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r-hued coloring of sparse graphs ☆

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ABSTRACT

For two positive integers k, r, a (k, r)-coloring (or r-hued k-coloring) of a graph G is a proper k-vertex-coloring such that every vertex v of degree $d_G(v)$ is adjacent to at least $\min\{d_G(v), r\}$ distinct colors. The r-hued chromatic number, $\chi_r(G)$, is the smallest integer k for which G has a (k, r)-coloring. The maximum average degree of G, denoted by mad (G), equals $\max\{2|E(H)|/|V(H)|: H \text{ is a subgraph of } G\}$.

In this paper, we prove the following results using the well-known discharging method. For a graph G, if $\text{mad}(G) < \frac{12}{5}$, then $\chi_3(G) \le 6$; if $\text{mad}(G) < \frac{7}{3}$, then $\chi_3(G) \le 5$; if G has no C_5-components and $\text{mad}(G) < \frac{8}{3}$, then $\chi_2(G) \le 4$.

1. Introduction

Graphs in this paper are simple and finite. Notations and terminology undefined here are referred to [1]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The set of neighbors of a vertex v is denoted by $N_G(v)$. We use $d_G(v)$ and $\Delta(G)$ to denote the *degree* of v and the maximum degree of G, respectively. A vertex of degree k (resp. at least k) is called a *k-vertex* (resp. *k^+-vertex*). The maximum average degree of G, denoted by $\text{mad}(G)$, equals $\max\{2|E(H)|/|V(H)| : H \text{ is a subgraph of } G\}$. A graph G is *r-regular* if each vertex of G has degree r. We use *cycles* to denote the connected 2-regular graphs and a cycle of length k is denoted by C_k.

A path $P = u_0u_1 \cdots u_ku_{k+1}$ is a *k-thread* of a graph G, if $u_1, \ldots, u_k$ are 2-vertices and u_0, u_{k+1} are 3^+-vertices. Vertices u_0 and u_{k+1} are called *endpoints* of P. The collection of l-threads with $l \ge k$ are *k^+-threads*. Two vertices u and v are *loosely adjacent* if u and v are contained in some k-thread P.

A *k-vertex-coloring* (or simply a *k-coloring*) of a graph G is a mapping $c : V(G) \to S$, where S is a set of k colors. In general, S is taken to be $\{1, \ldots, k\}$. If a vertex adjacent to u is colored i, then we say that *u sees i*. Otherwise, we say that *u misses i*. If $W \subseteq V(G)$, denote by $c(W)$ the set of colors received by at least one vertex of W. A k-coloring is *proper* if no two adjacent vertices receive the same color. As we are only concerned about the proper coloring, we refer to a proper coloring simply as a coloring. A *(k, r)-coloring* (or *r-hued k-coloring*) of a graph G is a k-coloring such that each vertex v is adjacent to at least $\min\{d_G(v), r\}$ distinct colors. The *r-hued chromatic number* of a graph G, denoted $\chi_r(G)$, is the minimum k for which G has a (k, r)-coloring. A list assignment L of a graph G is a function that assigns to every vertex v of G a set $L(v)$ of positive integers.

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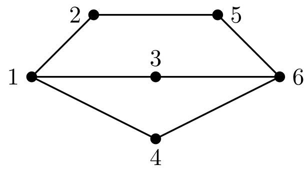

Fig. 1. G_0 with $\text{mad}(G) = \frac{7}{3}$ but $\chi_3(G_0) = 6$.

Given a list assignment L of G, a (L, r)-*coloring* of G is a coloring c such that each vertex v is adjacent to at least $\min\{d_G(v), r\}$ distinct colors and $c(v) \in L(v)$. The r-*hued choice number* of a graph G is the minimum k such that G has a (L, r)-coloring where $|L(v)| = k$ for each vertex $v \in V(G)$, and is denoted by $ch_r(G)$.

The concept of (k, r)-colorings was introduced by Lai et al. [5], and an upper bound of χ_2 was first studied in the same paper. In [6], Song et al. showed that, for K_4-minor free graphs, $\chi_r(G) \leq r + 3$ if $2 \leq r \leq 3$ and $\chi_r(G) \leq \lfloor 3r/2 \rfloor + 1$ if $r \geq 4$. Song et al. [7] proved that $\chi_r(G) \leq r + 5$ if G is a planar graph of girth at least 6. For any planar graph G, $\chi_2(G) \leq 5$ was proved by Chen et al. [2], and they conjectured that with the exception of C_5, $\chi_2(G) \leq 4$ for all planar graphs. Kim, et al. [3] verified this conjecture in 2013.

Motivated by above results, we use a discharging method and give upper bounds on the 2-hued and 3-hued chromatic numbers for graphs with different maximum average degree constraints in this paper.

Theorem 1.1. *If G is a graph with* $\text{mad}(G) < \frac{12}{5}$, *then* $\chi_3(G) \leq 6$.

In fact, we prove a slightly stronger result that $ch_3(G) \leq 6$ for graphs with $\text{mad}(G) < \frac{12}{5}$. See the remark at the end of Section 2.1.

Theorem 1.2. *If G is a graph with* $\text{mad}(G) < \frac{7}{3}$, *then* $\chi_3(G) \leq 5$.

Remark.

(1) The bound of $\text{mad}(G) < \frac{7}{3}$ is sharp since G_0 as shown in Fig. 1 satisfies that $\text{mad}(G_0) = \frac{7}{3}$ but $\chi_3(G_0) = 6$.

(2) The bound $\chi_3(G) \leq 5$ is the best possible bound for which there are infinitely many graphs satisfying $\text{mad}(G) < \frac{7}{3}$ and $\chi_3(G) = 5$. The following are two special cases and the construction of more such graphs.
(a) C_5 and a graph obtained from two edge-disjoint C_5 joining at exactly one vertex.
(b) In general, we define a family of connected graphs

$$\mathcal{F} = \{G\text{: } G \text{ contains a bridge } e \text{ such that } G - \{e\} \text{ has a } C_5 \text{ component}\}.$$

We claim that each member of $\mathcal{F}$ has 3-hued chromatic number at least 5. Assume $G \in \mathcal{F}$ has an edge xy such that $G - \{uv\}$ has a $C_5 = vxyzwv$ as a component. For any 3-hued coloring c of G, $|\{c(x), c(w), c(v), c(u)\}| = 4$ and $\{c(y), c(z)\} \cap \{c(x), c(w), c(v)\} = \emptyset$. Hence, $|c(C_5)| = 5$ and $\chi_3(G) \geq 5$. Combined with Theorem 1.2, each graph G of $\mathcal{F}$ with $\text{mad}(G) < \frac{7}{3}$ has $\chi_3(G) = 5$ and we have infinitely many of such graphs in $\mathcal{F}$.

In [4], Kim and Park submitted a proof that a graph G with $\text{mad}(G) < \frac{8}{3}$ satisfies $\chi_2(G) \leq 4$. Observe that $\chi_2(C_5) = 5$ while $\text{mad}(C_5) = 2 < \frac{8}{3}$, which reveals a gap in their results. In this paper, we also fix the proof in [4] and prove the following result.

Theorem 1.3. *Let G be a graph with no C_5-components. If* $\text{mad}(G) < \frac{8}{3}$, *then* $\chi_2(G) \leq 4$.

Remark. In [4], Kim and Park showed that the bound of $\text{mad}(G) < \frac{8}{3}$ is sharp. Let G be the graph obtained by subdividing every edge of K_5 once. It is easy to verify that $\text{mad}(G) = \frac{8}{3}$ but $\chi_2(G) = 5$.

2. 3-hued colorings

Lemma 2.1. *Let k be an integer where $k \geq 4$ and $m \geq 2$ be a real number. If a graph G is a graph with minimum number of vertices such that $\chi_3(G) \geq k + 1$ and* $\text{mad}(G) \leq m$, *then G is connected and has no 1-vertex.*

Proof. If G has two or more components, then each of the components of G has a $(3, k)$-coloring and so does G, a contradiction to the choice of G.

Suppose that G has a vertex u with $d_G(u) = 1$ and $uv \in E(G)$. Denote $G' = G - \{u\}$. Then $\text{mad}(G') \leq m$ and thus G' has a $(3, k)$-coloring c since $|V(G')| < |V(G)|$. If v sees three colors in G', we have $k - 1 \geq 3$ available options to color u. If v sees two or fewer colors, then there are at least $k - 3 \geq 1$ available options to color u. In both cases, we can extend the coloring c to u, a contradiction to the choice of G. ■

Lemma 2.2. *Let G be a graph with $\Delta \leq 2$, then $\chi_3(G) \leq 5$.*

Proof. Since the maximum degree of G is at most 2, G is a union of vertex-disjoint cycles and paths. It is easy to see that each path has a 3-hued coloring with three colors and each cycle has a 3-hued coloring with at most five colors. Thus $\chi_3(G) \leq 5$. ■

2.1. Proof of Theorem 1.1

Let G be a counterexample to Theorem 1.1 with $|V(G)|$ minimized.

Claim 2.1. *G has no two adjacent 2-vertices.*

Proof. Suppose that G has two adjacent 2-vertices x and y. Note that G is connected by Lemma 2.1 and $\Delta(G) \geq 3$ by Lemma 2.2. We can choose x and y with the property that x is adjacent to a 3^+-vertex u. Let v be the other neighbor of y and denote $G' = G - \{x, y\}$. Therefore, G' has 3-hued 6-coloring c since $|V(G')| < |V(G)|$ and $\text{mad}(G') \leq \text{mad}(G)$. Let us extend the coloring c to x first. If $d_G(u) \geq 4$, then $|c(N_{G'}(u))| \geq 3$ nd thus only $c(u)$ and $c(v)$ are the forbidden colors for x. If $d_G(u) = 3$, then $|c(N_{G'}(u))| \leq 2$, thus $c(N_{G'}(u)) \cup \{c(u), c(v)\}$ is the set of forbidden colors for x. Thus we first extend c to x. In the resulting coloring, y has at most five forbidden colors, $\{c(u), c(x), c(v)\} \cup c(N_{G'}(v))$ when $d_G(v) = 3$ or at most three forbidden color $\{c(u), c(v), c(x)\}$ if $d_G(u) \neq 3$. Hence, we can further extend c to y and the resulting coloring will contradict the assumption that G is a counterexample. ■

Initial Charge: *$M(x) = d(x) - 12/5$ for each vertex x in G.* Since $\text{mad}(G) < 12/5$, we have $\sum_{x\in V(G)} M(x) < 0$. It follows from Lemma 2.1 and Claim 2.1 that, G has no 1-vertices and each 2-vertex is adjacent to two 3^+-vertices. Note that each k-vertex where $k \geq 3$ is adjacent to at most k 2-vertices. Hence, we can redistribute the charge of the vertices of G as follows.

Discharging Rule: *Each 2-vertex receives 1/5 from each neighbor.*

Denote this new charge by $M'(x)$. Hence, $\sum_{x\in V(G)} M'(x) = \sum_{x\in V(G)} M(x) < 0$.

(1) For each 2-vertex u, $M'(u) = 2 - 12/5 + 2 \times 1/5 = 0$.
(2) For each k-vertex v where $k \geq 3$, $M'(v) \geq k - 12/5 - k \times 1/5 = (4k - 12)/5 \geq 0$.

Therefore, $M'(x) \geq 0$ for each $x \in V(G)$ and $0 > \sum_{x\in V(G)} M(x) = \sum_{x\in V(G)} M'(v) \geq 0$, a contradiction. This completes the proof of Theorem 1.1. ■

Remark. Note that in Claim 2.1, the choice of available colors for x and y do not depend on the set of colors. Therefore, the above result could be generalized to $ch_3(G) \leq 6$ for a graph G with $\text{mad}(G) < \frac{12}{5}$. That is, for every list assignment of size six, there is a 3-hued 6-coloring of G such that each vertex is assigned with a color from its list.

2.2. Proof of Theorem 1.2

Let G be a counterexample to Theorem 1.2 with $|V(G)|$ minimized.

Claim 2.2. *G has no 3^+-threads.*

Proof. Suppose that G has a 3^+-thread $u_0u_1 \cdots u_{k-1}u_k$ where $k \geq 4$. Let $G' = G - \{u_1, u_2, u_3\}$. Then G' has a 3-hued 5-coloring c since $|V(G')| < |V(G)|$ and $\text{mad}(G') \leq \text{mad}(G)$. Let us extend the coloring c to u_1 first. Observe that u_1 has at most three forbidden colors. Therefore we have at least two available options to color u_1. In the resulting coloring, u_3 has at most four forbidden colors and then we can further extend c to u_3. After that, u_2 has at most four forbidden colors $\{c(u_0), c(u_1), c(u_3), c(u_4)\}$. In the last step, we extend the coloring c to u_2 to obtain a 3-hued 5-coloring of G, a contradiction to the choice of G. ■

Claim 2.3. *If $P = uxyv$ is a 2-thread of G, then $d_G(u) = d_G(v) = 3$.*

Proof. Suppose that $P = uxyv$ be a 2-thread of G in which either $d_G(u) \geq 4$ or $d_G(v) \geq 4$. Without loss of generality, assume $d_G(u) \geq 4$. Let $G' = G - \{x, y\}$. So G' has a 3-hued 5-coloring c by the minimality of G. Let us color y first. The worst case is that v has degree three in G and then y would have at most four forbidden colors $\{c(u), c(v)\} \cup c(N_{G'}(v))$. Thus we can always extend the coloring c to y. In the resulting coloring, u has already seen three colors in c, so x has at most three forbidden colors. Hence, we can further extend the coloring c to x, a contradiction to the choice of G. ■

Claim 2.4. *Let $P = uxyv$ be a 2-thread of G and $G' = G - \{x, y\}$. If c is a 3-hued 5-coloring of G', then we can always extend c to G except when $c(N_{G'}(u)) = c(N_{G'}(v))$ and $c(u) \neq c(v)$.*

Proof. Suppose that c is a 3-hued 5-coloring of G' such that either $c(N(u)) \neq c(N(v))$ or $c(u) = c(v)$. Let us color x first. By Claim 2.3, $d_{G'}(u) = d_{G'}(v) = 2$. Thus x has at most 4 forbidden colors $c(N_{G'}(u)) \cup \{c(u), c(v)\}$ and we can color x with one of the available options. In the resulting coloring, the set of forbidden colors of y is $c(N_{G'}(v)) \cup \{c(u), c(x), c(v)\}$.

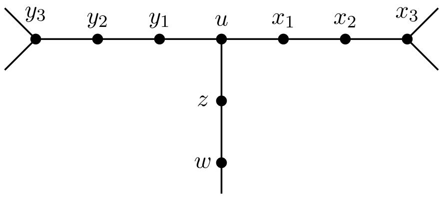

Fig. 2. Configuration of Claim 2.5.

If $c(u) = c(v)$, then $|c(N_{G'}(v)) \cup \{c(u), c(x), c(v)\}| \leq 4$. If $c(N_{G'}(u)) \neq c(N_{G'}(v))$, then we can recolor x such that $c(x) \in c(N_{G'}(v)) - c(N_{G'}(u))$, and therefore $|c(N_{G'}(v)) \cup \{c(u), c(x), c(v)\}| \leq 4$. In both cases, we can extend the coloring c to y, a contradiction to the choice of G. ■

Claim 2.5. *No 3-vertex is loosely adjacent to five or more 2-vertices.*

Proof. Let u be a 3-vertex of G such that u is loosely adjacent to at least five 2-vertices. Since G has no 3^+-threads by Claim 2.2, u is a common endpoint of either three 2-threads or two 2-threads and 1-thread (see Fig. 2). Hence, $d_G(x_1) = d_G(x_2) = d_G(y_1) = d_G(y_2) = d_G(z) = 2$. By Claim 2.3, $d_G(x_3) = d_G(y_3) = 3$. Let $G' = G - \{u, x_1, x_2, y_1, y_2\}$. Then G' has a 3-hued 5-coloring c by the minimality of G.

If $c(z) \notin c(N_{G'}(y_3))$, then we can extend the coloring c to u first since u has at most two forbidden colors. In the resulting coloring, x_2 has at most four forbidden colors, $\{c(u), c(x_3)\} \cup c(N_{G'}(x_3))$. Thus we can extend the coloring to x_2 with one of the available options. Then x_1 will have at most four forbidden colors $\{c(z), c(u), c(x_2), c(x_3)\}$, and we can further extend the coloring to x_1. After that, $c(N_{G'}(u)) \neq c(N_{G'}(y_3))$ since $c(z) \notin c(N_{G'}(y_3))$. By Claim 2.4, we can extend the coloring to $\{y_1, y_2\}$, a contradiction to the choice of G. If $c(z) \notin c(N_{G'}(x_3))$, we can extend the coloring to G by symmetry. Hence, we can assume that $c(z) \in c(N_{G'}(x_3)) \cap c(N_{G'}(y_3))$. Then $\{c(x_3), c(z)\} \cup c(N_{G'}(y_3)) = \{c(x_3)\} \cup c(N_{G'}(y_3))$.

We first extend the coloring c to x_1 by coloring x_1 with a color not in $\{c(x_3)\} \cup c(N_{G'}(y_3))$, then color x_2 with a color not in $\{c(x_1), c(x_3)\} \cup c(N_{G'}(x_3))$ and then further extend the coloring to u by coloring it with a color not in $\{c(x_1), c(x_2), c(z), c(w)\}$. Thus the resulting coloring is a 3-hued 5-coloring of $G - \{y_1, y_2\}$ and it satisfies $c(N_{G'}(u)) \neq c(N_{G'}(y_3))$ since $c(x_1) \notin c(N_{G'}(y_3))$. By Claim 2.4, we can finally extend the coloring to $\{y_1, y_2\}$, a contradiction to the choice of G. ■

Initial Charge: $M(x) = d(x) - 7/3$ *for each vertex x in G.*

Since $\text{mad}(G) < 7/3$, $\sum_{x \in V(G)} M(x) < 0$. G has no 1-vertices by Lemma 2.1. Claim 2.5 says that each 3-vertex is loosely adjacent to at most four 2-vertices. By Claims 2.2 and 2.3, each k-vertex where $k \geq 4$ can only be the endpoint of 1-thread and therefore is loosely adjacent to at most k 2-vertices. Now we can redistribute the charge as follows.

Discharging Rule: *Each 2-vertex u receives $1/6$ from each endpoint of the thread containing u.* Denote the new charge by $M'(x)$. Hence, $\sum_{x \in V(G)} M'(x) = \sum_{x \in V(G)} M(x) < 0$.

(1) For each 2-vertex u, $M'(u) = M(u) + 2 \times 1/6 = 2 - 7/3 + 1/3 = 2 - 6/3 = 0$.
(2) For each 3-vertex v, $M'(v) \geq M(v) - 4 \times 1/6 = 3 - 7/3 - 2/3 = 0$.
(3) For each k-vertex w with $k \geq 4$, $M'(w) \geq M(w) - k \times 1/6 = (5k - 14)/6 > 0$.

Hence, $M'(x) \geq 0$ for each $x \in V(G)$. So $\sum_{x \in V(G)} M(x) = \sum_{x \in V(G)} M'(x) \geq 0$, a contradiction. We complete the proof of Theorem 1.2. ■

3. Proof of Theorem 1.3

Let G be a counterexample to Theorem 1.3 with $|V(G)| + |E(G)|$ minimized. Then G must be connected. Otherwise, each component of G (not a C_5) has a 2-hued 4-coloring, and so does G. This would contradict the choice of G.

Claim 3.1. *G contains no cycle C as a subgraph such that $C = uwxyzu$ and w, x, y, z are 2-vertices of G.*

Proof. Suppose that G contains a cycle $C = uwxyzu$ where w, x, y, z are 2-vertices. Since $G \neq C_5$, $d_G(u) \geq 3$. Let $G' = G - \{w, x, y\}$. Since G is connected, so is G'. This implies that $G' \neq C_5$ since $d_{G'}(z) = 1$. Hence, G' has a 2-hued 4-coloring c by the minimality of G. Let us extend the coloring by assigning $c(w) = c(z)$, $c(x) = a$, $c(y) = b$ where $a \neq b$ and $a, b \notin \{c(u), c(z)\}$. It is easy to verify that the resulting coloring is a 2-hued 4-coloring of G. This contradicts the choice of G. ■

Claim 3.2. *G has no 1-vertex.*

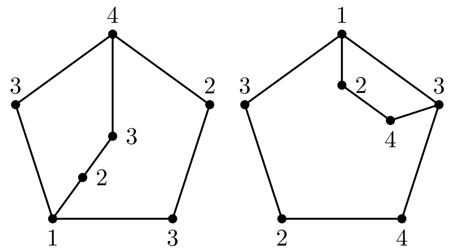

Fig. 3. Configurations when $G' = C_5$ in Claim 3.4.

Proof. Suppose that G has a vertex u with $d_G(u) = 1$ and $uv \in E(G)$. Denote $G' = G - \{u\}$. Then G' is connected and $G' \neq C_5$ for which would contradict Claim 3.1. Therefore, G' has a 2-hued 4-coloring c by the minimality of G. Note that u has at least two available colors. Thus we can extend the coloring c to u. This contradicts the assumption that G is a counterexample. ■

Claim 3.3. $\Delta(G) \geq 3$.

Proof. Suppose $\Delta(G) \leq 2$. Claim 3.2 says that G has no 1-vertex. Since G is connected, G must be a C_k where $k \neq 5$. Note that, except for C_5, every cycle can be 2-hued colored with four or fewer colors. This contradicts the choice of G. ■

Claim 3.4. *G has no two adjacent 2-vertices.*

Proof. Suppose that G has two adjacent 2-vertices x and y. Since $\Delta(G) \geq 3$ by Claim 3.3, we can choose x and y in a way that x is adjacent to a 3^+-vertex u. Let v be the other neighbor of y and denote $G' = G - \{x, y\}$. Now we consider the following two cases.

Case 1. $G' = C_5$.

By Claim 3.1, u and v are distinct vertices in C_5. G must be one of the configurations in Fig. 3. The corresponding 2-hued 4-colorings have been labeled in Fig. 3. This contradicts to the choice of G.

Case 2. $G' \neq C_5$.

If G' is disconnected, then G' has no C_5-components by Claim 3.1. If G' is connected, $G' \neq C_5$ by assumption. In both cases, G' has no C_5-components. By the minimality of G. G' has a 2-hued 4-coloring c. Let us color y first. Note that y has at most three forbidden colors and therefore we can extend c to y. Note that u has already seen at least two distinct colors in c since $d_{G'}(u) \geq 2$. Hence, x has at most three forbidden colors, $c(u)$, $c(y)$, and $c(v)$, and therefore we can further extend c to x. This contradicts the choice of G. ■

Claim 3.5. *Each 3-vertex in G is loosely adjacent to at most two 2-vertices.*

Proof. Suppose that G has a 3-vertex x which is loosely adjacent to at least three 2-vertices. By Claim 3.4, G has no 2^+-threads. Thus x is adjacent to three 2-vertices, say $\{y_1, y_2, y_3\}$, and each y_i is contained in a 1-thread xy_iv_i for each $i = 1, 2, 3$, where v_1, v_2, and v_3 are all 3^+-vertices.

We claim that x is not a cut-vertex. Otherwise, assume that x is a cut-vertex. Then at least one of $\{y_1, y_2, y_3\}$ is a cut-vertex. Without loss of generality, let y_1 be a cut-vertex of G. Then xy_1 is a cut-edge. By Claim 3.1 and since one component has minimum degree 1, no components of $G_1 = G - \{xy_1\}$ is a C_5. By the minimality of G, $G - \{xy_1\}$ has a 2-hued 4-coloring c. Note that $c(y_1) \neq c(v_1)$ and x is in a component that does not contain y_1 and v_1. Thus we may assume $c(x) \notin \{c(y_1), c(v_i)\}$. Observe that in G_1, both x and v_1 are 2^+-vertices. Thus c is a 2-hued 4-coloring of G, a contradiction to the choice of G. This proves that x is not a cut-vertex.

Let $G' = G - \{x, y_1, y_2, y_3\}$. Since G is connected and x is not a cut-vertex, G' is also connected. We consider the following two cases.

Case 1. $G' = C_5$.

If $v_1 = v_2 = v_3$, then G will satisfy the configuration in Claim 3.1, a contradiction. So $v_i \neq v_j$ for some $1 \leq i < j \leq 3$. Hence, G must be one of the configurations in Fig. 4. The corresponding 2-hued 4-coloring has been labeled in Fig. 4. This contradicts to the choice of G.

Case 2. $G' \neq C_5$.

Since G' is connected, G' has no C_5 components. Therefore, G' has a 2-hued 4-coloring c by the minimality of G. Since there are 4 colors, we can first extend c to x by coloring it with a color not in $\{c(v_1), c(v_2), c(v_3)\}$. Note that each of v_i has degree at least two in G' and thus sees at least two colors. We first color y_1 with a color not in $\{c(x), c(v_1)\}$ and then color y_2 with a color not in $\{c(x), c(y_1), c(v_2)\}$ and finally color y_3 with a color not in $\{c(x), c(v_3)\}$. It is easy to check that the extension of c is a 2-hued 4-coloring of G, a contradiction to the choice of G. ■

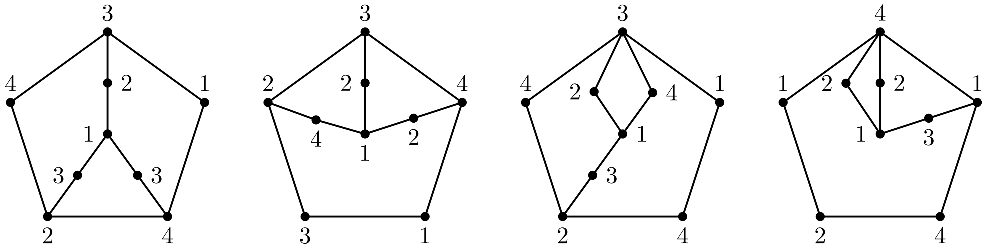

Fig. 4. Configurations when $G' = C_5$ in Claim 3.5.

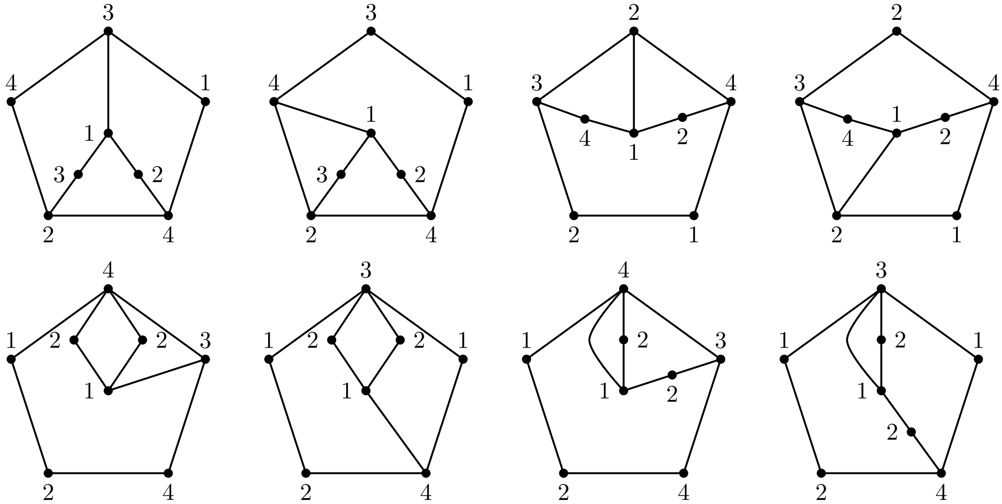

Fig. 5. Configurations when $G' = C_5$ in Claim 3.6.

Claim 3.6. *Each* 3*-vertex in G is loosely adjacent to at most one* 2*-vertex.*

Proof. By Claim 3.5, suppose that G has a 3-vertex x such that x is loosely adjacent to exactly two 2-vertices, say y_1 and y_2. Since G has no 2-threads, x is adjacent to y_1 and y_2, and each y_i is contained in a 1-thread xy_iv_i for each $i = 1, 2$. Let v_3 be the third neighbor of x. Thus v_1, v_2, v_3 are all 3^+-vertices.

With a similar argument as in Claim 3.5, we can show that x is not a cut-vertex. Let $G' = G - \{x, y_1, y_2\}$. Thus G' is connected. We consider the following two cases.

Case 1. $G' = C_5$.

If $v_1 = v_2 = v_3$, then G will satisfy the configuration in Claim 3.1, a contradiction. So $v_i \neq v_j$ for some $1 \leq i < j \leq 3$. Hence, G must be one of the configurations in Fig. 5. The corresponding 2-hued 4-coloring has been labeled in Fig. 5. This contradicts to the choice of G.

Case 2. $G' \neq C_5$.

Since G' is connected, G' has no C_5 components. Therefore, G' has a 2-hued 4-coloring c by the minimality of G. Note that for each $i = 1, 2, 3$, $d_{G'}(v_i) \geq 2$ and thus v_i sees at least two colors. We first color x with a color not in $\{c(v_1), c(v_2), c(v_3)\}$, then color y_1 with a color not in $\{c(v_1), c(x), c(v_3)\}$, and then color y_2 with a color not in $\{c(x), c(v_2)\}$. It is easy to see that this is a 2-hued 4-coloring of G, a contradiction to the choice of G. ■

Initial Charge: $M(x) = d(x) - 8/3$ *for each vertex x in G.*

Since mad$(G) < 8/3$, $\sum_{x\in V(G)} M(x) < 0$. By Claim 3.2, G has no 1-vertices. By Claim 3.4, each 2-vertex is adjacent to two 3^+-vertices. Claim 3.6 says that each 3-vertex is adjacent to at most one 2-vertex. By Claim 3.4, each k-vertex with $k \geq 4$ is adjacent to at most k 2-vertices. Now let us redistribute the charge as follows.

Discharging Rule: *Each* 2*-vertex receives* 1/3 *from its two neighbors.*

Denote the new charge by $M'(x)$. Hence, $\sum_{x\in V(G)} M(x) = \sum_{x\in V(G)} M'(x) < 0$.

(1) For each 2-vertex x, $M'(x) \geq 2 - 8/3 + 2 \times 1/3 = 0$.

(2) For each 3-vertex y, $M'(y) \geq 3 - 8/3 - 1/3 = 0$.
(3) For each k-vertex z with $k \geq 4$, $M'(z) \geq k - 8/3 - k \times 1/3 = (2k - 8)/3 \geq 0$.

For any $x \in V(G)$, $M'(x) \geq 0$ and therefore $\sum_{x \in V(G)} M(x) = \sum_{x \in V(G)} M'(x) \geq 0$, a contradiction. This completes the proof of Theorem 1.3. ■

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[2] Y. Chen, S. Fan, H.-J. Lai, H. Song, L. Sun, On dynamic coloring for planar graphs and graphs of higher genus, Discrete Appl. Math. 160 (2012) 1064–1071.
[3] S.-J. Kim, S.-J. Lee, W.-J. Park, Dynamic coloring and list dynamic coloring of planar graphs, Discrete Appl. Math. 161 (2013) 2207–2212.
[4] S.-J. Kim, W.-J. Park, List dynamic coloring of sparse graphs, in: Combinatorial Optimization and Applications, Springer Berlin Heidelberg, 2011, pp. 156–162.
[5] H.-J. Lai, B. Montgomery, H. Poon, Upper bounds of dynamic chromatic number, Ars Combin. 68 (2003) 193–201.
[6] H. Song, S. Fan, Y. Chen, L. Sun, H.-J. Lai, On r-hued coloring of K_4-minor free graphs, Discrete Math. 315 (2014) 47–52.
[7] H. Song, H.-J. Lai, J.L. Wu, On r-hued coloring of planar graphs with girth at least 6, Discrete Appl. Math. 198 (2016) 251–263.