An $O(\log_2(N))$ Algorithm for Reliability Evaluation of h-Extra Edge-Connectivity of Folded Hypercubes

Mingzu Zhang, Lianzhu Zhang, Xing Feng, and Hong-Jian Lai

Abstract—**Reliability analysis of an interconnection network is of great significance to the design and maintenance of multiprocessor systems. The h-extra edge-connectivity of a given interconnected network G with N processors, denoted by $\lambda_h(G)$, is the minimum cardinality of set of faulty links, such that whose removal will disconnect the network with all its resulting components having at least h processors for $h \leq N/2$. It gives a more refined quantitative analysis of indicators of the robustness of a multiprocessor system in the presence of failing links. The n-dimensional folded hypercube FQ_n, as one of potential interconnected networks, is a well-known variation of the hypercube structure with $N = 2^n$ processors. In this paper, the h-extra edge-connectivity of the network FQ_n, $\lambda_h(FQ_n)$, is first investigated for each well-defined positive integer $h \leq N/2$. We divide the interval $1 \leq h \leq N/2$ into some subintervals and obtain some properties of $\lambda_h(FQ_n)$ in these subintervals. Then, we deduce a recursive relation of $\lambda_h(FQ_n)$. Based on this recursion, an efficient $O(\log_2(N))$ algorithm is designed to totally determine the exact values and λ_h-optimality of $\lambda_h(FQ_n)$ for each $h \leq N/2$.**

Index Terms—**Algorithm, edge fault tolerance, folded hypercube, h-extra edge-connectivity, interconnection networks, multiprocessor, reliability evaluation.**

Notations

$G(V,E)$	Simple connected graph with the vertex set V and edge set E.
$V(G)$	Vertex set of G.
$E(G)$	Edge set of G.
$\lvert V\rvert$	Number of elements in the set V.
$G-F$	Subgraph after removing the edges in F from G for $F \subset E(G)$.
$\kappa(G)$	Connectivity of G.
$\lambda(G)$	Edge-connectivity of G.
$\kappa_h(G)$	h-extra connectivity of G.
$\lambda_h(G)$	h-extra edge-connectivity of G.
FQ^n	n-dimensional folded hypercube network.
$\sum_{i=0}^{s} 2^{t_i}$	Binary representation of a positive integer h, with $t_0 > t_1 > \cdots > t_s$.
$\lceil x \rceil$	Ceiling function of x.
$\lfloor x \rfloor$	Floor function of x.
$[X,\overline{X}]$	Set of edges of G with one end in X and the other in $\overline{X} = V(G)\setminus X$ for $X \subset V(G)$.
$\lvert[X,\overline{X}]\rvert$	Number of elements of $[X,\overline{X}]$.
$G[X]$	Subgraph induced by X in G.
$ex_m(G)$	Twice of the maximum number edges of the subgraph of G induced by m vertices.
$\xi_m(G)$	$\min\{\lvert[X,\overline{X}]\rvert : \ \lvert X\rvert = m \leq \lfloor \lvert V(G)\rvert/2 \rfloor$, and $G[X]$ is connected$\}$.

Manuscript received February 28, 2017; revised May 14, 2017 and September 23, 2017; accepted November 24, 2017. Date of publication January 9, 2018; date of current version March 1, 2018. This work was supported by the National Natural Science Foundation of China under Grant 11471273 and Grant 11671186. Associate Editor: M. Grottke. *(Corresponding author: Lianzhu Zhang.)*

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Digital Object Identifier 10.1109/TR.2017.2779130

I. Introduction

Rapid technological innovations in very large scale integration technology and Wafer-scale integration technology make it possible for research specialists to design and produce multiprocessor systems with hundreds or even thousands of processors integrated through an interconnection network on a single chip. Their research has found wide applications in daily life and is prospective. Usually, the underlying topology of an interconnection network is modeled by a connected graph $G(V,E)$, where vertices (or nodes) represent processors and edges (or links) represent communication links between processors. Thus, in this paper, graphs and networks are interchangeable.

Once a catastrophe occurs, the fragile topological structures of interconnection networks of parallel computing systems will be destroyed more or less because of malfunction of some links or processors. By the classical Menger's theorem, the connectivity or edge-connectivity of a connected graph G, denoted by $\kappa(G)$ or $\lambda(G)$, is an important parameter to evaluate the reliability and fault tolerance of a network, which is the minimum number of vertices or edges whose deletion from G makes the resulting graph disconnected. Generally, the bigger the $\kappa(G)$ or $\lambda(G)$ is, the more reliable the network is. However, these parameters tacitly assume that all vertices or all edges that are incident to the same vertex can potentially fail simultaneously. This is practically impossible in large network applications [4]. Moreover, the further properties of every disconnected component are not considered. To address this deficiency, the notion of conditional connectivity and conditional edge-connectivity, introduced by F. Harary in 1983 [12], generalized the theories of connectivity in both vertex and edge versions. Not only does it meet the increasing need of a more accurate measure of reliability of large-scale parallel processing systems, but also theoretically enriches the theory of network connectedness [1], [4], [7], [13], [22], [25], [32], [37], [38], [40]–[43], [45], and so on. In this paper, the emphasis is placed on the edge version, namely, the conditional edge-connectivity. The relationship between the vertex version and various diagnosability has been investigated [3], [14], [19]–[23], [36].

Let $\mathscr{P}$ be any property of a graph G and $F \subset E(G)$. The $\mathscr{P}$ edge-connectivity of G, denoted by $\lambda(G;P)$ in [12], if exists, is the minimum cardinality of F, such that $G-F$ is disconnected and every component of $G-F$ has property $\mathscr{P}$. In particular, given a positive integer h, when $\mathscr{P}$ denotes the property that a graph contains at least h vertices, this gives the h-extra edge-connectivity, which was first introduced by J. Fàbrega and M. A. Foil in 1996 [10] (also called h-restricted edge-connectivity [24]–[26], [40]). An edge subset F of a connected graph G is called as an h-extra edge-cut of G, if $G-F$ is no more connected, and each remaining component of $G-F$ has at least h vertices. The h-extra edge-connectivity of G, if any, denoted by $\lambda_h(G)$, is the minimum cardinality of an h-extra edge-cut of G. Given a vertex set $X \subset V(G)$, we denote $G[X]$ the subgraph induced by X, and denote $[X, \overline{X}]$ the set of edges of G in which each edge contains exactly one end in X and the other in $\overline{X} = V(G) \setminus X$. Let $\xi_m(G) = \min\{|[X,\overline{X}]| : |X| = m \le \lfloor |V(G)|/2 \rfloor$, and $G[X]$ is connected$\}$. If $\lambda_h(G) = \xi_h(G)$, we say that G is λ_h-optimal. For terminology and notation not defined here, we refer the reader to [2].

Note that $\lambda(G) = \lambda_1(G)$. The two-extra edge-connectivity is specifically called restricted edge-connectivity [7], [8]. Several authors have investigated the h-extra edge-connectivity for various classes of networks [1], [4], [7], [8], [10], [18], [24], [25], [27], [37], [38], [40], [41], [43]–[45]. The focus of these results is either on some classes of graphs for several h or on some special graphs for linearly and even exponentially many values of h [4], [18], [24], [25], [37], [38], [40], [41], [43], [45]. However, seldom do the researchers pay their attentions on thoroughly solving this problem for some classes of networks for each $h \le \lfloor |V(G)|/2 \rfloor$, except for trees, cycles, and Harary graphs. It is known that if G is a tree, if any, $\lambda_h(G) = 1$; if G is a cycle, $\lambda_h(G) = 2$. Liu *et al.* [24] have studied the optimality of the h-extra edge-connectivity of Harary graph. Regarding the computational complexity of the problem, Esfahanian and Hakimi [8] presented a polynomial-time algorithm for the computation of $\lambda_2(G)$. Very recently, in 2017, Montejano and Sau [26] discussed the complexity of computing the h-restricted edge-connectivity (also for h-extra edge-connectivity) for general cases. It is quite difficult for general h and G. Given a graph G with N vertices, they proved that the problem of determining that whether there exists an h-extra edge-cut or not for $h \le N/2$ is NP-complete, even when the maximum degree of the G is at most 5. And for a given integer l, the problem of determining that whether the h-extra edge-connectivity of G is at most l is NP-hard. More related results can be found in [26].

As a variant of the hypercube, the n-dimensional folded hypercube FQ_n, proposed first by El-Amawy and Latifi [6], is one of the most potential interconnected networks (see Fig. 1). Compared with an n-dimensional hypercube, the n-dimensional folded hypercube with the same number of vertices has larger vertex degree, half the diameter, highly symmetric property, and vertex-transitivity. In parallel computing, they have been used as underlying topologies of several parallel systems, such as ATM switches [28], [29], three-dimensional (3-D) FolH-NOC networks [5], [9], and PM2I networks [16], [17].

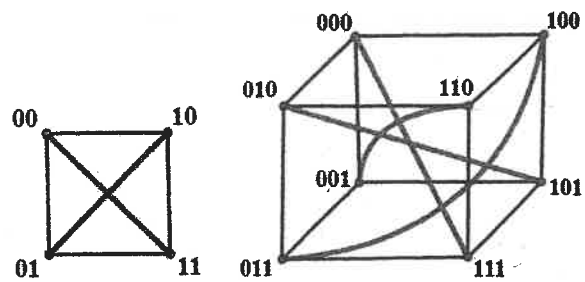

Fig. 1. Folded hypercube FQ_2 and FQ_3.

The exact values of the h-extra edge-connectivity $\lambda_h(FQ_n)$ of FQ_n for some h are given in the several literatures. El-Amawy and Latifi [6] proved that $\lambda_1(FQ_n) = n+1, n \ge 2$; Zhu and Xu [44] proved that $\lambda_2(FQ_n) = 2n, n \ge 2$; Xu *et al.* [45] proved that $\lambda_3(FQ_n) = 3n-1, n \ge 4$; Chang *et al.* [4] proved that $\lambda_4(FQ_n) = 4n-4, n \ge 5$; Yang and Li [37] gave the exact values of $\lambda_h(FQ_n)$ for $h \le n, n \ge 6$; Zhang *et al.* [41] gave the exact values of $\lambda_h(FQ_n)$ for exponentially many values of h, $n \ge 4$, $h \le 2^{\lceil \frac{n}{2} \rceil+1}$ and $2^{\lceil \frac{n}{2} \rceil+r} - l_r \le h \le 2^{\lceil \frac{n}{2} \rceil+r}$, where $r = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, $l_r = \frac{2^{2r+2}-1}{3}$ if n is odd, $l_r = \frac{2^{2r+1}-2}{3}$ if n is even.

The $\lambda_h(FQ_n)$ highly relies on the monotonic intervals and fractal-like structure of function $\xi_m(FQ_n)$. In [41], given a fixed n, there are exactly one monotone increasing interval and one monotone decreasing interval on the function $\xi_h(FQ_n)$ in term of the integer h for $h \le 2^{\lceil \frac{n}{2} \rceil+1}$, which makes it possible to give the exact values of $\lambda_h(FQ_n)$ by two formulations. Efforts have been made to show that the h-extra edge-connectivity of FQ_n is a constant for $2^{\lceil \frac{n}{2} \rceil+r} - l_r \le h \le 2^{\lceil \frac{n}{2} \rceil+r}$. The problem for the rest of the values of h becomes more complicated, and we fail to give general formulations by previous methods. For a fixed n, both the number of 2^{n-1} and the number of monotonic intervals of function $\xi_h(FQ_n)$ in term of the integer h for $h > 2^{\lceil \frac{n}{2} \rceil+1}$ explode, and the outline of the image of $\xi_h(FQ_n)$ has the fractal structure (self-similarity). An example is shown in Fig. 2, where $g(n)$ denotes the number of monotonic intervals of function $\xi_h(FQ_n)$ in term of the integer h for $h > 2^{\lceil \frac{n}{2} \rceil+1}$.

Because $2^{n-1} - (2^{\lceil \frac{n}{2} \rceil+1} + \sum_{r=2}^{\lfloor \frac{n}{2} \rfloor-1} l_r + \lfloor \frac{n}{2} \rfloor - 3) = \frac{5}{9}2^{n-1} - O(2^{\lceil \frac{n}{2} \rceil+1})$, and $\frac{5}{9} \approx 0.5556$, there are still about 55.56% values of h whose $\lambda_h(FQ_n)$ are undetermined.

In this paper, the problem on the h-extra edge-connectivity of folded hypercubes is completely solved for each well-defined $h \le 2^{n-1}$, and of course for the rest of 55.56% values of h. We redivide the interval $1 \le h \le 2^{n-1}$ into some subintervals, and each subinterval will be further divided, to obtain some properties of $\xi_h(FQ_n)$. Furthermore, we deduce a recursive relation on $\lambda_h(FQ_n)$. Based on them and some known results, an efficient $O(\log_2(N))$ algorithm is designed to totally determine the exact values and λ_h-optimality of $\lambda_h(FQ_n)$ for each $h \le 2^{n-1}$.

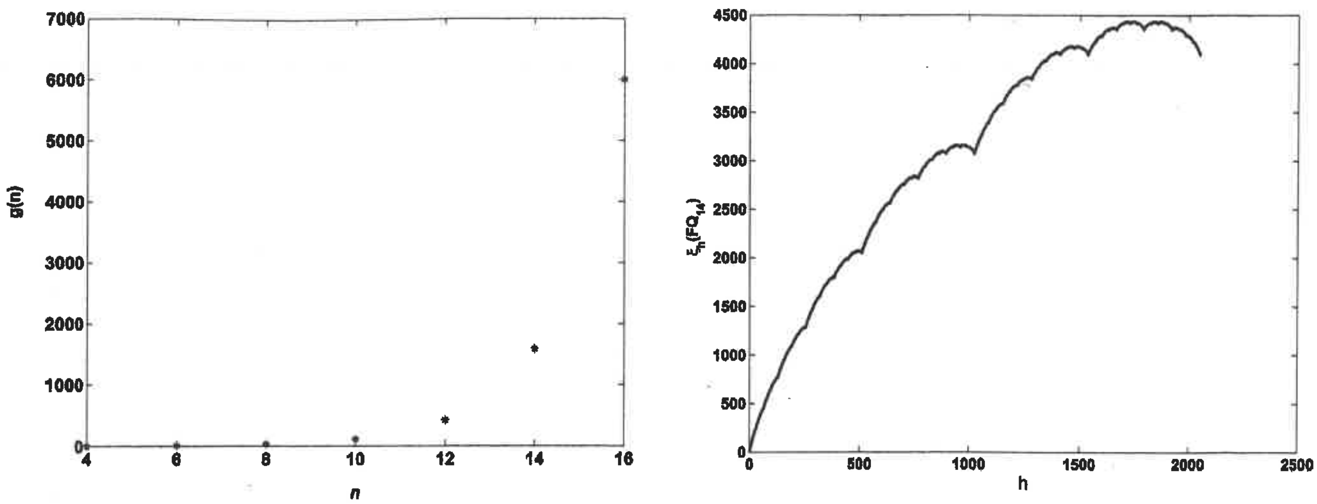

Fig. 2. Plots of functions $g(n)$ (left) and $\xi_h(FQ_{14})$ (right).

The rest of this paper is organized as follows. In Section II, some preliminaries, terminologies, and known results will be introduced. In Section III, the main results about the h-extra edge-connectivity of folded hypercubes will be given, from which we obtain an algorithm to calculate $\lambda_h(FQ_n)$. A simulation experiment is offered to discuss how our theoretical results can be applied to a specific real-world example in Section IV. In Section V, we give the proofs of our main results. The paper is concluded in Section VI.

II. Preliminaries and Terminologies

Let $G(V, E)$ be a graph G with vertex set $V(G)$ and edge set $E(G)$, and let $|V(G)|$ and $|E(G)|$ be the numbers of vertices and edges of G, respectively. The components of the graph G are its maximal connected subgraphs.

Recall that $\lambda_h(G)$ is the minimum number of an edge set of the graph G whose removal disconnects the graph G with all its components having at least h vertices. Suppose F is an h-extra edge-cut, it can be deduced that the optimal number is obtained only when there are exactly two components produced in $G - F$. In fact, if there exists a minimum h-extra edge-cut F_0, whose removal disconnects the connected graph G with all its components $H_1, H_2, \ldots, H_z$, $z > 2$ having at least h vertices, then $[V(H_1), \overline{V(H_1)}]$ is also an h-extra edge-cut, and $|[V(H_1), \overline{V(H_1)}]| < |F_0|$, a contradiction.

If we can find $X \subset V(G)$ such that $\xi_m(G) = |[X, \overline{X}]|$, with $|X| = m \leq \lfloor |V(G)|/2 \rfloor$, and both $G[X]$ and $G[\overline{X}]$ are connected, then, for each $1 \leq h \leq \lfloor |V(G)|/2 \rfloor$

$$\lambda_h(G) = \min\{\xi_m(G) : h \leq m \leq \lfloor |V(G)|/2 \rfloor\}. \quad (1)$$

For a d-regular graph, it follows that

$$\xi_m(G) = dm - ex_m(G) \quad (2)$$

where $ex_m(G)$ is twice of the maximum number edges of $G[X]$ with m vertices in G.

Definition 2.1: For an integer $n \geq 1$, the n-dimensional hypercube, denoted by Q_n, is a graph with 2^n vertices. The vertex set $V(Q_n) = \{x_n x_{n-1} \ldots x_1 : x_i \in \{0, 1\}, 1 \leq i \leq n\}$ is the set of all n-bit binary strings. Two vertices $u = u_n u_{n-1} \ldots u_1$ and $v = v_n v_{n-1} \ldots v_1$ of Q_n are adjacent if and only if they differ in exact one position.

Definition 2.2: Two vertices $x = x_n x_{n-1} \ldots x_1$ and $y = y_n y_{n-1} \ldots y_1$ of Q_n are complementary if and only if the bits of x and y are complements of each other, that is, $y_i = \overline{x_i} = 1 - x_i$ for each $i = 1, 2, \ldots, n$. The n-dimensional folded hypercube, denoted by FQ_n, is a graph obtained from Q_n by adding extra 2^{n-1} edges with two vertices of every edge complementary each other. These edges are called complementary edges, denoted by M.

Note that Q_n is n-regular n-connected and FQ_n is $(n+1)$-regular $(n+1)$-connected [6], [15]. The graphs shown in Fig. 1 are the folded hypercubes FQ_2 and FQ_3, respectively, where the complementary edges are represented by thick red lines (grey in print version). For every positive integer m, there exists an unique binary representation $\sum_{i=0}^{s} 2^{t_i}$ satisfying that $t_i < t_{i-1}$ for each $i > 0$. Let S_m be the set $\{0, 1, 2, \ldots, m-1\}$ (under decimal representation) and L_m the corresponding set represented by n-binary strings.

Let $FQ_n[L_m]$ be the subgraph induced by L_m in FQ_n. Rajasingh and Arockiaraj obtained an interesting result in [31], for $m \leq 2^{n-1}$

$$ex_m(FQ_n) = 2|E(FQ_n[L_m])| = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}.$$

For example, assume that $n = 4$ and $m = 7$. We have $S_7 = \{0, 1, 2, \ldots, 5, 6\}$ and $L_7 = \{0000, 0001, 0010, 0011, 0100, 0101, 0110\}$. Since $7 = 2^2 + 2^1 + 2^0$, we have $t_0 = 2, t_1 = 1$ and $t_2 = 0$, and $ex_7(FQ_4) = 2|E(FQ_4[L_7])| = 2 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 2 \times 0 \times 2^2 + 2 \times 1 \times 2^1 + 2 \times 2 \times 2^0 = 18$. The subgraph induced by L_7 is shown in Fig. 3.

M. Zhang, L. Zhang, and X. Feng have applied their result to determine h-extra edge-connectivity of the n-dimensional folded hypercubes for some values of h and obtained in the following theorems.

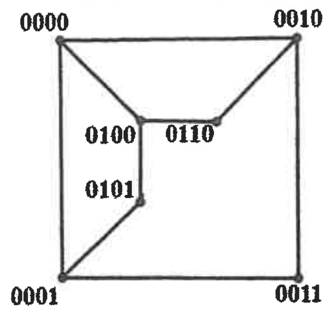

Fig. 3. Induced subgraph $FQ_4[L_7]$.

Theorem 2.3 ([41]): If $h \leq m = \sum_{i=0}^{s} 2^{t_i} \leq 2^{n-1}$, then
(a) $\xi_m(FQ_n) = (n+1)m - \sum_{i=0}^{s} t_i 2^{t_i} - \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$;
(b) $\lambda_h(FQ_n) = \min\{\xi_m(FQ_n) : h \leq m \leq 2^{n-1}\}$.

On the basis of the above results, after deeply mining the properties of function $\xi_m(FQ_n)$, an algorithm to determine the exact values of $\lambda_h(FQ_n)$ for each $h \leq 2^{n-1}$ is designed in the next section.

III. Main Results and Algorithm

To begin with, we redivide the integer interval $1 \leq h \leq 2^{n-1}$ into $\lfloor \frac{n}{2} \rfloor$ subintervals $1 \leq h \leq 2^{\lceil \frac{n}{2} \rceil}$ and $2^{\lceil \frac{n}{2} \rceil + r - 1} < h \leq 2^{\lceil \frac{n}{2} \rceil + r}$ for $r = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$. For $1 \leq h \leq 2^{\lceil \frac{n}{2} \rceil}$, we have the following results.

Theorem 3.1 ([41]): If $n \geq 2$ and $1 \leq h \leq 2^{\lceil \frac{n}{2} \rceil}$, then

$$\lambda_h(FQ_n) = \xi_h(FQ_n). \tag{3}$$

To deal with the rest intervals, $2^{\lceil \frac{n}{2} \rceil + r - 1} < h \leq 2^{\lceil \frac{n}{2} \rceil + r}$ for $r = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, let $w_{r,j} = \sum_{i=0}^{j} 2^{\lceil \frac{n}{2} \rceil + r - 1 - i}, j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - r$. Thus, $w_{r,0} = 2^{\lceil \frac{n}{2} \rceil + r - 1}, w_{r+1,0} = 2^{\lceil \frac{n}{2} \rceil + r}$, and

$$w_{r,0} < w_{r,1} < \cdots < w_{r,j} < \cdots < w_{r, \lfloor \frac{n}{2} \rfloor - r} < w_{r+1,0}.$$

Each interval of $2^{\lceil \frac{n}{2} \rceil + r - 1} < h \leq 2^{\lceil \frac{n}{2} \rceil + r}$ for $r = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, is further divided into $\lfloor \frac{n}{2} \rfloor - r + 1$ subintervals: $w_{r,j} < h \leq w_{r,j+1}$ with $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - r - 1$ and $w_{r, \lfloor \frac{n}{2} \rfloor - r} < h \leq 2^{\lceil \frac{n}{2} \rceil + r}$.

Theorem 3.2: If $n \geq 4$ and $2^{\lceil \frac{n}{2} \rceil + r - 1} \leq h \leq 2^{\lceil \frac{n}{2} \rceil + r}$ for $r = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, then

$$\lambda_h(FQ_n) = \xi_{w_{r, \lfloor \frac{n}{2} \rfloor - r}}(FQ_n) = \left(\lfloor \frac{n}{2} \rfloor - r + 1\right) 2^{\lceil \frac{n}{2} \rceil + r} \tag{4}$$

for $w_{r, \lfloor \frac{n}{2} \rfloor - r} \leq h \leq 2^{\lceil \frac{n}{2} \rceil + r}$; and

$$\lambda_h(FQ_n) = \xi_{w_{r,j}}(FQ_n) + \lambda_{h - w_{r,j}}(FQ_{n-2j-2}) \tag{5}$$

for $w_{r,j} < h \leq w_{r,j+1}$ with $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - r - 1$.

The proof of Theorem 3.2 is more cumbersome. It will be given in Section V.

According to Theorem 2.3(a)

$$\xi_m(FQ_n) = (n+1)m - \sum_{i=0}^{s} t_i 2^{t_i} - \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$$

for $0 < m = \sum_{i=0}^{s} 2^{t_i} \leq 2^{n-1}$, an $O(\log_2(N))$ algorithm to calculate $\xi_m(FQ_n)$, can be designed by the above formula.

Furthermore, for given two positive integers n and h, $2 \leq n, h \leq 2^{n-1}$, by Theorems 2.3, 3.1, and 3.2, an algorithm can be designed to calculate h-extra edge-connectivity $\lambda_h(FQ_n)$, for $1 \leq h \leq 2^{n-1}$. Actually, it is not necessary to compute the value of $\xi_{w_{r,j}}(FQ_n)$ for each the recursion of Theorem 3.2. One can just accumulate the subscript $w_{r,j}$ by the following (6).

By Theorem 2.3(b)

$$\lambda_h(FQ_n) = \min\{\xi_m(FQ_n) : h \leq m \leq 2^{n-1}\}$$

there exists at least one integer m such that $\lambda_h(FQ_n) = \xi_m(FQ_n)$. A desire to crack this nut by redesigning an algorithm urges us to discuss some more properties of $\xi_{w_{r,j}}(FQ_n)$.

For $w_{r,j} < h = w_{r,j} + h' \leq w_{r,j+1}$ and $0 < h' \leq w_{r,j+1} - w_{r,j} = 2^{\lceil \frac{n}{2} \rceil + r - j - 2}$, let $h' = \sum_{i=0}^{s} 2^{t_i}$. Then, $h = w_{r,j} + h' = \sum_{i=0}^{j} 2^{\lceil \frac{n}{2} \rceil + r - 1 - i} + \sum_{i=j+1}^{j+s+1} 2^{t_{i-j-1}}$ and so the following equation

$$\begin{aligned} \xi_h(FQ_n) &= \xi_{w_{r,j} + h'}(FQ_n) \\ &= \xi_{w_{r,j}}(FQ_n) + \xi_{h'}(FQ_{n-2j-2}) \end{aligned} \tag{6}$$

holds and $n - 2j - 2 \geq 2$. In fact, by Theorem 2.3(a), we have

$$\begin{aligned} &\xi_h(FQ_{n'}) \\ &= (n+1)(w_{r,j} + h') - \Bigg[\sum_{i=0}^{j} \left(\left\lceil \frac{n}{2} \right\rceil + r - 1 - i\right) 2^{\lceil \frac{n}{2} \rceil + r - 1 - i} \\ &\quad + \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{j} 2 \cdot i \cdot 2^{\lceil \frac{n}{2} \rceil + r - 1 - i} \\ &\quad + \sum_{i=j+1}^{j+s+1} 2i 2^{t_{i-j-1}} \Bigg] \\ &= \xi_{w_{r,j}}(FQ_n) + (n - 2j - 2 + 1)h' - \sum_{i=0}^{s} (t_i + 2i) 2^{t_i} \\ &= \xi_{w_{r,j}}(FQ_n) + \xi_{h'}(FQ_{n-2j-2}) \end{aligned}$$

since $(n - 2j - 2) - \lceil \log_2 h' \rceil \geq n - 2j - 2 - (\lceil \frac{n}{2} \rceil + r - j - 2) = \lfloor \frac{n}{2} \rfloor - r - j \geq 1$ for $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - r - 1$.

Based on (3), (4), (5), and (6), for given integers n and h with $n \geq 2$ and $h \leq 2^{n-1}$, we can redesign an algorithm to determine the exact values of $\lambda_h(FQ_n)$ and to find integer m such that $\lambda_h(FQ_n) = \xi_m(FQ_m)$ as follows.

Step 1: Let $m_0 = 0$.

Step 2: If $h \leq 2^{\lceil \frac{n}{2} \rceil}$, then let $m = m_0 + h$, $\lambda_h(FQ_n) = \xi_m(FQ_n)$, and stop.

Step 3: If $w_{r, \lfloor \frac{n}{2} \rfloor - r} < h \leq 2^{\lceil \frac{n}{2} \rceil + r}$ for some r, then let $m = m_0 + 2^{\lceil \frac{n}{2} \rceil + r}$, $\lambda_h(FQ_n) = \xi_m(FQ_n)$, and stop.

Step 4: If $w_{r,j} < h \leq w_{r,j+1}$ for some r and j, let $m_0' = m_0 + w_{r,j}$, $n' = n - 2j - 2$, and $h' = h - w_{r,j}$ instead of m_0, n, and h, respectively. Then go to Step 2.

Steps 2 and 3 are based on (3) and (4), respectively. And Step 4 can be deduced by (5) and (6). The process described above will be terminated on Steps 2 or 3, since in each Step 4, both n and h are strictly reduced, $n - 2j - 2 \geq \lceil \log_2 h' \rceil + 1 \geq 2$ and $2^{n-1} = 2^{\lceil \frac{n}{2} \rceil}$ if $n = 2$ or $n = 3$. Specifically, if the

Algorithm 1: The h-extra edge-connectivity of FQ_n and λ_h-optimality.

Result: Calculating the h-extra edge-connectivity of FQ_n.
Input : Two positive integers n and h.
Output: n, h, S and m.

1 **if** $h > 2^{n-1}$ *or* $n \le 1$ **then**
  Output: "$h > 2^{n-1}$." break
2 **else**
3   $H \leftarrow h$
4   $N \leftarrow n$
5   $m \leftarrow 0$
6   $b \leftarrow \lceil \frac{n}{2} \rceil$
7   $r \leftarrow \lceil log_2 h \rceil - b$
8   **while** $2^b < h$ **do**
9     $w_0 \leftarrow 2^{b+r-1}$
10     $w \leftarrow w_0 + 2^{b+r-2}$
11     $j \leftarrow 1$
12   **while** $w < h$ **do**
13   **if** $j < n - b - r$ **then**
14         $j \leftarrow j + 1$
15         $w_0 \leftarrow w$
16         $w \leftarrow w + 2^{b+r-1-j}$
17   **else**
18         $m \leftarrow m + 2^{b+r}$
19   **goto** 30
20   **end**
21   **end**
22     $m \leftarrow m + w_0$
23     $n \leftarrow n - 2j$
24     $h \leftarrow h - w_0$
25     $b \leftarrow b - j$
26     $r \leftarrow \lceil log_2(h) \rceil - b$
27   **end**
28   $m \leftarrow m + h$
29   $i \leftarrow 0$
30   $S \leftarrow (N+1)m$
31   $M \leftarrow m$
32   **while** $m \neq 0$ **do**
33     $t \leftarrow \lfloor log_2 m \rfloor$
34     $S \leftarrow S - (t + 2i)2^t$
35     $m \leftarrow m - 2^t$
36     $i \leftarrow i + 1$
37   **end**
38   $n \leftarrow N$
39   $h \leftarrow H$
40   $m \leftarrow M$
41 **end**

process terminates in Step 2, then $m = h$ and it is λ_h-optimal. Otherwise, it is not λ_h-optimal.

Combining with the equation to calculate $\xi_m(FQ_n)$ [see Theorem 2.3(a)], the flowchart and the pseudocode of the algorithm are given in Fig. 4 and Algorithm 1, respectively, where $\lambda_h(FQ_n) = \xi_m(FQ_n) = S$.

The time complexity of the algorithm above is $O(n) = O(\log_2 2^n)$. In fact, the time complexity of Algorithm 1 is the same with the subprogram to find m such that $\lambda_h((FQ_n) = \xi_m(FQ_n)$ from the sentence 8 to the sentence 28, and also is the same with the subprogram to calculate $\xi_m(FQ_n)$ from the sentence 29 to the sentence 40. In the subprogram from the sentence 8 to the sentence 28, suppose that it needs t times to repeat the circulation. Then $t \le \lfloor \frac{n}{2} \rfloor - 1$, $b_1 = \lceil \frac{n}{2} \rceil$, $r_1 = r \le \lfloor \frac{n}{2} \rfloor - 1$, and $j_1 = j \le b_1 - r_1$. For $i > 1$ In the ith time, the time complexity is at most $4 + 3j_i + 5$, where $b_i = b_{i-1} - j_{i-1}$, $r_i = r_{i-1} - i + 1$, and $j_i \le b_i - r_i$. Hence, $j_t \le b_1 - j_1 - \cdots - j_{t-1} - r_t$. Thus $j_1 + j_2 + \cdots + j_t \le b_1 - r_t \le \lceil \frac{n}{2} \rceil$ and

$$\sum_{i=1}^{t}(4 + 3j_i + 5) \le 12 \left\lceil \frac{n}{2} \right\rceil .$$

In the subprogram from the sentence 29 to the sentence 40, the time complexity is at most $2n$. Therefore, the time complexity of the algorithm is $O(\log_2(N))$, where $N = 2^n$.

IV. Application

In parallel computing, the n-dimensional folded hypercubes have been used as underlying topologies of several parallel systems, such as ATM switches, 3-D-FolH-NOC networks, and PM2I networks.

Our theoretical results offer a more refined quantitative analysis of indicators of the robustness of a folded hypercube based multiprocessor system in the presence of failing links. For the n-dimensional folded hypercubes network with $N = 2^n$ processors, the h-extra edge-connectivity of FQ_n is the minimum cardinality of set of links, whose removal disconnects the network with all its resulting components having at least h processors for each $h \le N/2$. In other words, at least $\lambda_h(G)$ number of links must be deleted to disconnect this network, provided that the deletion of these links does not isolate any subnetwork with at most $h - 1$ processors.

By Theorems 2.3, 3.1, and 3.2 and (6), the values of the $\lambda_h(FQ_n)$ have close relationship with $\xi_h(FQ_n)$ for $1 \le m \le h \le 2^{n-1}$. Our algorithm is based on the fact that $\xi_m(FQ_n) = |[L_m, \overline{L_m}]| = (n+1)m - \sum_{i=0}^{s} t_i 2^{t_i} - \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$ for $m \le 2^{n-1}$.

For example, assume that $n = 4$ and $h = 4$. We have $S_4 = \{0, 1, 2, 3\}$ and $L_4 = \{0000, 0001, 0010, 0011\}$. Since $4 = 2^2$, we have $t_0 = 2$, $s = 0$, and $\xi_4(FQ_4) = |[L_4, \overline{L_4}]| = (4+1)4 - ex_4(FQ_4) = 20 - 2 \times 2^2 + 0 \times 2^0 = 12$. But if $L_4^1 = \{0000, 0010, 0011, 0111\}$ and $L_4^2 = \{0000, 0001, 0010, 0100\}$ then both $[L_4^1, \overline{L_4^1}]$ and $[L_4^2, \overline{L_4^2}]$ are four-extra edge-cuts of FQ_4 with $|[L_4^1, \overline{L_4^1}]| = |[L_4^2, \overline{L_4^2}]| = 13 > 12 = |[L_4, \overline{L_4}]|$. These three kinds of four-extra edge cuts in FQ_4 are shown in the first three figures of Fig. 5, where the four-extra edge cuts marked in purple lines (light grey in print version).

If $h = 7 = 2^2 + 2^1 + 2^0$, then $L_7 = \{0000, 0001, 0010, 0011, 0100, 0101, 0110\}$. We have $\xi_7(FQ_4) = |[L_7, \overline{L_7}]| = 5 \times 7 - (2 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 2 \times 0 \times 2^2 + 2 \times 1 \times 2^1 + 2 \times 2 \times 2^0) = 17$ shown in the last figure of Fig. 5. We make a simulation experiment for determining the sizes of all the h-extra edge-cuts of FQ_4 for each $h \le 2^3$ in Fig. 6.

By our algorithm the exact values of $\lambda_h(FQ_4)$ for each $h \le 2^{n-1}$ will be given in Table I. For $1 \le h \le 2^{\lceil \frac{n}{2} \rceil} = 2^2$, $\lambda_h(FQ_4) = \xi_h(FQ_4) = (n+1)h - \sum_{i=0}^{s} t_i 2^{t_i} - \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$.

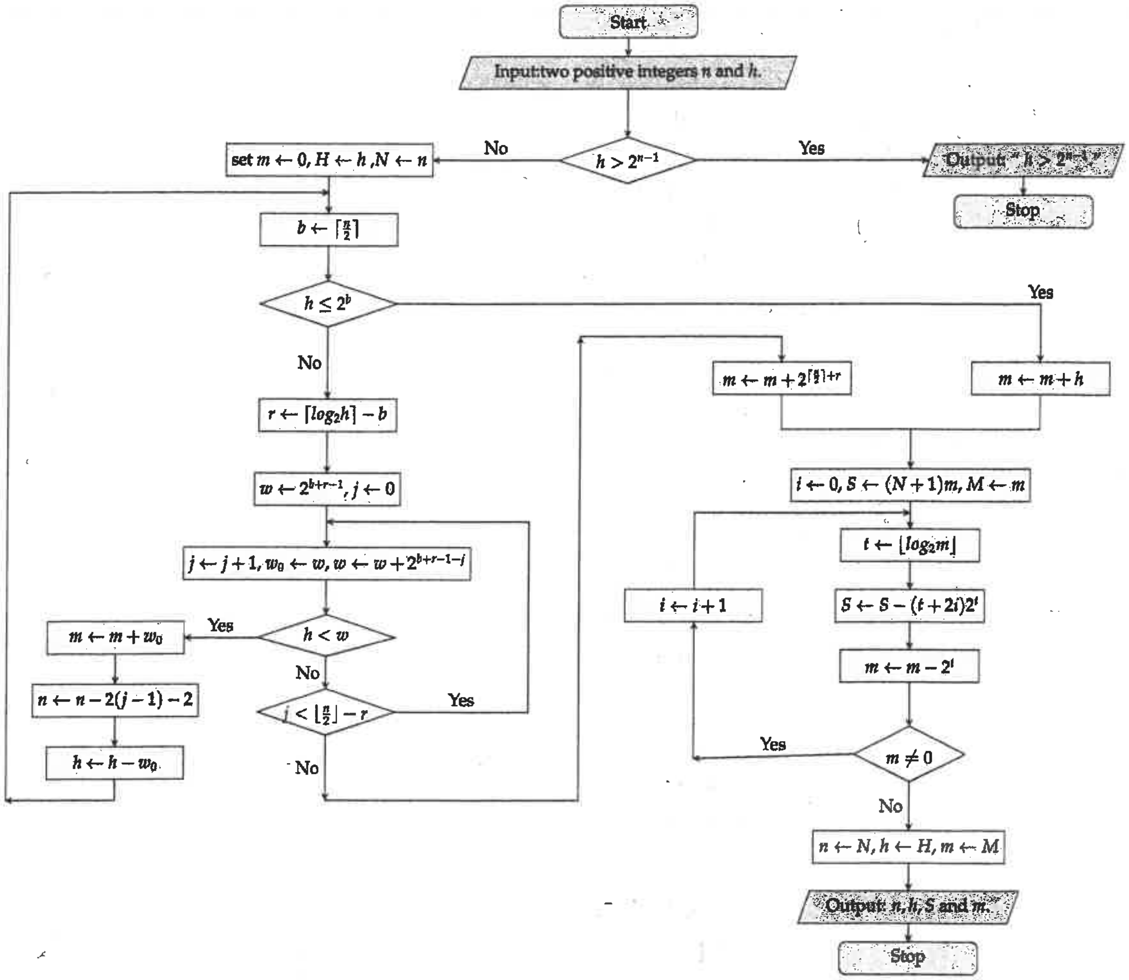

Fig. 4. Flowchart on the algorithm of calculating $\lambda_h(FQ_n)$.

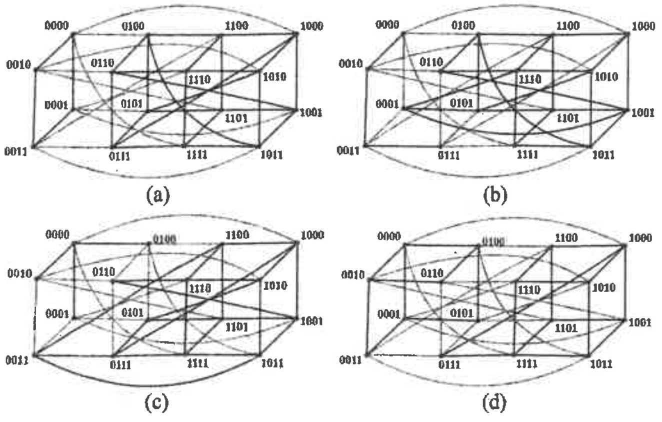

Fig. 5. Four-extra edge-cuts in FQ_4 with one component 4 vertices and $[L_7, \overline{L_7}]$.

For $5 \leq h \leq 6$, let $r = \lceil \log_2 h \rceil - \lceil \frac{n}{2} \rceil = 3 - 2 = 1$, $w_{r,0} = w_{1,0} = 2^2 = 4$, and $w_{r,\lfloor \frac{n}{2} \rfloor - r} = w_{1,1} = 2^2 + 2^1 = 6$. So $w_{1,0} < h \leq w_{1,1}$, $j = 0$, $n' = n - 2j - 2 = 4 - 2 = 2$, $h' = h - w_{1,0} = h - 4 \leq 2$, and $h' \leq 2^{\lceil \frac{n'}{2} \rceil} = 2^1$. We have $m = w_{1,0} + h' = h$ and $\lambda_h(FQ_4) = \xi_h(FQ_4)$. For $6 < h \leq 8 = 2^{n-1} = 2^3$, $6 = w_{r,\lfloor \frac{n}{2} \rfloor - r} = w_{1,1} < h \leq 2^{\frac{4}{2}+1} = 2^{\lceil \frac{n}{2} \rceil + r} = 8$, we have $\lambda_h(FQ_4) = \xi_{2^{\lceil \frac{n}{2} \rceil + r}}(FQ_4) = \xi_{2^3}(FQ_4) = 16$.

After processing Algorithm 1 for some small cases $n \leq 7$ and $h \leq 2^{n-1}$, the data of $\lambda_h(FQ_n)$ have been presented in Table II, where the values of $\lambda_h(FQ_n)$ not satisfying the equality $\lambda_h(FQ_n) = \xi_h(FQ_n)$ are marked in blue (bold in print version), otherwise are marked in black. Based on these data, the following visual images of both $\lambda_h(FQ_n)$ and $\xi_h(FQ_n)$ are shown in Fig. 7. One can see that as the integer n increases, the number of h with $\lambda_h(FQ_n) \neq \xi_h(FQ_n)$ also increases.

V. Proof of Theorem 3.2

We first present a known result about the function $\xi_m(FQ_n)$ in the following.

Lemma 5.1 ([41]): Let c and n be two integers with $0 \leq c \leq n - 2$. If an integer m satisfies that $2^c < m \leq 2^{n-1}$, then

$$\xi_m(FQ_n) \geq \xi_{2^c}(FQ_n).$$

For given an integer h, $2^{\lceil \frac{n}{2} \rceil} < h \leq 2^{n-1}$, there exists an integer r, $1 \leq r \leq \lfloor \frac{n}{2} \rfloor - 1$ such that $2^{\lceil \frac{n}{2} \rceil + r - 1} < h \leq 2^{\lceil \frac{n}{2} \rceil + r}$. By Theorem 2.3(b) and Lemma 4.1, it can be obtained that

$$\lambda_h(FQ_n) = \min\{\xi_m(FQ_n) : h \leq m \leq 2^{n-1}\}$$
$$= \min\{\xi_m(FQ_n) : h \leq m \leq 2^{\lceil \frac{n}{2} \rceil + r}\}. \quad (7)$$

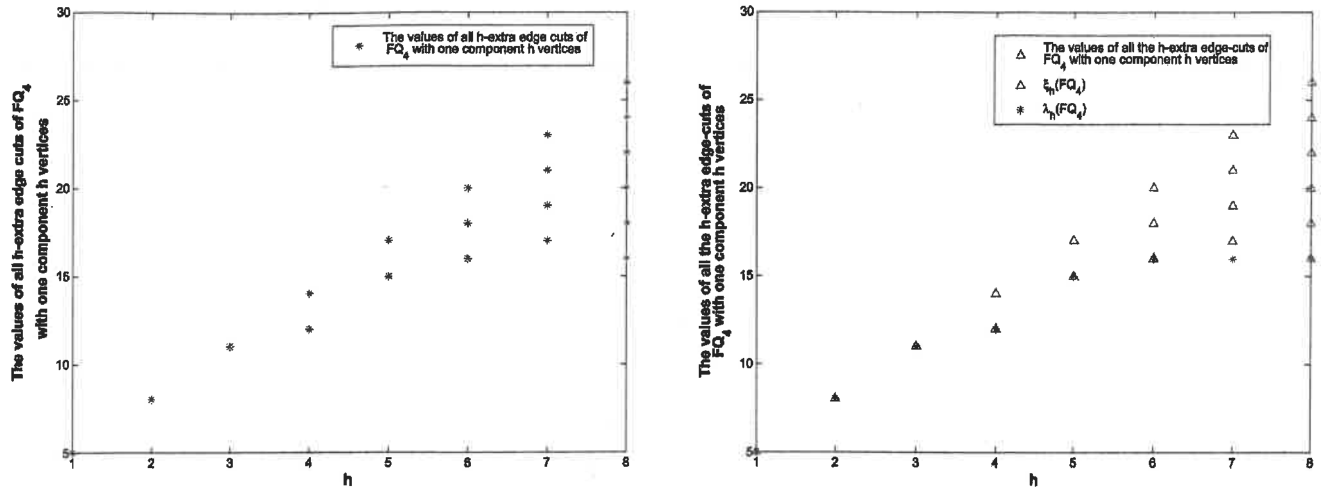

Fig. 6. The comparison between the simulation and our results.

TABLE I
$\lambda_h(FQ_4)$ FOR $h \le 2^3$

h	Decomposition of h	$ex_h(FQ_4)$	$\xi_h(FQ_4)$	$\lambda_h(FQ_4)$	Step
1	2^0	$0 \cdot 2^0 + 2 \cdot 0 \cdot 2^0 = 0$	$5 \times 1 - 0 = 5$	$\lambda_1(FQ_4) = \xi_1(FQ_4) = 5$	Step 1, Step 1
2	2^1	$1 \cdot 2^1 + 2 \cdot 0 \cdot 2^1 = 2$	$5 \times 2 - 2 = 8$	$\lambda_2(FQ_4) = \xi_2(FQ_4) = 8$	Step 1, Step 1
3	$2^1 + 2^0$	$1 \cdot 2^1 + 0 \cdot 2^0 + 2 \cdot 0 \cdot 2^1 + 2 \cdot 1 \cdot 2^0 = 4$	$5 \times 3 - 4 = 11$	$\lambda_3(FQ_4) = \xi_3(FQ_4) = 11$	Step 1, Step 2
4	2^2	$2 \cdot 2^2 + 2 \cdot 0 \cdot 2^2 = 8$	$5 \times 4 - 8 = 12$	$\lambda_4(FQ_4) = \xi_4(FQ_4) = 12$	Step 1, Step 2
5	$2^2 + 2^0$	$2 \cdot 2^2 + 0 \cdot 2^0 + 2 \cdot 0 \cdot 2^2 + 2 \cdot 1 \cdot 2^0 = 10$	$5 \times 5 - 10 = 15$	$\lambda_5(FQ_4) = \xi_5(FQ_4) = 15$	Step 1, Step 4, Step 2
6	$2^2 + 2^1$	$2 \cdot 2^2 + 1 \cdot 2^1 + 2 \cdot 0 \cdot 2^2 + 2 \cdot 1 \cdot 2^1 = 14$	$5 \times 6 - 14 = 16$	$\lambda_6(FQ_4) = \xi_6(FQ_4) = 16$	Step 1, Step 4, Step 2
7	$2^2 + 2^1 + 2^0$	$2 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 + 2 \cdot 0 \cdot 2^2 + 2 \cdot 1 \cdot 2^1 + 2 \cdot 2 \cdot 2^0 = 18$	$5 \times 7 - 18 = 17$	$\boldsymbol{\lambda_7(FQ_4) = \xi_8(FQ_4) = 16}$	Step 1, Step 4, Step 3
8	2^3	$3 \cdot 2^3 + 2 \cdot 0 \cdot 2^3 = 24$	$5 \times 8 - 24 = 16$	$\lambda_8(FQ_4) = \xi_8(FQ_4) = 16$	Step 1, Step 3, Step 2

TABLE II
EXAMPLE OF $\lambda_h(FQ_n)$ AND $\xi_h(FQ_n)$

h	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\xi_h(FQ_2)$	3	4														
$\lambda_h(FQ_2)$	3	4														
$\xi_h(FQ_3)$	4	6	8	8												
$\lambda_h(FQ_3)$	4	6	8	8												
$\xi_h(FQ_4)$	5	8	11	12	15	16	17	16								
$\lambda_h(FQ_4)$	5	8	11	12	15	16	16	16								
$\xi_h(FQ_5)$	6	10	14	16	20	22	24	24	28	30	32	32	34	34	34	32
$\lambda_h(FQ_5)$	6	10	14	16	20	22	24	24	28	30	32	32	32	32	32	32
$\xi_h(FQ_6)$	7	12	17	20	25	28	31	32	37	40	43	44	47	48	49	48
$\lambda_h(FQ_6)$	7	12	17	20	25	28	31	32	37	40	43	44	47	48	48	48
$\xi_h(FQ_7)$	8	14	20	24	30	34	38	40	46	50	54	56	60	62	64	64
$\lambda_h(FQ_7)$	8	14	20	24	30	34	38	40	46	50	54	56	60	62	64	64
h	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$\xi_h(FQ_6)$	53	56	59	60	63	64	65	64	67	68	69	68	69	68	67	64
$\lambda_h(FQ_6)$	53	56	59	60	63	64	64	64	64	64	64	64	64	64	64	64
$\xi_h(FQ_7)$	70	74	78	80	84	86	88	88	92	94	96	96	98	98	98	96
$\lambda_h(FQ_7)$	70	74	78	80	84	86	88	88	92	94	96	96	98	98	98	96
h	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
$\xi_h(FQ_7)$	102	106	110	112	116	118	120	120	124	126	128	128	130	130	130	128
$\lambda_h(FQ_7)$	102	106	110	112	116	118	120	120	124	126	128	128	128	128	128	128
h	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
$\xi_h(FQ_7)$	132	134	136	136	138	138	138	136	138	138	138	136	136	134	132	128
$\lambda_h(FQ_7)$	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128	128

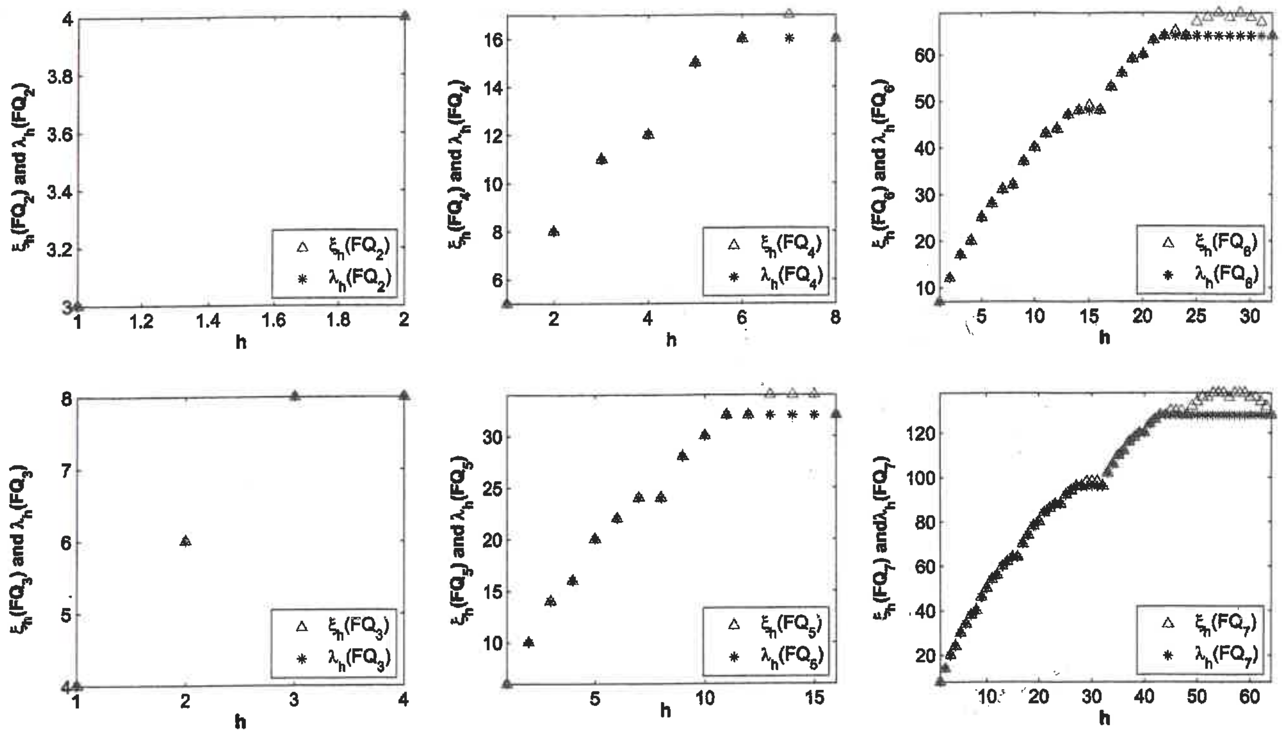

Fig. 7. Examples of plots of functions $\lambda_h(FQ_n)$ and $\xi_h(FQ_n)$ for cases $n \le 7$.

Now, we divide the interval $2^{\lceil \frac{n}{2}\rceil + r - 1} < m \le 2^{\lceil \frac{n}{2}\rceil + r}$ into $\lfloor \frac{n}{2}\rfloor - r + 1$ subintervals: $w_{r,j} < m \le w_{r,j+1}$, $j = 0, 1, \ldots, \lfloor \frac{n}{2}\rfloor - r - 1$, and $w_{r,\lfloor \frac{n}{2}\rfloor - r} < h \le 2^{\lceil \frac{n}{2}\rceil + r}$, where $w_{r,j} = \sum_{i=0}^{j} 2^{\lceil \frac{n}{2}\rceil + r - 1 - i}$. We have the following results.

Lemma 5.2: (a) For $w_{r,\lfloor \frac{n}{2}\rfloor - r} < m < 2^{\lceil \frac{n}{2}\rceil + r} \xi_m(FQ_n) > \xi_{w_{r,\lfloor \frac{n}{2}\rfloor - r}}(FQ_n)$.

(b) $\xi_{w_{r,\lfloor \frac{n}{2}\rfloor - r}}(FQ_n) = \xi_{2^{\lceil \frac{n}{2}\rceil + r}}(FQ_n) = (\lfloor \frac{n}{2}\rfloor - r + 1)\, 2^{\lceil \frac{n}{2}\rceil + r}$.

Proof: (a) Let $m = w_{r,\lfloor \frac{n}{2}\rfloor - r} + m'$ and $m' = \sum_{i=0}^{s} 2^{t_i}$. Since

$$w_{r,\lfloor \frac{n}{2}\rfloor - r} = \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor - r} 2^{\lceil \frac{n}{2}\rceil + r - 1 - i} = \sum_{i=2r-1+e}^{\lceil \frac{n}{2}\rceil + r - 1} 2^{i} = 2^{\lceil \frac{n}{2}\rceil + r} - 2^{2r-1+e}$$

where $e = 1$ if n is odd, $e = 0$ if n is even, we have $0 < m' = \sum_{i=0}^{s} 2^{t_i} < 2^{\lceil \frac{n}{2}\rceil + r} - w_{r,\lfloor \frac{n}{2}\rfloor - r} = 2^{2r-1+e}$ and

$$m = \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor - r} 2^{\lceil \frac{n}{2}\rceil + r - 1 - i} + \sum_{i=0}^{s} 2^{t_i} = \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor - r} 2^{\lceil \frac{n}{2}\rceil + r - 1 - i} + \sum_{\lfloor \frac{n}{2}\rfloor - r + 1}^{\lfloor \frac{n}{2}\rfloor - r + s + 1} 2^{t_{i - \lfloor \frac{n}{2}\rfloor + r - 1}}.$$

By Theorem 2.3(a), we have

$$\begin{aligned}
\xi_m(FQ_n) &= (n+1)m - \left[\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor - r} \left(\lceil \frac{n}{2}\rceil + r - 1 - i\right) 2^{\lceil \frac{n}{2}\rceil + r - 1 - i} + \sum_{i=0}^{s} t_i 2^{t_i}\right] - \left(\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor - r} 2i2^{\lceil \frac{n}{2}\rceil + r - 1 - i} + \sum_{\lfloor \frac{n}{2}\rfloor - r + 1}^{\lfloor \frac{n}{2}\rfloor - r + s + 1} 2i2^{t_{i - \lfloor \frac{n}{2}\rfloor + r - 1}}\right) \\
&= \xi_{w_{r,\lfloor \frac{n}{2}\rfloor - r}}(FQ_n) + (n+1)m' - \sum_{i=0}^{s} t_i 2^{t_i} - \sum_{\lfloor \frac{n}{2}\rfloor - r + 1}^{\lfloor \frac{n}{2}\rfloor - r + s + 1} 2i2^{t_{i - \lfloor \frac{n}{2}\rfloor + r - 1}} \\
&= \xi_{w_{r,\lfloor \frac{n}{2}\rfloor - r}}(FQ_n) + (n+1)m' - \sum_{i=0}^{s} t_i 2^{t_i} - \sum_{i=0}^{s} 2it_i 2^{t_i} - 2\left(\lfloor \frac{n}{2}\rfloor + r - 1\right) m' \\
&= \xi_{w_{r,\lfloor \frac{n}{2}\rfloor - r}}(FQ_n) + (2r - 1 + e)m' - \sum_{i=0}^{s} (t_i + 2i) 2^{t_i} \\
&> \xi_{w_{r,\lfloor \frac{n}{2}\rfloor - r}}(FQ_n).
\end{aligned}$$

The last inequality will hold which is based on the fact $(2r-1+e)m' - \sum_{i=0}^{s} t_i 2^{t_i} - \sum_{i=0}^{s} 2it_i 2^{t_i} > 0$ [31].

(b) Note that $w_{r,\lfloor \frac{n}{2}\rfloor - r} = 2^{\lceil \frac{n}{2}\rceil + r} - 2^{2r-1+e}$ is obtained in the proof of (a). On one hand

$$
\begin{aligned}
&\xi_{w_{r,\lfloor \frac{n}{2}\rfloor - r}}(FQ_n)\\
&= (n+1)w_{r,\lfloor \frac{n}{2}\rfloor - r} - ex_{w_{r,\lfloor \frac{n}{2}\rfloor - r}}(FQ_n)\\
&= (n+1)\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor - r} 2^{\lceil \frac{n}{2}\rceil + r - 1 - i}\\
&\quad - \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor - r}\left(\left\lceil \frac{n}{2}\right\rceil + r - 1 + i\right)2^{\lceil \frac{n}{2}\rceil + r - 1 - i}\\
&= \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor - r}[n+1-\left(\left\lceil \frac{n}{2}\right\rceil + r - 1 + i\right)]2^{\lceil \frac{n}{2}\rceil + r - 1 - i}\\
&= \left(\left\lfloor \frac{n}{2}\right\rfloor - r + 2\right)\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor - r} 2^{\lceil \frac{n}{2}\rceil + r - 1 - i}\\
&\quad - \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor - r} i \times 2^{\lceil \frac{n}{2}\rceil + r - 1 - i}\\
&= \left(\left\lfloor \frac{n}{2}\right\rfloor - r + 2\right)\left(2^{\lceil \frac{n}{2}\rceil + r} - 2^{2r-1+e}\right)\\
&\quad - \sum_{i=1}^{\lfloor \frac{n}{2}\rfloor - r} i \times 2^{\lceil \frac{n}{2}\rceil + r - 1 - i}\\
&= \left(\left\lfloor \frac{n}{2}\right\rfloor - r + 2\right)\left(2^{\lceil \frac{n}{2}\rceil + r} - 2^{2r-1+e}\right)\\
&\quad - \sum_{i=1}^{\lfloor \frac{n}{2}\rfloor - r}\sum_{j=i}^{\lfloor \frac{n}{2}\rfloor - r} 2^{\lceil \frac{n}{2}\rceil + r - 1 - j}\\
&= \left(\left\lfloor \frac{n}{2}\right\rfloor - r + 2\right)\left(2^{\lceil \frac{n}{2}\rceil + r} - 2^{2r-1+e}\right)\\
&\quad - \sum_{i=1}^{\lfloor \frac{n}{2}\rfloor - r}\left(2^{\lceil \frac{n}{2}\rceil + r - i} - 2^{2r-1+e}\right)\\
&= \left(\left\lfloor \frac{n}{2}\right\rfloor - r + 2\right)\left(2^{\lceil \frac{n}{2}\rceil + r} - 2^{2r-1+e}\right)\\
&\quad - \left[\left(2^{\lceil \frac{n}{2}\rceil + r} - 2^{2r+e}\right) - \left(\left\lfloor \frac{n}{2}\right\rfloor - r\right)2^{2r-1+e}\right]\\
&= \left(\left\lfloor \frac{n}{2}\right\rfloor - r + 1\right)2^{\lceil \frac{n}{2}\rceil + r}.
\end{aligned}
$$

On the other hand, by Theorem 2.3(a)

$$
\begin{aligned}
\xi_{2^{\lceil \frac{n}{2}\rceil + r}}(FQ_n) &= (n+1)2^{\lceil \frac{n}{2}\rceil + r} - \left(\left\lceil \frac{n}{2}\right\rceil + r\right)2^{\lceil \frac{n}{2}\rceil + r}\\
&= \left(\left\lfloor \frac{n}{2}\right\rfloor - r + 1\right)2^{\lceil \frac{n}{2}\rceil + r}.
\end{aligned}
$$

The proof of (b) is completed. ■

Lemma 5.3: $\xi_m(FQ_n) \ge \xi_{w_{r,j}}(FQ_n)$, for any $w_{r,j} \le m \le w_{r,\lfloor \frac{n}{2}\rfloor - r}$ with $j = 0, 1, \ldots, \lfloor \frac{n}{2}\rfloor - r - 1$ and $r = 1, 2, \ldots, \lfloor \frac{n}{2}\rfloor - 1$.

Proof: The proof is divided into the following two steps.

For $r = 1, 2, \ldots, \lfloor \frac{n}{2}\rfloor - 1$, $j = 0, 1, \ldots, \lfloor \frac{n}{2}\rfloor - r - 1$, $w_{r,j} = \sum_{i=0}^{j} 2^{\lceil \frac{n}{2}\rceil + r - 1 - i} < 2^{\lceil \frac{n}{2}\rceil + r} \le 2^{n-1}$, and $2^{\lceil \frac{n}{2}\rceil + r - j - 2} = w_{r,j+1} - w_{r,j}$, by (6) in Section III, we have

$$
\begin{aligned}
&\xi_{w_{r,j+1}}(FQ_n) - \xi_{w_{r,j}}(FQ_n)\\
&= \xi_{2^{\lceil \frac{n}{2}\rceil + r - j - 2}}(FQ_{n-2j-2})\\
&= \left(n - 2j - 2 + 1 - \left\lceil \frac{n}{2}\right\rceil - r + j + 2\right)2^{\lceil \frac{n}{2}\rceil + r - j - 2}\\
&= \left(\left\lfloor \frac{n}{2}\right\rfloor - r - j + 1\right)2^{\lceil \frac{n}{2}\rceil + r - j} > 0.
\end{aligned}
$$

For $j = 0, 1, \ldots, \lfloor \frac{n}{2}\rfloor - r - 1$ and $w_{r,j+1} - w_{r,j} = 2^{\lceil \frac{n}{2}\rceil + r - 2 - j} > m' = \sum_{i=0}^{s} 2^{t_i}$, similarly, by (6), we have

$$
\xi_{w_{r,j}+m'}(FQ_n) - \xi_{w_{r,j}}(FQ_n) = \xi_{m'}(FQ_{n-2j-2}) > 0
$$

where $(\lceil \frac{n}{2}\rceil + r - 2 - j) - \lceil \frac{n-2j-2}{2}\rceil = r - 1$.

Based on the above analysis, the results hold. ■

By Lemma 5.3, $\xi_m(FQ_n) \ge \xi_{w_{r,j}}(FQ_n)$, for any $w_{r,j} \le m \le w_{r,\lfloor \frac{n}{2}\rfloor - r}$ with $j = 0, 1, \ldots, \lfloor \frac{n}{2}\rfloor - r - 1$ and $r = 1, 2, \ldots, \lfloor \frac{n}{2}\rfloor - 1$, we can deduce that $\min\{\xi_m(FQ_n) : w_{r,j+1} \le m \le w_{r,\lfloor \frac{n}{2}\rfloor - r}\} = \xi_{w_{r,j+1}}(FQ_n)$. So for any $w_{r,j} \le h \le m \le w_{r,j+1}$, $\min\{\xi_m(FQ_n) : h \le m \le w_{r,\lfloor \frac{n}{2}\rfloor - r}\} = \min\{\xi_m(FQ_n) : h \le m \le w_{r,j+1}\}$.

The proof of Theorem 3.2: For given h, $2^{\lceil \frac{n}{2}\rceil + r - 1} < h \le 2^{\lceil \frac{n}{2}\rceil + r}$, there exists an integer j, such that $w_{r,j} < h \le w_{r,j+1}$, $j = 0, 1, \ldots, \lfloor \frac{n}{2}\rfloor - r - 1$, or $w_{r,\lfloor \frac{n}{2}\rfloor - r} < h \le 2^{\lceil \frac{n}{2}\rceil + r}$.

If $w_{r,\lfloor \frac{n}{2}\rfloor - r} \le h \le 2^{\lceil \frac{n}{2}\rceil + r}$, then by formula (7) and Lemma 5.2

$$
\begin{aligned}
&\lambda_h(FQ_n)\\
&= \min\{\xi_m(FQ_n) : h \le m \le 2^{n-1}\}\\
&= \min\{\xi_m(FQ_n) : h \le m \le 2^{\lceil \frac{n}{2}\rceil + r}\}\\
&= \xi_{2^{\lceil \frac{n}{2}\rceil + r}}(FQ_n)\\
&= \left(\left\lfloor \frac{n}{2}\right\rfloor - r + 1\right)2^{\lceil \frac{n}{2}\rceil + r}.
\end{aligned}
$$

Therefore, we obtain (4) in Theorem 3.2.

If $w_{r,j} < h \le w_{r,j+1}$ with $j = 0, 1, \ldots, \lfloor \frac{n}{2}\rfloor - r - 1$, by (6) $\xi_h(FQ_n) = \xi_{w_{r,j}}(FQ_n) + \xi_{h - w_{r,j}}(FQ_{n-2j-2})$.

Specially

$$
\begin{aligned}
&\xi_{w_{r,j+1}}(FQ_n)\\
&= \xi_{w_{r,j}}(FQ_n) + \xi_{w_{r,j+1} - w_{r,j}}(FQ_{n-2j-2})\\
&= \xi_{w_{r,j}}(FQ_n) + \xi_{2^{\lfloor \frac{n}{2}\rfloor - r - j - 2}}(FQ_{n-2j-2})\\
&> \xi_{w_{r,j}}(FQ_n).
\end{aligned}
$$

Therefore, by formula (7) and $m - w_{r,j} \le w_{r,j+1} - w_{r,j} = 2^{\lfloor \frac{n}{2}\rfloor - r - j - 2}$, we have

$$
\begin{aligned}
&\lambda_h(FQ_n)\\
&= \min\{\xi_m(FQ_n) : h \le m \le 2^{\lceil \frac{n}{2}\rceil + r}\}
\end{aligned}
$$

$$
\begin{aligned}
&= \min\{\xi_m(FQ_n) : h \le m \le w_{r,j+1}\} \\
&= \min\{\xi_{w_{r,j}}(FQ_n) + \xi_{m-w_{r,j}}(FQ_{n-2j-2}) : \\
&\qquad h - w_{r,j} \le m - w_{r,j} \le 2^{\lfloor \frac{n}{2} \rfloor - r - j - 2}\} \\
&= \xi_{w_{r,j}}(FQ_n) + \min\{\xi_{m-w_{r,j}}(FQ_{n-2j-2}) : \\
&\qquad h - w_{r,j} \le m - w_{r,j} \le 2^{\lfloor \frac{n}{2} \rfloor - r - j - 2}\} \\
&= \xi_{w_{r,j}}(FQ_n) + \lambda_{h-w_{r,j}}(FQ_{n-2j-2}).
\end{aligned}
$$

Hence, (5) holds in Theorem 3.2 and the proof of Theorem 3.2 is completed. ■

VI. Conclusion

Reliability evaluation of an interconnection network is important to the design and maintenance of multiprocessor systems. The h-extra edge-connectivity of a given network G, denoted by $\lambda_h(G)$, gives a quantitative analysis of indicators of the robustness of a multiprocessor system in presence of failing links. For a network with N processors, it is the minimum cardinality of set of links, whose removal disconnects the network with all its resulting components having at least h processors for $h \le N/2$. In other words, at least $\lambda_h(G)$ number of links must be deleted to disconnect this network, provided that the deletion of these links does not isolate any subnetwork with at most $h-1$ processors. This paper deals with the each h-extra edge-connectivity of folded hypercubes $\lambda_h(FQ_n)$ for $1 \le h \le 2^{n-1}$. Since $\lambda_h(FQ_n) = \min\{\xi_m(FQ_n) : h \le m \le 2^{n-1}\}$, deeply mining the properties of function $\xi_m(FQ_n)$ offers a method to effectively determine the exact values of $\lambda_h(FQ_n)$. We divide the interval $1 \le h \le 2^{n-1}$ into some subintervals and investigate some properties of $\xi_m(FQ_n)$ in these subintervals. A recurrence relation of $\lambda_h(FQ_n)$ is found. Based on them and some known results, an efficient $O(\log_2(N))$ algorithm to determine the exact values of $\lambda_h(FQ_n)$ is obtained. It also serves a positive integer $h \le m \le 2^{n-1}$ such that $\lambda_h(FQ_n) = \xi_m(FQ_n)$, for each $h \le 2^{n-1}$. The λ_h-optimality of $\lambda_h(FQ_n)$ is also determined. The problem on the h-extra edge-connectivity of folded hypercubes is completely solved.

Acknowledgment

The authors would like to thank the anonymous referees and the editors for their help. Their valuable comments and suggestions help them to improve the quality of this paper.

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