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Connectivity keeping stars or double-stars in 2-connected graphs☆

Yingzhi Tian [a,*], Jixiang Meng [a], Hong-Jian Lai [b], Liqiong Xu [c]

[a] College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, PR China
[b] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA
[c] School of Science, Jimei University, Xiamen, Fujian 361021, PR China

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ABSTRACT

In Mader (2010), Mader conjectured that for every positive integer k and every finite tree T with order m, every k-connected, finite graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor + m - 1$ contains a subtree T' isomorphic to T such that $G - V(T')$ is k-connected. In the same paper, Mader proved that the conjecture is true when T is a path. Diwan and Tholiya (2009) verified the conjecture when $k = 1$. In this paper, we will prove that Mader's conjecture is true when T is a star or double-star and $k = 2$.

1. Introduction

In this paper, *graph* always means a finite, undirected graph without multiple edges and without loops. For graph-theoretical terminologies and notation not defined here, we follow [1]. For a graph G, the vertex set, the edge set, the minimum degree and the connectivity number of G are denoted by $V(G)$, $E(G)$, $\delta(G)$ and $\kappa(G)$, respectively. The *order* of a graph G is the cardinality of its vertex set, denoted by $|G|$. k and m always denote positive integers.

In 1972, Chartrand, Kaugars, and Lick proved the following well-known result.

Theorem 1.1 ([2]). *Every k-connected graph G of minimum degree $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor$ has a vertex u with $\kappa(G - u) \geq k$.*

Fujita and Kawarabayashi proved in [4] that every k-connected graph G with minimum degree at least $\lfloor \frac{3}{2}k \rfloor + 2$ has an edge $e = uv$ such that $G - \{u, v\}$ is still k-connected. They conjectured that there are similar results for the existence of connected subgraphs of prescribed order $m \geq 3$ keeping the connectivity.

Conjecture 1 ([4]). *For all positive integers k, m, there is a (least) non-negative integer $f_k(m)$ such that every k-connected graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor - 1 + f_k(m)$ contains a connected subgraph W of exact order m such that $G - V(W)$ is still k-connected.*

They also gave examples in [4] showing that $f_k(m)$ must be at least m for all positive integers k, m. In [5], Mader proved that $f_k(m)$ exists and $f_k(m) = m$ holds for all k, m.

Theorem 1.2 ([5]). *Every k-connected graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor + m - 1$ for positive integers k, m contains a path P of order m such that $G - V(P)$ remains k-connected.*

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* Corresponding author.
E-mail addresses: tianyzhxj@163.com (Y. Tian), mjx@xju.edu.cn (J. Meng), hjlai@math.wvu.edu (H.-J. Lai), 200661000016@jmu.edu.cn (L. Xu).

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In the same paper, Mader [5] asked whether the result is true for any other tree T instead of a path, and gave the following conjecture.

Conjecture 2 (*[5]*). *For every positive integer k and every finite tree T, there is a least non-negative integer $t_k(T)$, such that every k-connected, finite graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor - 1 + t_k(T)$ contains a subgraph $T' \cong T$ with $\kappa(G - V(T')) \geq k$.*

Mader showed that $t_k(T)$ exists in [6].

Theorem 1.3 (*[6]*). *Let G be a k-connected graph with $\delta(G) \geq 2(k-1+m)^2 + m - 1$ and let T be a tree of order m for positive integers k, m. Then there is a tree $T' \subseteq G$ isomorphic to T such that $G - V(T')$ remains k-connected.*

Mader further conjectured that $t_k(T) = |T|$.

Conjecture 3 (*[5]*). *For every positive integer k and every tree T, $t_k(T) = |T|$ holds.*

Theorem 1.2 showed that Conjecture 3 is true when T is a path. Diwan and Tholiya [3] proved that the conjecture holds when $k = 1$. In the next section, we will verify that Conjecture 3 is true when T is a star and $k = 2$. It is proved in the last section that Conjecture 3 is true when T is a double-star and $k = 2$.

A *block* of a graph G is a maximal connected subgraph of G that has no cut vertex. Note that any block of a connected graph of order at least two is 2-connected or isomorphic to K_2.

For a vertex subset U of a graph G, $G[U]$ denotes the subgraph induced by U and $G-U$ is the subgraph induced by $V(G)-U$. The *neighborhood* $N_G(U)$ of U is the set of vertices in $V(G) - U$ which are adjacent to some vertex in U. If $U = \{u\}$, we also use $G - u$ and $N_G(u)$ for $G - \{u\}$ and $N_G(\{u\})$, respectively. The *degree* $d_G(u)$ of u is $|N_G(u)|$. If H is a subgraph of G, we often use H for $V(H)$. For example, $N_G(H)$, $H \cap G$ and $H \cap U$ mean $N_G(V(H))$, $V(H) \cap V(G)$ and $V(H) \cap U$, respectively. If there is no confusion, we always delete the subscript, for example, $d(u)$ for $d_G(u)$, $N(u)$ for $N_G(u)$, $N(U)$ for $N_G(U)$ and so on. A *tree* is a connected graph without cycles. A *star* is a tree that has exact one vertex with degree greater than one. A *double-star* is a tree that has exact two vertices with degree greater than one.

2. Connectivity keeping stars in 2-connected graphs

Theorem 2.1. *Let G be a 2-connected graph with minimum degree $\delta(G) \geq m + 2$, where m is a positive integer. Then for a star T with order m, G contains a star T' isomorphic to T such that $G - V(T')$ is 2-connected.*

Proof. If $m \leq 3$, then T is a path, and the theorem holds by Theorem 1.2. Thus we assume $m \geq 4$ in the following.

Since $\delta(G) \geq m+2$, there is a star $T' \subseteq G$ with $T' \cong T$. Assume $V(T') = \{u, v_1, \ldots, v_{m-1}\}$ and $E(T') = \{uv_i | 1 \leq i \leq m-1\}$. We say T' is a star rooted at u or with root u. Let $G' = G - T'$. Let B be a maximum block in G' and let l be the number of components of $G' - B$. If $l = 0$, then $B = G'$ is 2-connected. So we may assume that $l \geq 1$. Let $H_1, \ldots, H_l$ be the components of $G' - B$ with $|H_1| \geq \cdots \geq |H_l|$.

Take such a star T' so that

(P1) $|B|$ is as large as possible,

(P2) $(|H_1|, \ldots, |H_l|)$ is as large as possible in lexicographic order, subject to (P1).

We will complete the proof by a series of claims.

Claim 1. *$|N(H_i) \cap B| \leq 1$ and $|N(H_i) \cap V(T')| \geq 1$ for each $i \in \{1, \ldots, l\}$.*

Since B is a block of G', we have $|N(H_i) \cap B| \leq 1$ for each $i \in \{1, \ldots, l\}$. Since G is 2-connected, $|N(H_i) \cap V(T')| \geq 1$ for each $i \in \{1, \ldots, l\}$.

Claim 2. *$l = 1$.*

Assume $l \geq 2$. By Claim 1, there is an edge th between T' and H_1, where $t \in T'$ and $h \in H_1$. Choose a vertex $x \in H_l$. Since $\delta(G) \geq m + 2$ and $|N(H_l) \cap B| \leq 1$ (by Claim 1), we have $|N(x) \setminus (B \cup \{t\})| \geq m + 2 - 1 - 1 = m$. Thus we can choose a star $T'' \cong T$ with root x such that $V(T'') \cap (B \cup \{t\}) = \emptyset$. But then either there is a larger block than B in $G - T''$, or $G - T'' - B$ contains a larger component than H_1 ($H_1 \cup \{t\}$ is contained in a component of $G - T'' - B$), which contradicts to (P1) or (P2).

Claim 3. *$|N(t) \cap B| \leq 1$ and $|N(t) \cap H_1| \geq 2$ for any vertex $t \in V(T')$.*

Assume $|N(t) \cap B| \geq 2$. Choose a vertex $x \in H_1$. Since $\delta(G) \geq m + 2$ and $|N(H_1) \cap B| \leq 1$, we have $|N(x) \setminus (B \cup \{t\})| \geq m + 2 - 1 - 1 = m$. Thus we can choose a star $T'' \cong T$ with root x such that $V(T'') \cap (B \cup \{t\}) = \emptyset$. But $G - T''$ has a block containing $B \cup \{t\}$ as a subset, which contradicts to (P1). Thus $|N(t) \cap B| \leq 1$ holds. By $d(t) \geq m + 2$ and $|N(t) \cap B| \leq 1$, we have $|N(t) \cap H_1| = d(t) - |N(t) \cap B| - |N(t) \cap T'| \geq m + 2 - 1 - (m - 1) = 2$.

Claim 4. *For any edge $t_1t_2 \in E(T')$, $|N(\{t_1, t_2\}) \cap B| \leq 1$ holds.*

By contradiction, assume $|N(\{t_1, t_2\}) \cap B| \geq 2$. Because $|N(t_1) \cap B| \leq 1$ and $|N(t_2) \cap B| \leq 1$, we can assume that there are two distinct vertices $b_1, b_2 \in B$ such that $t_1b_1, t_2b_2 \in E(G)$. Choose a vertex $x \in H_1$. Since $\delta(G) \geq m+2$ and $|N(H_1) \cap B| \leq 1$, we have $|N(x) \setminus (B \cup \{t_1, t_2\})| \geq m + 2 - 1 - 2 = m - 1$. Thus we can choose a star $T'' \cong T$ with root x such that $V(T'') \cap (B \cup \{t_1, t_2\}) = \emptyset$. But then $G - T''$ has a block containing $B \cup \{t_1, t_2\}$ as a subset, which contradicts to (P1).

Because $|N(H_1) \cap B| \leq 1$ and G is 2-connected, we have $|N(T') \cap B| \geq 1$. The following claim further shows that $|N(T') \cap B| = 1$.

Claim 5. $|N(T') \cap B| = 1$.

By contradiction, assume $|N(T') \cap B| \geq 2$. If $N(u) \cap B \neq \emptyset$, say $N(u) \cap B = \{u'\}$, then we have $N(\{v_1, \ldots, v_{m-1}\}) \cap B \subseteq \{u'\}$ by Claim 4. That is, $N(T') \cap B = \{u'\}$, a contradiction. Thus $N(u) \cap B = \emptyset$. Assume, without loss of generality, that there are two distinct vertices w and w' in B such that $v_1w, v_2w' \in E(G)$. If $N(v_3) \cap B = \emptyset$ or $|N(v_3) \cap \{v_1, v_2\}| \leq 1$, then we can choose a star T'' with order m and root v_3 such that $V(T'') \cap (B \cup \{u, v_1, v_2\}) = \emptyset$. But then $B \cup \{u, v_1, v_2\}$ is contained in a block of $G - T''$, contradicting to (P1). Thus we assume v_3 is adjacent to a vertex y in B and is adjacent to both v_1 and v_2. Without loss of generality, assume y is distinct from w. Then we can choose a star T'' with order m and root u such that $V(T'') \cap (B \cup \{v_1, v_3\}) = \emptyset$. But $B \cup \{v_1, v_3\}$ is contained in a block of $G - T''$, contradicting to (P1). Thus $|N(T') \cap B| = 1$.

By Claim 5, $|N(T') \cap B| = 1$. Assume $N(T') \cap B = \{w\}$. Since G is 2-connected, we have $|N(H_1) \cap B| \geq 1$. By Claim 1, $|N(H_1) \cap B| = 1$. Assume $N(H_1) \cap B = \{z\}$. Let P be a shortest path from z to w going through H_1 and T'. Assume $P := p_1p_2 \cdots p_{q-1}p_q$, where $p_1 = z$, $p_q = w$ and $p_i \in H_1 \cup T'$ for each $i \in \{2, \ldots, q-1\}$. Since P is a shortest path, $|N(p_i) \cap P| = 2$ for each $2 \leq i \leq q-1$. By $N(T') \cap B = \{w\}$ and $N(H_1) \cap B = \{z\}$, $N(p_i) \cap B \subseteq \{w, z\} \subseteq V(P)$ for each $2 \leq i \leq q-1$. Thus $|N(p_i) \cap (B \cup P)| = 2$ and $|N(p_i) \cap (V(G) \setminus (B \cup P))| \geq m$ for each $2 \leq i \leq q-1$. This implies $G - (B \cup P)$ is not empty. For any vertex x in $G - (B \cup P)$, we have $|N(x) \cap P| \leq 3$. For otherwise, we can find a path P' containing x from z to w going through H_1 and T' shorter than P, a contradiction. By $\delta(G) \geq m+2$, $|N(x) \cap (G - (B \cup P))| \geq m + 2 - 3 = m - 1$. Then we can find a star $T'' \cong T$ with root x such that $T'' \cap (B \cup P) = \emptyset$. But then $B \cup P$ is contained in a block of $G - T''$, a contradiction. The proof is thus complete. □

3. Connectivity keeping double-stars in 2-connected graphs

Lemma 3.1. *Let G be a graph and T be a double-star with order m. If there is an edge $e = uv \in E(G)$ such that $|N(u) \setminus v| \geq \lfloor \frac{m}{2} \rfloor - 1$, $|N(v) \setminus u| \geq m - 3$ and $|(N(u) \cup N(v)) \setminus \{u, v\}| \geq m - 2$, then there is a double-star $T' \subseteq G$ isomorphic to T.*

Proof. Since T is a double-star, we have $m \geq 4$. Assume the double-star T is constructed from an edge $e' = u'v'$ by adding r leaves to u' and s leaves to v', where $1 \leq r \leq s$ and $r + s = m - 2$. Then $1 \leq r \leq \lfloor \frac{m}{2} \rfloor - 1$ and $\lceil \frac{m}{2} \rceil - 1 \leq s \leq m - 3$. Since $|N(u) \setminus v| \geq \lfloor \frac{m}{2} \rfloor - 1$, $|N(v) \setminus u| \geq m - 3$ and $|(N(u) \cup N(v)) \setminus \{u, v\}| \geq m - 2$, we can find a double-star $T' \cong T$ in G with center-edge $e = uv$, where u is adjacent to r leaves and v is adjacent to s leaves. □

The main idea of the proof of Theorem 3.2 is similar to that of Theorem 2.1, with much more complicated and different details.

Theorem 3.2. *Let T be a double-star with order m and G be a 2-connected graph with minimum degree $\delta(G) \geq m + 2$. Then G contains a double-star T' isomorphic to T such that $G - V(T')$ is 2-connected.*

Proof. Since T is a double-star, we have $m \geq 4$. If $m = 4$, then T is a path, and the theorem holds by Theorem 1.2. Thus we assume $m \geq 5$ in the following.

Since $\delta(G) \geq m + 2$, there is a double-star $T' \subseteq G$ with $T' \cong T$. Assume $V(T') = \{u, v, u_1, \ldots, u_r, v_1, \ldots, v_s\}$ and $E(T') = \{uv\} \cup \{uu_i | 1 \leq i \leq r\} \cup \{vv_j | 1 \leq j \leq s\}$, where $1 \leq r \leq s$ and $r + s = m - 2$. We say T' is a double-star with center-edge uv. Let $G' = G - T'$. Let B be a maximum block in G' and let l be the number of components of $G' - B$. If $l = 0$, then $B = G'$ is 2-connected. So we may assume that $l \geq 1$. Let $H_1, \ldots, H_l$ be the components of $G' - B$ with $|H_1| \geq \cdots \geq |H_l|$.

Take such a double-star T' so that

(P1) $|B|$ is as large as possible,

(P2) $(|H_1|, \ldots, |H_l|)$ is as large as possible in lexicographic order, subject to (P1).

We will complete the proof by a series of claims.

Claim 1. $|N(H_i) \cap B| \leq 1$ *and* $|N(H_i) \cap T'| \geq 1$ *for each* $i \in \{1, \ldots, l\}$.

Since B is a block of G', we have $|N(H_i) \cap B| \leq 1$ for each $i \in \{1, \ldots, l\}$. Since G is 2-connected, $|N(H_i) \cap T'| \geq 1$ for each $i \in \{1, \ldots, l\}$.

Claim 2. $|H_i| \geq 2$ *for each* $i \in \{1, \ldots, l\}$.

This claim holds because $|N(h_i) \cap H_i| = d(h_i) - |N(h_i) \cap T'| - |N(h_i) \cap B| \geq m + 2 - m - 1 = 1$ for any vertex $h_i \in H_i$, where $1 \leq i \leq l$.

Claim 3. $l = 1$.

Assume $l \geq 2$. By Claim 1, there is an edge th between T' and H_1, where $t \in T'$ and $h \in H_1$. By Claim 2, we can choose an edge $xy \in E(H_l)$. Since $\delta(G) \geq m+2$ and $|N(H_l) \cap B| \leq 1$ (by Claim 1), we have $|N(x) \setminus (B \cup \{y, t\})| \geq m+2-1-2 = m-1$ and $|N(y) \setminus (B \cup \{x, t\})| \geq m+2-1-2 = m-1$. Thus, by Lemma 3.1, we can choose a double-star $T'' \cong T$ with center-edge xy such that $V(T'') \cap (B \cup \{t\}) = \emptyset$. But then either there is a larger block than B in $G - T''$, or $G - T'' - B$ contains a larger component than H_1 ($H_1 \cup \{t\}$ is contained in a component of $G - T'' - B$), which contradicts to (P1) or (P2).

Claim 4. *$|N(t) \cap B| \leq 1$ and $|N(t) \cap H_1| \geq 2$ for any vertex $t \in V(T')$.*

Assume $|N(t) \cap B| \geq 2$. Choose an edge $xy \in E(H_1)$. Since $\delta(G) \geq m+2$ and $|N(H_1) \cap B| \leq 1$, we have $|N(x) \setminus (B \cup \{y, t\})| \geq m+2-1-2 = m-1$ and $|N(y) \setminus (B \cup \{x, t\})| \geq m+2-1-2 = m-1$. Thus, by Lemma 3.1, we can choose a double-star $T'' \cong T$ with center-edge xy such that $V(T'') \cap (B \cup \{t\}) = \emptyset$. But then $B \cup \{t\}$ is contained in a block of $G - T''$, which contradicts to (P1). Thus $|N(t) \cap B| \leq 1$ holds for any vertex $t \in V(T')$. By $d(t) \geq m+2$ and $|N(t) \cap B| \leq 1$, we have $|N(t) \cap H_1| = d(t) - |N(t) \cap B| - |N(t) \cap T'| \geq m+2-1-(m-1) = 2$.

Claim 5. *For any edge $t_1t_2 \in E(T')$, $|N(\{t_1, t_2\}) \cap B| \leq 1$ holds.*

By contradiction, assume $|N(\{t_1, t_2\}) \cap B| \geq 2$. Because $|N(t_1) \cap B| \leq 1$ and $|N(t_2) \cap B| \leq 1$, we can assume that there are two distinct vertices $b_1, b_2 \in B$ such that $t_1b_1, t_2b_2 \in E(G)$. Choose an edge $xy \in E(H_1)$. Since $\delta(G) \geq m+2$ and $|N(H_1) \cap B| \leq 1$, we have $|N(x) \setminus (B \cup \{y, t_1, t_2\})| \geq m+2-1-3 = m-2$ and $|N(y) \setminus (B \cup \{x, t_1, t_2\})| \geq m+2-1-3 = m-2$. Thus, by Lemma 3.1, we can choose a double-star $T'' \cong T$ with center-edge xy such that $V(T'') \cap (B \cup \{t_1, t_2\}) = \emptyset$. But then $G - T''$ has a block containing $B \cup \{t_1, t_2\}$ as a subset, which contradicts to (P1).

Claim 6. *For any 3-path $t_1t_2t_3$ in T', $|N(\{t_1, t_2, t_3\}) \cap B| \leq 1$ holds.*

By contradiction, assume $|N(\{t_1, t_2, t_3\}) \cap B| \geq 2$. Then we have $|N(t_2) \cap B| = 0$. For otherwise, if $|N(t_2) \cap B| = 1$, then we have $|N(\{t_1, t_2, t_3\}) \cap B| \leq 1$ by $|N(\{t_1, t_2\}) \cap B| \leq 1$ and $|N(\{t_2, t_3\}) \cap B| \leq 1$, a contradiction. Because $|N(t_1) \cap B| \leq 1$ and $|N(t_3) \cap B| \leq 1$, we can assume that there are two distinct vertices $b_1, b_3 \in B$ such that $t_1b_1, t_3b_3 \in E(G)$. Choose any edge $xy \in E(H_1)$. Since $\delta(G) \geq m+2$ and $|N(H_1) \cap B| \leq 1$, we have $|N(x) \setminus (B \cup \{y, t_1, t_2, t_3\})| \geq m+2-1-4 = m-3 > \lfloor \frac{m}{2} \rfloor - 1$ (by $m \geq 5$) and $|N(y) \setminus (B \cup \{x, t_1, t_2, t_3\})| \geq m+2-1-4 = m-3$.

If $|N(x) \setminus (B \cup \{y, t_1, t_2, t_3\})| > m-3$ or $|N(y) \setminus (B \cup \{x, t_1, t_2, t_3\})| > m-3$, then by Lemma 3.1, we can choose a double-star $T'' \cong T$ with center-edge xy such that $V(T'') \cap (B \cup \{t_1, t_2, t_3\}) = \emptyset$. But then $G - T''$ has a block containing $B \cup \{t_1, t_2, t_3\}$ as a subset, which contradicts to (P1). Thus we assume $|N(x) \setminus (B \cup \{y, t_1, t_2, t_3\})| = m-3$ and $|N(y) \setminus (B \cup \{x, t_1, t_2, t_3\})| = m-3$, which imply $|N(x) \cap B| = 1$ and $|N(y) \cap B| = 1$. Since $|N(H_1) \cap B| \leq 1$, we can assume $N(x) \cap B = N(y) \cap B = \{z\}$. Without loss of generality, assume $z \neq b_1$.

If $N(x) \setminus y \neq N(y) \setminus x$, then $|(N(x) \cup N(y)) \setminus (B \cup \{x, y, t_1, t_2, t_3\})| \geq m-2$. So we can choose a double-star $T'' \cong T$ with center-edge xy disjoint from $B \cup \{t_1, t_2, t_3\}$. But then $G - T''$ contains a larger block than B, a contradiction. Thus $N(x) \setminus y = N(y) \setminus x$. Because we choose the edge xy in H_1 arbitrarily, we conclude that H_1 is a complete graph and each vertex not in H_1 is adjacent to all vertices in H_1 if it is adjacent to one vertex in H_1. In particular, every vertex t in T' is adjacent to all vertices in H_1 by Claim 4 and the vertex z in B is adjacent to all vertices in H_1.

Let t_4h_4 be an edge of graph G, where $t_4 \in V(T') \setminus \{t_1, t_2, t_3\}$ and $h_4 \in V(H_1)$. Let h_1 be a vertex in H_1 distinct from h_4. Then $t_1h_1, h_1z \in E(G)$. Thus we can choose a double-star $T'' \cong T$ with center-edge t_4h_4 disjoint from $B \cup \{t_1, h_1\}$. But then $B \cup \{t_1, h_1\}$ is contained in a block of $G - T''$, contradicting to (P1).

Because $|N(H_1) \cap B| \leq 1$ and G is 2-connected, we have $|N(T') \cap B| \geq 1$. The following claim further shows that $|N(T') \cap B| = 1$.

Claim 7. $|N(T') \cap B| = 1$.

By contradiction, assume $|N(T') \cap B| \geq 2$. If $N(u) \cap B \neq \emptyset$, say $N(u) \cap B = \{u'\}$, then we have $N(\{u_1, \ldots, u_r, v\}) \cap B \subseteq \{u'\}$ by Claim 5 and $N(\{v_1, \ldots, v_s\}) \cap B \subseteq \{u'\}$ by Claim 6. That is, $N(T') \cap B = \{u'\}$, a contradiction. Thus $N(u) \cap B = \emptyset$. Similarly, we have $N(v) \cap B = \emptyset$. Since $|N(\{u_1, \ldots, u_r\}) \cap B| \leq 1$ and $|N(\{v_1, \ldots, v_s\}) \cap B| \leq 1$ (by Claim 6), we have $|N(T') \cap B| = 2$. Assume, without loss of generality, that there are two distinct vertices w and w' in B such that $u_1w, v_1w' \in E(G)$.

We first show that any vertex x in $\{u_1, \ldots, u_r, v_1, \ldots, v_s\} \setminus \{u_1, v_1\}$ has no neighbors in B. By contradiction, assume there is a vertex in $\{u_1, \ldots, u_r, v_1, \ldots, v_s\} \setminus \{u_1, v_1\}$, say v_j for some $j \in \{2, \ldots, s\}$ (the case u_i for some $i \in \{2, \ldots, r\}$ can be proved similarly), such that $N(v_j) \cap B = \{w'\}$. If v_j is adjacent to u (or u_1), then for any edge vv' (v' is a neighbor of v in H_1), we have $|N(v) \setminus (B \cup \{u, u_1, v_j, v'\})| \geq m+2-4 = m-2$ (or $|N(v) \setminus (B \cup \{u_1, v_j, v'\})| \geq m+2-3 = m-1$) and $|N(v') \setminus (B \cup \{u, v, u_1, v_j\})| \geq m+2-1-4 = m-3$ (or $|N(v') \setminus (B \cup \{v, u_1, v_j\})| \geq m+2-1-3 = m-2$). By Lemma 3.1, we can find a double-star $T'' \cong T$ with center-edge vv' such that T'' is disjoint from $B \cup \{u, u_1, v_j\}$ (or $B \cup \{u_1, v_j\}$). But then $G - T''$ contains a larger block than B, a contradiction. Thus neither u nor u_1 is adjacent to v_j. Choose a neighbor v_j' of v_j in H_1. Since $|N(v_j) \setminus (B \cup \{u, v, u_1, v_1, v_j'\})| \geq m+2-1-3 = m-2$ and $|N(v_j') \setminus (B \cup \{u, v, u_1, v_1, v_j\})| \geq m+2-1-5 = m-4 \geq \lfloor \frac{m}{2} \rfloor - 1$ (by $m \geq 5$), we can find a double-star $T'' \cong T$ with center-edge v_jv_j' such that T'' is disjoint from $B \cup \{u, v, u_1, v_1\}$. But then $G - T''$ contains a larger block than B, a contradiction. Thus we have $N(\{u_1, \ldots, u_r, v_1, \ldots, v_s\} \setminus \{u_1, v_1\}) \cap B = \emptyset$.

Let $v_2v_2' \in E(G)$, where v_2' is a neighbor of v_2 in H_1. Since $\delta(G) \geq m+2$ and $N(v_2) \cap B = \emptyset$, we have $|N(v_2) \setminus (B \cup \{u, v, u_1, v_1, v_2'\})| \geq m+2-5 = m-3$ and $|N(v_2') \setminus (B \cup \{u, v, u_1, v_1, v_2\})| \geq m+2-1-5 = m-4 \geq \lfloor \frac{m}{2} \rfloor - 1$ (by $m \geq 5$). If $|N(v_2) \setminus (B \cup \{u, v, u_1, v_1, v_2'\})| \geq m-2$, then, by Lemma 3.1, we can find a double-star $T'' \cong T$ with center-edge v_2v_2' such that T'' avoids $B \cup \{u, v, u_1, v_1\}$. But then $G - T''$ contains a larger block than B, a contradiction. Thus assume $|N(v_2) \setminus (B \cup \{u, v, u_1, v_1, v_2'\})| = m-3$, which implies v_2 is adjacent to both u_1 and v_1. For the edge uv, we can verify that $|N(u) \setminus (B \cup \{v, u_1, v_1, v_2\})| \geq m+2-4 = m-2$ and $|N(v) \setminus (B \cup \{u, u_1, v_1, v_2\})| \geq m+2-4 = m-2$. By Lemma 3.1, we can find a double-star $T'' \cong T$ with center-edge uv such that T'' avoids $B \cup \{u_1, v_1, v_2\}$. But then $B \cup \{u_1, v_1, v_2\}$ is contained in a block of $G - T''$, contradicting to (P1). Thus Claim 7 holds.

By Claim 7, $|N(T') \cap B| = 1$. Assume $N(T') \cap B = \{w\}$. Since G is 2-connected, we have $|N(H_1) \cap B| = 1$ by Claim 1. Let $N(H_1) \cap B = \{z\}$. Let P be a shortest path from z to w going through H_1 and T''. Assume $P := p_1p_2 \cdots p_{q-1}p_q$, where $p_1 = z$, $p_q = w$ and $p_i \in H_1 \cup T'$ for each $i \in \{2, \ldots, q-1\}$. Since P is a shortest path, $N(p_i) \cap P = \{p_{i-1}, p_{i+1}\}$ for $2 \leq i \leq q-1$. Because $\delta(G) \geq m+2$ and $N(p_i) \cap B \subseteq \{w, z\} \subseteq P$ for each $2 \leq i \leq q-1$, we know p_i has at least m neighbors not in $B \cup P$, that is, $G-(B \cup P)$ is not empty. For any vertex x in $G-(B \cup P)$, we have $|N(x) \cap P| \leq 3$. For otherwise, we can find a path P' containing x from z to w going through H_1 and T'' shorter than P, a contradiction. By $\delta(G) \geq m+2$, $|N(x) \cap (G-(B \cup P))| \geq m+2-3 = m-1$. Choose an edge xy in $G-(B \cup P)$. Since $|N(x) \setminus (B \cup P \cup \{y\})| \geq m+2-4 = m-2$ and $|N(y) \setminus (B \cup P \cup \{x\})| \geq m+2-4 = m-2$, we can find a double-star $T'' \cong T$ with center-edge xy such that $T'' \cap (B \cup P) = \emptyset$. But then $B \cup P$ is contained in a block of $G - T''$, a contradiction. The proof is thus complete. □

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