x_2, say) each of which is adjacent to a vertex of degree at most 13 other than x. By Lemma 4.1(iii), $d_G(x_3) \ge d'(x_3) \ge r-2$. Hence by $R_{3.2}$, x receives $1-\epsilon$ from x_3, and by R_1, gives away at most $2 \times \frac{3}{5}$. Thus $w(x) \ge d_G(x) + 1 - \epsilon - \frac{6}{5} = \frac{14}{5} - \epsilon$.
Case 2 $N_G(x) \cap D_2(G)$ contains just one vertex (x_1, say) which is adjacent to a vertex of degree at most 13 other than x. As $r \ge 66$ and by Lemma 4.1(iii), we may assume that $d_G(x_2) \ge d'(x_2) \ge \lceil \frac{r}{2} \rceil \ge 33$. Hence either $d_G(x_2) \le r-4$, then $w(x) \ge d_G(x) + \frac{4}{5} - \frac{3}{5} \ge \frac{14}{5} - \epsilon$ by R2 and R1; or $d_G(x_2) \ge r-3$ by $R_{3.2}$ and R_1, $w(x) \ge d_G(x) + 1 - \epsilon - \frac{3}{5} \ge \frac{14}{5} - \epsilon$.
Case 3 No vertex in $N_G(x) \cap D_2(G)$ is adjacent to a vertex of degree at most 13. As $d_G(x) \le 13$, by R_1, $w(x) = d_G(x) \ge \frac{14}{5} - \epsilon$. □

Claim 3. *Let $x \in D_d(G)$ for some integer d with $4 \le d \le 6$. Then $w(x) \ge \frac{14}{5} - \epsilon$.*

Proof of Claim 3. If no vertex in $N_G(x) \cap D_2(G)$ is adjacent to a vertex of degree at most 13, then by R_1, $w(x) = d \ge \frac{14}{5} - \epsilon$. Hence we assume that $N_G(x) \cap D_2(G)$ contains a vertex (x_1, say) which is adjacent to a vertex of degree at most 13 other than x. As $r \ge 66$ and by Lemma 4.1(iii), $N_G(x) - D_2(G)$ contains a vertex x_2 such that $d_G(x_2) \ge d'(x_2) \ge \lceil \frac{r}{5} \rceil \ge 14$. Hence either $d_G(x_2) \le r-4$ and by R_2 and R_1, $w(x) \ge d + \frac{4}{5} - (d-1) \times \frac{3}{5} \ge \frac{14}{5} - \epsilon$; or $d_G(x_2) \ge r-3$ by $R_{3.2}$ and R_1, $w(x) \ge d + 1 - \epsilon - (d-1) \times \frac{3}{5} \ge \frac{14}{5} - \epsilon$. □

Claim 4. *Let $x \in D_d(G)$ for some integer $d \ge 7$. Then $w(x) \ge \frac{14}{5} - \epsilon$.*

Proof of Claim 4. If $7 \le d \le 13$, then by R_1, $w(x) \ge d - \frac{3d}{5} \ge \frac{2d}{5} \ge \frac{14}{5}$. If $14 \le d \le r-4$, then by R_2, $w(x) = d - \frac{4d}{5} = \frac{d}{5} \ge \frac{14}{5}$. Finally we assume that $d \ge r-3$. By R_3, R_g and by $d \ge r-3 \ge \frac{16}{5\epsilon} - 1$, we have $w(x) \ge d - d(1-\epsilon) - \frac{2}{5} = d\epsilon - \frac{2}{5} \ge \epsilon(\frac{16}{5\epsilon} - 1) - \frac{2}{5} = \frac{14}{5} - \epsilon$. □

By Claims 1–4 and by (4), we conclude that $w(x) \ge \frac{14}{5} - \epsilon$ for any $x \in V(G)$ and $w(p_0) \ge 0$. It follows by the assumption of Theorem 1.6 that

$$\frac{14}{5} - \epsilon \le \frac{\sum_{x \in V(G)} w(x)}{|V(G)|} \le \frac{\sum_{x \in V(G)} d_G(x)}{|V(G)|} \le mad(G) < \frac{14}{5} - \epsilon.$$

This contradiction justifies Theorem 1.6.

As remarked by Bonamy, Lévêque, and Pinlouin in [3], the limitation of this method lies in the configuration when a vertex of degree 3 is 2-linked to two vertices of degree r, and is adjacent to a vertex of degree r. Assume that for some real number α, every vertex has a new charge at least $2+\alpha$ after recharging. Then, vertices in $D_2(G) \cup D_3(G)$ need to receive at least $4\alpha - (1-\alpha) = 5\alpha - 1$. It means that if $\alpha \ge \frac{4}{5}$, then a vertex $v \in D_r(G)$ will have to be discharged at least 1 for each such configuration it is adjacent to. However, the current hypothesis cannot forbid the existence of such configurations. This indicates a barricade when attempting to improve the result in this direction.

5. Proof of Theorem 1.7

Let ϵ be a real number with $1 > \epsilon > 0$. Define $M = \frac{8}{\epsilon} - 2$ and $h(\epsilon) = 5M - 9$. Thus

$$M - (4-\epsilon) = M \times (1 - \frac{\epsilon}{2}), \text{ and } h(\epsilon) \ge 2M + 1. \tag{5}$$

To prove Theorem 1.7 by contradiction, we assume the for some integer $k \ge r + h(\epsilon) = r + 5M - 9$, there exists a graph G with $mad(G) < 4 - \epsilon$ such that G does not have a (k, r)-coloring and such that

$$|V(G)| + |E(G)| \text{ is minimized.} \tag{6}$$

A vertex $v \in D_2(G) \cup D_3(G)$ is **weak** if v has at most one neighbor of modified degree more than M. Some useful properties of such a minimum counterexample G will be investigated in Lemma 5.1.

Lemma 5.1. *For the integer k and a counterexample G satisfying (6), each of the following holds.*
(i) G is connected and $\delta(G) \ge 2$.
(ii) If $u \in V(G)$ with $d_G(u) \le M$ and with a weak vertex $x \in N_G(u)$, then either $|\{v \in N_G(u) : d'(v) \ge 4\}| \ge 4$ or $|\{v \in N_G(u) : d'(v) > M\}| \ge 2$.

Proof. (i) follows from Lemma 3.5(i) with $\ell = k$.

(ii) Given a vertex $u \in V(G)$ with $d_G(u) \le M$ and with a weak vertex $x \in N_G(u)$, let $N' = \{v \in N_G(u) : d'(v) \ge 4\}$, $N'' = N_G(u) - N'$, and $N''' = \{v \in N_G(u) : d'(v) > M\}$. By contradiction, we assume that both $|N'| \le 3$ and $|N'''| \le 1$ (see Fig. 4 C2). By (6), $G - \{ux\}$ has a (k, r)-coloring c. Let c_0 denote the restriction of c to $V(G) - \{u, x\}$. Since $|N'| + |N''| = d_G(u) \le M$ and since every $v \in N''$ satisfies $d'(v) \le 3$, we have

$$\begin{aligned} |\bigcup_{v \in N_G(u)} c_0[v]| &\le \sum_{v \in N'''} |c_0[v]| + \sum_{v \in N' - N'''} |c_0[v]| + \sum_{v \in N'' - \{x\}} |c_0[v]| + |c_0[x]| \\ &\le r + 2M + 3(M-4) + 2 = r + 5M - 10 \le k - 1. \end{aligned}$$

Hence there exists $\eta_1 \in \overline{k} - (\cup_{v \in N_G(u)} c_0[v])$. Since $x \in N_G(u)$, we have $\eta_1 \notin c_0[x]$. As $d_G(x) \le 3$, we have $|\bigcup_{v \in N_G(x)} c_0[v] \cup \{\eta_1\}| \le r + 2M + 1 < k$, we can pick $\eta_2 \in \overline{k} - (\bigcup_{v \in N_G(x)} c_0[v] \cup \{\eta_1\})$. Define

$$c_1(z) = \begin{cases} c(z) & \text{if } z \in V(G) - \{u, x\} \\ \eta_1 & \text{if } z = u \\ \eta_2 & \text{if } z = x. \end{cases}$$

By definition, c_1 is a (k, r)-coloring of G, contrary to (6). This proves (ii) and completes the proof of the lemma. □

To complete the proof of Theorem 1.7, we again apply charge and discharge method to obtain a contradiction. For every $x \in V(G)$, define the initial charge of x as $d_G(x)$. Let $N_1(x) = \{v \in N_G(x) : v \text{ is a weak vertex}\}$ and $N_2(x) = N_G(x) - N_1(x)$. The rules to recharge x are given below. (See Fig. 5 for an illustration.)

(R_1) If $d_G(x) > M$, for each vertex $x' \in N_G(x)$, x gives $(1 - \frac{\epsilon}{2})$ to x'.

(R_2) If $d_G(x) \le M$, for each $x' \in N_1(x)$, x gives $(1 - \frac{\epsilon}{2})$ to x'.

Let $w(u)$ denote the final charge of each $u \in V(G)$ after all the operations of charge transferring are complete on it. We shall show that, for any vertex $u \in V(G)$, $w(u) \ge 4 - \epsilon$ by justifying Claims 1 and 2. By Lemma 5.1(i), $\delta(G) \ge 2$.

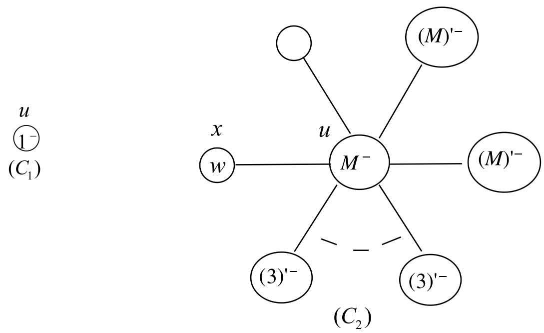

Fig. 4. Forbidden configurations for Theorem 1.7.

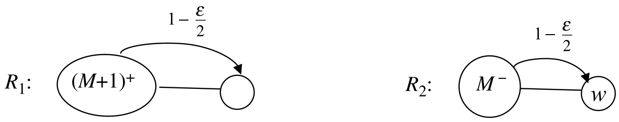

Fig. 5. Discharging rules R_1, R_2 for Theorem 1.7.

Claim 1. *If in the recharging process, u discharges some weight away, then* $w(u) \geq 4-\epsilon$.

Proof of Claim 1. If $d_G(u) > M$, then by R_1 and by definition of M, $w(u) \geq d_G(u) - d_G(u)(1-\frac{\epsilon}{2}) = \frac{d_G(u)\epsilon}{2} > M\frac{\epsilon}{2} = 4-\epsilon$. Hence we assume that $d_G(u) \leq M$. By R_2, u discharges if and only if $|N_1(u)| > 0$. By Lemma 5.1(ii), either $|\{v \in N_G(u) : d'(v) \geq 4\}| \geq 4$ or $\{v \in N_G(u) : d'(v) > M\}| \geq 2$. If $v_1, v_2, v_3, v_4 \in \{v \in N_G(u) : d'(v) \geq 4\}$, then as $d_G(v_i) \geq d'(v_i) \geq 4$, for each i with $1 \leq i \leq 4$, we have $|N_1(u)| \leq |N_G(u) - \{v_1, v_2, v_3, v_4\}| = d_G(u) - 4$. By R_2, $w(u) \geq 4 + (d(u) - 4) \times (1-\frac{\epsilon}{2}) > 4-\epsilon$. Hence we assume that $u_1, u_2 \in \{v \in N_G(u) : d'(v) > M\}$, and so $d_G(u_1) \geq d'(u_1) > M$ and $d_G(u_2) \geq d'(u_2) > M$. By R_1, each of u_1 and u_2 discharges $1-\frac{\epsilon}{2}$ to u. As $|N_1(u)| \leq |N_G(u) - \{u_1, u_2\}| = d_G(u) - 2$, by R_1 and R_2, we have $w(u) \geq 2 + 2(1-\frac{\epsilon}{2}) + (d_G(u) - 2) \times (1-\frac{\epsilon}{2}) \geq 4-\epsilon$. $\square$

Claim 2. *If in the recharging process, u never discharges, then* $w(u) \geq 4-\epsilon$.

Proof of Claim 2. If $d_G(u) \geq 4$, then as u never discharges, we have $w(u) \geq d_G(u) > 4-\epsilon$. Hence we assume that $d_G(u) \leq 3 < M$ and $N_G(u)$ contains no weak vertices.

If u is a weak vertex, then u gives nothing away and receives $1-\frac{\epsilon}{2}$ from each of its neighbors. By the definition of a weak vertex, $2 \leq d_G(u) \leq 3$ and so $w(u) \geq d_G(u) + d_G(u)(1-\frac{\epsilon}{2}) \geq 2 + 2(1-\frac{\epsilon}{2}) = 4-\epsilon$. Hence we assume that u is not a weak vertex. By the definition of weak vertices, there are at least two vertices $u', u'' \in N_G(u)$ with $d_G(u') \geq d'(u') > M$ and $d_G(u'') \geq d'(u'') > M$. By R_1, both u' and u'' discharge $1-\frac{\epsilon}{2}$ to u, and so $w(u) \geq d_G(u) + d_G(u)(1-\frac{\epsilon}{2}) \geq 2 + 2\times(1-\frac{\epsilon}{2}) = 4-\epsilon$. $\square$

By Claims 1 and 2, every vertex u of G has a final charge $w(u)$ at least $4-\epsilon$. It follows that

$$4-\epsilon \leq \frac{\sum_{u \in V(G)} w(u)}{|V(G)|} = \frac{\sum_{u \in V(G)} d_G(u)}{|V(G)|} \leq mad(G) < 4-\epsilon.$$

This contradiction establishes Theorem 1.7.

Remark. We choose to present a simple proof despite the fact that it means the function h is probably not best possible. However, in some sense, it is optimal up to a constant factor as the graph family presented in Fig. 1 shows that the function cannot be reduced to $\frac{2}{\epsilon}$. Indeed, the family $\{G(p, 1)\}_{p \in \mathbb{N}^*}$ satisfies $\chi_r(G(p, 1)) \geq r + \frac{2}{4-mad(G(p,1))}$.

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