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Locally dense supereulerian digraphs

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ABSTRACT

Let $\lambda(D)$ be the arc strong-connectivity of a digraph D, and $k > 0$ be an integer, and α, β be rational numbers. A strong digraph D is locally $(\alpha, \beta)^+$-arc-connected if $\forall v \in V(D)$, $\lambda(D[N^+(v)]) \geq \alpha|N^+(v)| + \beta$. A locally $(0, k)^+$-arc-connected digraph is also called k^+-locally-arc-connected. We show that for any integer k, a strong, k^+-locally-arc-connected digraph may not be supereulerian, and we also show that every locally $(\frac{2}{3}, 0)^+$-arc-connected strong digraph is supereulerian.

1. Introduction

We consider finite graphs and finite digraphs without loops and parallel arcs (called simple digraphs in the paper). Usually, we use G to denote a graph and D to denote a digraph. Undefined terms and notations will follow [8] for graphs and [4] for digraphs. For a digraph D, let $G(D)$ denote underlying undirected graph of D, obtained from D by erasing the directions on each arc of D. In particular, $\lambda(D)$ denotes the arc-strong connectivity of a digraph D, $c(D)$ denotes the number of components of $G(D)$. We use the notation (u, v) to denote an arc oriented from u to v in a digraph; For graphs H and G, by $H \subseteq G$ we mean that H is a subgraph of G. Similarly, for digraphs H and D, by $H \subseteq D$ we mean that H is a subdigraph of D. If $X \subseteq V(D)$ or $X \subseteq A(D)$, then $D[X]$ denotes the subdigraph induced by X. For a subdigraph S of a digraph D, and for subsets $X \subseteq A(S)$ and $Y \subseteq A(D) - A(S)$, we use $H - X + Y$ to denote $D[(A(S) - X) \cup Y]$, the subdigraph induced by the arc subset $(A(S) - X) \cup Y$.

A digraph D is strong if $\lambda(D) > 0$. For $X, Y \subseteq V(D)$, define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X, y \in Y\},$$

and

$$\partial_D^+(X) = (X, V(D) - X)_D \text{ and } \partial_D^-(X) = (V(D) - X, X)_D.$$

For any vertex $v \in V(D)$, we define the **out neighbors** of v by $N^+(v) = \{u : u \in V(D) \text{ and } (v, u) \in A(D)\}$, the **in neighbors** of v by $N^-(v) = \{u : u \in V(D) \text{ and } (u, v) \in A(D)\}$ and the neighbors of v by $N(v) = N^+(v) \cup N^-(v)$. Accordingly, $d_D^+(v) = |\partial_D^+(\{v\})|$ and $d_D^-(v) = |\partial_D^-(\{v\})|$ are the **out-degree** and the **in-degree** of v in D, respectively. Let

$$\partial_D(\{v\}) = \partial_D^+(\{v\}) \cup \partial_D^-(\{v\}), \text{ and } d_D(v) = d_D^+(v) + d_D^-(v).$$

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When the digraph D is understood from the context, we often omit the subscript D. By the definition of $\lambda(D)$ in [4], we note that for any integer $k \geq 0$ and a digraph D,

$$\lambda(D) \geq k \text{ if and only if for any } \emptyset \neq X \subset V(D), |\partial^+_D(X)| \geq k. \quad (1)$$

Following the definition in [4] (page 11), for a subdigraph H of a digraph D, an (x, y)-path P is an (H, H)**-path** if $x, y \in V(H)$ and $V(P) \cap V(H) = \{x, y\}$.

Motivated by the Chinese Postman Problem, Boesch, Suffel, and Tindell [7] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning Eulerian subgraphs, and they [7] indicated that this problem would be very difficult. Pulleyblank [20] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been lots of researches on this topic. Catlin [10] in 1992 presented the first survey on supereulerian graphs. Later Chen et al. [11] gave an update in 1995, specifically on the reduction method associated with the supereulerian problem. A recent survey on supereulerian graphs can be found in [18].

A natural problem to consider is to investigate supereulerian digraphs. A strong digraph D is **eulerian** if for any $v \in V(D)$, $d^+_D(v) = d^-_D(v)$. A digraph D is **supereulerian** if D contains a spanning eulerian subdigraph. The main problem under consideration is to characterize supereulerian digraphs. Several efforts have been made. The earlier studies were done by Gutin [13,14]. Recently, quite a few researches have been done on this topic, as seen in [1–3,5,16,17], among others. In particular, the following have been obtained.

Theorem 1.1 (*Hong et al. [16]*). *Let D be a strong simple digraph on n vertices. If $\delta^+(D) + \delta^-(D) \geq n - 4$, then either D is supereulerian, or D belongs to a class of well characterized digraphs.*

Theorem 1.2 (*Hong et al. [17]*). *Let D be a strong simple digraph on $n \geq 11$ vertices. If $\delta^+(x) + \delta^-(y) \geq n - 4$ for any pair of vertices $\{x, y\}$ such that $(x, y) \notin A(D)$, then either D is supereulerian, or D belongs to a class of well characterized digraphs.*

Theorem 1.3 (*J. Bang-Jensen and A. Maddaloni [5]*). *Let D be a strong simple digraph on n vertices. If $d(x) + d(y) \geq 2n - 3$ for any pair of non adjacent vertices x and y, then D is supereulerian.*

Following [8], we use $\alpha(H)$ and $\alpha'(H)$ to denote the independence number and maximum size of a matching of a graph H; and define $\alpha(D) = \alpha(G(D))$ and $\alpha'(D) = \alpha'(G(D))$.

Thomassen in [21] suggested that it is natural to consider if some of the theorems concerning hamiltonicity in graphs can be extended to digraphs. He considered to extend the following Chvátal–Erdös Theorem to digraphs.

Theorem 1.4 (*Chvátal and Erdös [12]*). *If $\kappa(G) \geq \alpha(G)$, then G is hamiltonian.*

There have been many studies to investigate the Chvatal–Erdos type sufficient conditions for supereulerian graphs, as seen in [15,19,22,23], among others. Thomassen [21] presented examples to show that Theorem 1.4 cannot be extended to digraphs. This motivated Bang-Jensen and S. Thomass'e to propose the following conjecture.

Conjecture 1.5 (*J. Bang-Jensen and Thomassé, [6], see also [5]*). *If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

Towards this direction, the following have been obtained.

Theorem 1.6 (*Theorem (2.4) of J. Bang-Jensen and A. Maddaloni [5]*). *Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D has a spanning subdigraph H such that for any $v \in V(H)$, $d^+_H(v) = d^-_H(v)$.*

Theorem 1.7 (*Algefari and Lai [2]*). *Let D be a digraph. If $\lambda(D) \geq \alpha'(D)$, then D is supereulerian.*

A graph G is **locally connected** if for every vertex $v \in V(G)$, the vertices adjacent to v induce a connected subgraph in G. A vertex $v \in V(D)$ is k^+**-locally-arc-connected**, (or k^-**-locally-arc-connected**, or k**-locally-arc-connected**, respectively) if $\lambda(D[N^+(v)]) \geq k$ ($\lambda(D[N^-(v)]) \geq k$, or $\lambda(D[N(v)]) \geq k$, respectively). A digraph D is k^+-locally-arc-connected, (or k^--locally-arc-connected, or k-locally-arc-connected, respectively) if every vertex of D is k^+-locally-arc-connected, (or k^--locally-arc-connected, or k-locally-arc-connected, respectively).

It is well known (for example, Corollary 1 of [9]) that every connected, locally-connected graph other than K_2 is supereulerian. The problem investigated in this paper is whether strong and locally strong digraphs will behave similarly, and how local structure in digraph will warrant the existence of a spanning eulerian subdigraph. In the next section, we shall show that for any integer $k > 0$, there exists an infinite family of strong, k^+-locally-arc-connected non-supereulerian digraphs. This family is also an infinite family of strong, k-locally-arc-connected non-supereulerian digraphs. These suggest that local connectivity may not be sufficient for supereulerian digraphs. This motivates us to seek sufficient locally dense conditions for supereulerian digraphs.

For rational numbers α and β, define a strong digraph D to be **locally** $(\alpha, \beta)^+$**-dense** if $\forall v \in V(D)$, $d_{D[N^+(v)]}(u) \geq \alpha|N^+(v)| + \beta$, $\forall u \in N^+(v)$; and D is **locally** $(\alpha, \beta)^+$**-arc-connected** if $\forall v \in V(D)$, $\lambda(D[N^+(v)]) \geq \alpha|N^+(v)| + \beta$; as well as ; D is **locally** (α, β)**-dense** if $\forall v \in V(D)$, $d_{D[N(v)]}(u) \geq \alpha|N(v)| + \beta$, $\forall u \in N(v)$; and D is **locally** (α, β)**-arc-connected** if $\forall v \in V(D)$, $\lambda(D[N(v)]) \geq \alpha|N(v)| + \beta$. In Section 3, we will prove the following main results in this paper.

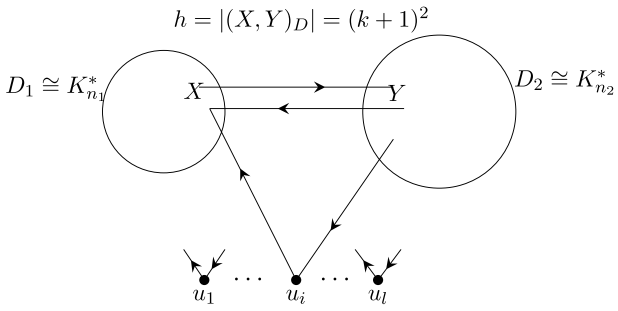

Fig. 1. The digraph $D = D(n_1, n_2, l)$, with $n_1, n_2 \geq k+2$, and $l > (k+1)^2$.

Theorem 1.8. *Every locally* $(\frac{4}{3}, \frac{-7}{3})^+$*-dense strong simple digraph is supereulerian.*

Corollary 1.9. *Every locally* $(\frac{2}{3}, 0)^+$*-arc-connected strong simple digraph is supereulerian.*

In the next section, we will present examples to show that there exist infinitely many strong, locally strong digraphs that may not be supereulerian. We will also present examples to show that for some fractional numbers α and β, there exist strong simple nonsupereulerian digraphs D satisfying the condition that for any $v \in V(D)$, it holds that $d_{D[N^+(v)]}(u) \geq \alpha|N^+(v)| + \beta - 1, \forall u \in N^+(v)$. This shows that the main result Theorem 1.8 is best possible in some sense.

2. Examples

In this section, we shall first show that for any integer $k > 0$, a strong, k^+-locally-arc-connected digraph and a strong, k-locally-arc-connected digraph may not be supereulerian. For this purpose, we shall display a necessary condition for supereulerian digraphs, established in [16].

Let D be a strong digraph and $U \subset V(D)$. Then in $D[U]$, the digraph induced by U, we can find some ditrails $P_1, \ldots, P_t$ such that $\bigcup_{i=1}^{t} V(P_i) = U$ and $A(P_i) \cap A(P_j) = \emptyset$ for any $i \neq j$. Let $\tau(U)$ be the minimum value of such t. Then $c(G(D[U])) \leq \tau(U) \leq |U|$, where $c(G(D[U]))$ is the number of components of the underlying graph of $D[U]$. For any $A \subseteq V(D) - U$, denote $B := V(D) - U - A$ and let

$$h(U, A) : = \min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + \min\{|(U, B)_D|, |(B, U)_D|\} - \tau(U), \text{ and} \quad (2)$$
$$h(U) : = \min\{h(U, A) : A \cap U = \emptyset\}.$$

In [16], Hong et al. give the following proposition, and use it to find some classes of digraphs which are not supereulerian.

Proposition 2.1 (*Proposition (2.1) of Hong et al. [16]*)**.** *If D has a spanning eulerian subdigraph, then for any $U \subset V(D)$, $h(U) \geq 0$.*

This proposition can be applied to show that there exists a family of strong and locally k^+-arc-connected non-supereulerian digraphs which is also locally k-arc-connected non-supereulerian digraphs.

Example 2.2. Let $k > 0, l > (k+1)^2$ and $n_1 \geq n_2 \geq k+2$ be integers, D_1 and D_2 be two vertex disjoint complete digraphs on n_1 and n_2 vertices, respectively, $X \subset V(D_1)$ and $Y \subset V(D_2)$ with $|X| = |Y| = k+1$ and let U be a set of independent vertices disjoint from $V(D_1) \cup V(D_2)$ with $|U| = l$. Let $\mathcal{D}(k, l)$ denote the family of digraphs such that $D \in \mathcal{D}(k, l)$ if and only if D is the digraph obtained from the disjoint union $D_1 \cup D_2 \cup U$ by adding all arcs directed from every vertex in U and D_2 to every vertex in D_1, and all arcs directed from every vertex in D_2 to every vertex in U, and then by adding $(k+1)^2$ arcs from X to Y. (See Fig. 1.) We shall justify that D is a locally k^+-arc-connected digraph below. Let $A = V(D_1)$. Then

$$h(U, A) = |\partial_D^+(A)| + |(U, V(D) - U - A)_D| - \tau(U) = (k+1)^2 + 0 - l < 0.$$

By Proposition 2.1, D is not supereulerian. D is strong because for any two vertices $v, z \in V(D)$, there exists a (v, z)-dipath.

Let $h = (k+1)^2$. We will verify our claims above that D is both a locally k^+-arc-connected digraph and a locally k-arc-connected digraph. The observations below follow from the definition of D.

Observation 2.3. *Let $V_1 = V(D_1)$ and $V_2 = V(D_2)$.*
(i) $|(V_1, V_2)_D| = (k+1)^2$ *and* $|(V_2, V_1)_D| = n_1 n_2 \geq (k+2)^2$.
(ii) $|(V_1, U)_D| = 0$ *and* $|(U, V_1)_D| = ln_1$.
(iii) $|(V_2, U)_D| = ln_2$ *and* $|(U, V_2)_D| = 0$.
(iv) If $v \in V_2$*, then* $N^+(v) = V(D) - \{v\}$.

(v) If $v \in V_1 \cup U - X$, then $N^+(v) = V_1 - \{v\}$; if $v \in X$, then $N^+(v) = V_1 \cup Y - \{v\}$.
(vi) If $v \in V_1 \cup V_2$, then $N(v) = V(D) - \{v\}$; if $v \in U$, then $N(v) = V_1 \cup V_2$.
(vii) If two vertices r and s are contained in a complete subdigraph K with $|V(K)| \geq k+1$, then K has k arc-disjoint (r, s)-dipaths.

Claim 1. *For any $v \in V(D)$, and for any two vertices $r, s \in V(D) - \{v\}$, $D - \{v\}$ has k arc-disjoint (r, s)-dipaths.*

Assume first that $r \in V_1 \cup U$. Since $D_1 \cong K^*_{n_1}$ with $n_1 \geq k+2$, it follows by Observation 2.3(vii) that $|\{r, s\} \cap V_1| \leq 1$. If $|\{r, s, v\} \cap X| \leq 1$, then since $|X| = |Y| = k+1$, there exist distinct vertices $x_1, x_2, \dots, x_k \in X - \{r, s, v\}$ and $y_1, y_2, \dots, y_k \in Y - \{v\}$ such that if $s \in Y$, then $y_k = s$. Since $|(X, Y)_D| = (k+1)^2$ and by Observation 2.3(i)–(iii), if $y_k \neq s$, then the dipaths rx_iy_is, $(1 \leq i \leq k)$, are k arc-disjoint (r, s)-dipaths in $D - \{v\}$; if $y_k = s$, then the dipaths rx_is, $(1 \leq i \leq k)$, are k arc-disjoint (r, s)-dipaths in $D - \{v\}$. Since $|(X, Y)_D| = (k+1)^2$ and by Observation 2.3(i), (iii), (vii), if $r, v \in X$, then for distinct vertices $y_1, y_2, \dots, y_k \in Y - \{s\}$, ry_is, are k arc-disjoint (r, s)-dipaths in $D - \{v\}$. Since $|V_1| \geq k+2$, there exist distinct vertices $h_1, h_2, \dots, h_k \in V_1 - \{r, s, v\}$. Thus by Observation 2.3 (ii), (vii), if $s, v \in X$, rh_is, $(1 \leq i \leq k)$, are k arc-disjoint (r, s)-dipaths in $D - \{v\}$.

Therefore, we assume that $r \in V_2$. Since $D_2 \cong K^*_{n_2}$ with $n_2 \geq k+2$, it follows by Observation 2.3(vii) that $s \in V(D) - V_2$. Since $|V_2| \geqslant k+2$, there exist distinct vertices $d_1, d_2, \dots, d_k \in V_2 - \{r, v\}$. Thus by Observation 2.3(i), the dipaths rd_is, are k arc-disjoint (r, s)-dipaths in $D - \{v\}$. This proves Claim 1.

Claim 2. *D is locally k^+-arc-connected.*

We shall show that for any vertex $v \in V(D)$, and for any two vertices $r, s \in N^+(v)$, $D[N^+(v)]$ contains k arc-disjoint (r, s)-dipaths. By Observation 2.3(vii), we may assume that

$$D[N^+(v)] \text{ has no complete subdigraphs } K \text{ with } |V(K)| \geq k+1 \text{ and } r, s \in V(K). \tag{3}$$

By Observation 2.3(v), if $v \in V_2$, then $N^+(v) = V(D) - \{v\}$. By Claim 1, there are k arc-disjoint (r, s)-dipaths in $D[N^+(v)]$. Hence we may assume that $v \in V_1 \cup U$. By Observation 2.3(v), if $v \in V_1 \cup U - X$, then $N^+(v) = V_1 - \{v\}$; if $v \in X$, then $N^+(v) = V_1 \cup Y - \{v\}$.

Therefore by (3), we may assume that $v \in X$, $N^+(v) = V_1 \cup Y - \{v\}$. As $|(X, Y)_D| = (k+1)^2$ and by Observation 2.3, $D[X \cup Y - \{v\}]$ is a complete digraph with $2k+1$ vertices. By (3), we must have $|(V_1 - X) \cap \{r, s\}| = |Y \cap \{r, s\}| = 1$ and $X \cap \{r, s\} = \emptyset$. Since $|X| = k+1$, $|(X, Y)_D| = (k+1)^2$ and since $X \cap \{r, s\} = \emptyset$, we have $r \in (V_1 - X) \cup Y$ and there exist distinct vertices $x_1, x_2, \dots, x_k \in X - \{v\}$ such that rx_is, $1 \leq i \leq k$, are k arc-disjoint (r, s)-dipaths in $D[N^+(v)]$. This proves Claim 2.

Claim 3. *D is locally k-arc-connected.*

Again we shall show that for any vertex $v \in V(D)$, and for any two vertices $r, s \in N(v)$, $D[N(v)]$ contains k arc-disjoint (r, s)-dipaths. By Observation 2.3(vii), we may assume that

$$D[N(v)] \text{ has no complete subdigraphs } K \text{ with } |V(K)| \geq k+1 \text{ and } r, s \in V(K). \tag{4}$$

If $v \in V_1 \cup V_2$, then by Observation 2.3(vi), $N(v) = V(D) - \{v\}$, and so by Claim 1, there exist k arc-disjoint (r, s)-dipaths in $D[N(v)]$. Hence we only need to show Claim 3 when $v \in U$. By Observation 2.3(vi), $N(v) = V_1 \cup V_2$.

By (4), $|V_1 \cap \{r, s\}| = |V_2 \cap \{r, s\}| = 1$. Since $|X| = |Y| = k+1$, there exist distinct vertices $x_1, x_2, \dots, x_k \in X - \{r, s\}$ and $y_1, y_2, \dots, y_k \in Y - \{r, s\}$. Since $|(X, Y)_D| = (k+1)^2$ and by Observation 2.3(i)–(iii), the dipaths rx_iy_is, $(1 \leq i \leq k)$, are k arc-disjoint (r, s)-dipaths in $D[N(v)]$. This justifies Claim 3.

Remark 2.4. By Claims 2 and 3, we conclude that any $D = D(n_1, n_2, l)$ with $n_1 \geq n_2 \geq k+2$ and $l > (k+1)^2$ is locally k^+-arc-connected and locally k-arc-connected, but it is not supereulerian.

Next, we will present an example to address the sharpness of Theorem 1.8.

Example 2.5. We will first present a building block digraph, and then we will use the building block to build an infinite family of digraphs to show that Theorem 1.8 is best possible in some sense.
(i) Let L be a digraph with $V(L) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $A(L) = \{(v_1, v_2), (v_2, v_1), (v_1, v_4), (v_3, v_2), (v_2, v_4), (v_3, v_1), (v_4, v_3), (v_3, v_6), (v_3, v_5), (v_6, v_4), (v_5, v_4), (v_5, v_6), (v_6, v_5)\}$ as depicted in Fig. 2.
(ii) Using the notation in Fig. 2, we can fix the vertex set of one of the two K^*_2's in L, $L[\{v_1, v_2\}]$ and $L[\{v_5, v_6\}]$, and define it as a distinguished pair. Let $\{a, b\}$ be a distinguished pair of L, and denote L by $L(a, b)$. Let $L_1(a', b')$ and $L_2(a'', b'')$ be two copies of $L(v_1, v_2)$. Obtain a new digraph $L_1 \oplus L_2$ from L_1 and L_2 by identifying a' and a'' to form a new vertex a, and identifying b' and b'' to form a new vertex b. See Fig. 3 for a drawing of $L_1 \oplus L_2$ with $a = v_1$ and $b = v_2$.

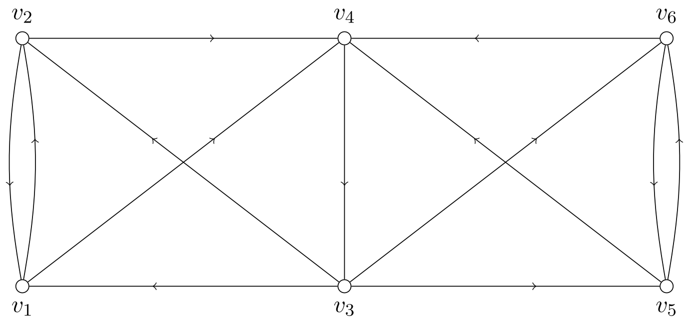

Fig. 2. The graph L in Example 2.5.

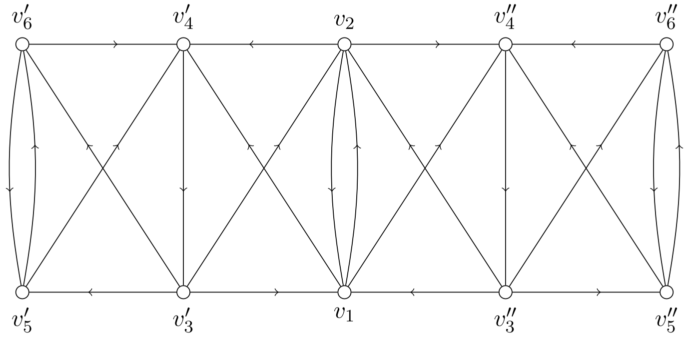

Fig. 3. The digraph $L_1 \oplus L_2$ in Example 2.5.

Observation 2.6. *We make the following observations on the digraph L defined in Example 2.5.*

(i) D is strong. This can be verified by observing that each of these subdigraphs $D[\{v_1, v_2\}]$, $D[\{v_2, v_4, v_3\}]$, $D[\{v_4, v_3, v_6\}]$ *and* $D[\{v_5, v_5\}]$ *is a directed circuit.*

(ii) D is not supereulerian. This can be seen by letting $U = \{v_1, v_2\}$ *and* $A = \{v_4\}$. *By (2), direct computation yields that* $|\partial_D^+(A)| = 1$, $|(U, B)_D| = 0$ *and* $\tau(U) = 2$. *Thus* $h(U) < 0$, *and so by Proposition 2.1, D is not supereulerian.*

(iii) For any i with $i \in \{1, 2, 4, 5, 6\}$, *direct computation shows that* $\forall u \in N^+(v_i)$, $d_{D[N^+(v_i)]}(u) \geq \frac{4}{3}|N^+(v_i)| - \frac{7}{3}$.

(iv) For $i = 3$, *we have* $|N^+(v_3)| = 4$. *For any* $z \in N^+(v_3)$, *we have* $d_{D[N^+(v_3)]}(z) = 2 = \frac{4}{3}|N^+(v_3)| - \frac{7}{3} - 1$.

Remark 2.7. By Observation 2.6, L is a strong nonsupereulerian digraph. Furthermore, for any $v \in V(D)$, it holds that $d_{D[N^+(v)]}(u) \geq \frac{4}{3}|N^+(v)| + \frac{7}{3} - 1 = \frac{13}{12}|N^+(v_3)| - \frac{7}{3}$, $\forall u \in N^+(v)$. Thus smaller value of $\frac{4}{3}$ or $\frac{7}{3}$ would not warrant the existence of a spanning ditrail in D. To construct an infinite family of such digraphs, we can use the $\oplus$ operation to build large strong but nonsupereulerian digraphs satisfying the same local density conditions. This shows that Theorem 1.8 is in some sense, best possible. However, we do not have examples to show for any sufficiently small real numbers $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exist strong non-supereulerian digraphs which are locally $(\frac{4}{3} - \epsilon_1, \frac{-7}{3})^+$-dense or locally $(\frac{4}{3}, \frac{-7}{3} - \epsilon_2)^+$-dense.

3. Proof of Theorem 1.8

We will present the proof of Theorem 1.8 in this section. Assume that

$$D \text{ is a locally } \left(\frac{4}{3}, \frac{-7}{3}\right)^+\text{-dense strong simple digraph.} \tag{5}$$

Since D is strong, D must have an eulerian subdigraph. Let S be an eulerian subdigraph of D such that among all eulerian subdigraphs of D

$$|V(S)| \text{ being maximized.} \tag{6}$$

If $|V(S)| = |V(D)|$, then S is a spanning eulerian subdigraph of D and we are done. Assume by contradiction that $|V(D)| > |V(S)| > 1$. Hence $V(D) - V(S) \neq \emptyset$. Since D is strong and S is a maximum eulerian subdigraph, there exists an (S, S)-path Q

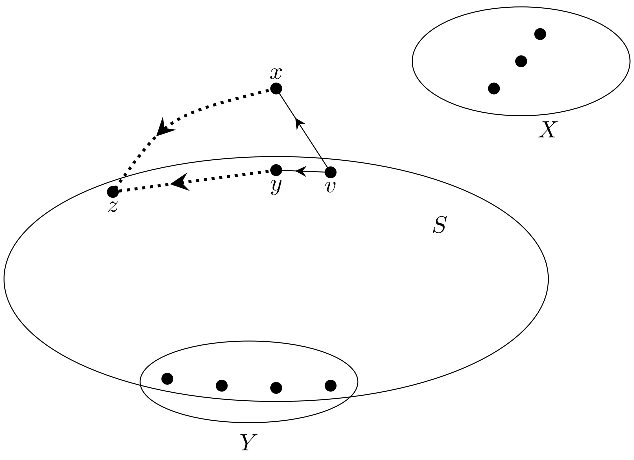

Fig. 4. Proof of Theorem 1.8.

on at least three vertices. Let Q be chosen so that:

$$\text{the length of a shortest dipath } P \text{ in } S \text{ between the endpoints of } Q \text{ is minimized.} \tag{7}$$

Assume that $V(Q) \cap V(S) = \{v, z\}$ with $(v, x) \in A(Q)$. If $(v, z) \in A(S)$, then $S - (v, z) + A(Q)$ is an eulerian subdigraph with at least one more vertex than S, contrary to (6). Therefore, there must be a vertex $y \in V(P) - \{v, z\}$ such that $(v, y) \in A(P)$. (See Fig. 4.) Let

$$X = N^+(v) - (\{x, y\} \cup V(S)) \text{ and } Y = N^+(v) \cap (V(S) - \{y\}), \tag{8}$$
$$\text{and} \quad |X| = k, |Y| = h, |X| + |Y| = m = k + h, \ \ |N^+(v)| = n_v.$$

We make the following observations.

Claim 1. *Each of the following holds.*

(i) For each $x' \in X$ with $(x, x'), (x', x) \in A(D)$, we must have $|\{(y, x'), (x', y)\} \cap A(D)| = 0$.

(ii) If $h > 0$, then for any $y' \in Y$, we cannot have $\{(y', x), (x, y')\} \subset A(D)$.

(iii) $\{(x, y), (y, x)\} \cap A(D) = \emptyset$.

We argue by contradiction to prove Claim 1(i), and assume that for some vertex $x' \in X$ with $(x, x'), (x', x) \in A(D)$, either $(y, x') \in A(D)$ or $(x', y) \in A(D)$. If $(y, x') \in A(D)$, then as $D[(A(Q) - \{v, x\}) \cup \{(y, x'), (x', x)\}]$ is a dipath of D intersecting S only at y, z, and as the distance from y to z in S is shorter than that from v to z in S, a contradiction to (7) is obtained. If $(x', y) \in A(D)$, then $S - (v, y) + \{(v, x'), (x', y)\}$ is an eulerian subdigraph of D with more vertices than S, contrary to (6). This proves (i).

We argue by contradiction to prove Claim 1(ii), and assume that for an $y' \in Y$, we have $\{(y', x), (x, y')\} \subset A(D)$. Then $S + \{(x, y'), (y', x)\}$ is an eulerian subdigraph of D with $|V(S)| + 1$ vertices, contrary to (6). This justifies Claim 1(ii).

To prove (iii), assume that either $(x, y) \in A(D)$, whence $S - (v, y) + \{(v, x), (x, y)\}$ is an eulerian subdigraph of D with more vertices than S, contrary to (6); or $(y, x) \in A(D)$, whence (7) is violated. Thus (iii) must hold.

Claim 2. *If $d_{D[N^+(v)]}(x) \geq 2k + h$, then $d_{D[N^+(v)]}(y) \leq 2h$.*

By contradiction, we assume that $d_{D[N^+(v)]}(x) \geq 2k + h$ and $d_{D[N^+(v)]}(y) \geq 2h + 1$. By Claim 1(ii), $|(x, Y)_D \cup (Y, x)_D| \leq h$. Since $d_{D[N^+(v)]}(x) \geq 2k + h$, for each $x' \in X$, we must have both $(x, x').(x'x) \in A(D)$. Since $d_{D[N^+(v)]}(y) \geq 2h + 1$ and since $|(y, Y)_D \cup (Y, y)_D| \leq 2h$, there must be an $x' \in X$ such that $|\{(x', y), (y, x')\} \cap A(D)| \geq 1$, contrary to Claim 1(i). Hence Claim 2 must hold.

Claim 3. *$m \geq 2$.*

If $m = 0$, then by (8), we have $|N^+(v)| = 2$, and so by (5), $d_{D[N^+(v)]}(x) \geq \frac{4(2)-7}{3} = \frac{1}{3}$. Thus $d_{D[N^+(v)]}(x) \geq 1$. Similarly, $d_{D[N^+(v)]}(y) \geq 1$. Therefore either (x, y) or (y, x) must be in $A(D)$, contrary to Claim 1(i), and so the case $m = 0$ is not possible.

Therefore, we must have $m = 1$. By Claim 1(iii), we may assume that $\{(x, y), (y, x)\} \cap A(D) = \emptyset$. By (8), we have $|N^+(v)| = 3$, and so by (5), $d_{D[N^+(v)]}(x) \geq \frac{4(3)-7}{3} = \frac{5}{3}$. Thus $d_{D[N^+(v)]}(x) \geq 2$. Similarly, $d_{D[N^+(v)]}(y) \geq 2$. As $m = 1$ and as $\{(x, y), (y, x)\} \cap A(D) = \emptyset$, either $X = \{x'\}$ and $Y = \emptyset$, whence $\{(y, x'), (x', y)\} \subset A(D)$, contrary to Claim 1(i); or $X = \emptyset$ and $Y = \{y'\}$, whence $\{(x, y'), (y', x)\} \subset A(D)$, contrary to Claim 1(ii). This completes the proof of Claim 3.

By Claim 3, we assume that $m \geq 2$. For some integers $s \geq 1$ and t with $t \in \{-1, 0, 1\}$, we write $m = h + k = 3s + t$. By (8), $n_v = |N^+(v)| = m + 2 = 3s + t + 2$. By (5), for each $z \in N^+(v)$ $d_{D[N^+(v)]}(z) \geq \frac{4(n_v)-7}{3} = 4s + t + \frac{t+1}{3} \geq 4s + t$.

Case 1. $s > k$.

Then for some integer $k' \geq 1$, $s = k + k'$. as $h + k = m = 3(k + k') + t$, we have $h = 2k + 3k' + t$. By (5), $d_{D[N^+(v)]}(x) \geq 4s + t = 4k + 4k' + t = 2k + h + k'$. Since $|(x, X)_D \cup (X, x)_D| \leq 2k$ and as $k' > 0$, it follows that for some $y' \in Y$, both (x, y') and (y', x) are in $A(D)$ or $\{(x, y), (y, x)\} \cap A(D) \neq \emptyset$, contrary to Claim 1(ii) or (iii). This proves that Case 1 cannot occur.

Case 2. $s = k$.

As $m = k + h$, this is equivalent to $h = 2k + t$, and so $2k + h = 2h - t$. By (8), $n_v = m + 2 = 3k + t + 2$. By (5), $d_{[N^+(v)]}(x) \geq \frac{4(3k+t+2)-7}{3} = 2k + h + \frac{t+1}{3}$. Similarly, $d_{[N^+(v)]}(y) \geq 2h - t + \frac{t+1}{3} = 2h + \frac{1-2t}{3}$. For $t \in \{-1, 0\}$, this is a violation to Claim 2. And for $t = 1$, this is a violation to Claim 1(ii) or 1(iii). This proves that Case 2 cannot occur.

Case 3. $s < k$, and either $k'' \geq 2$ or both $k'' = 1$ and $t \neq 1$.

Then for some integer $k \geq k'' \geq 1$, $s = k - k''$. as $h + k = m = 3(k - k'') + t$, we have $h = 2k - 3k'' + t$. By (5), $d_{D[N^+(v)]}(y) \geq 4s + t = 4k - 4k'' + t = (4k - 6k'' + 2t) + k'' + (k'' - t) = 2h + k'' + (k'' - t) \geq 2h + k'' + 1$; and $d_{D[N^+(v)]}(x) \geq 4s + t = 2k + (2k - 3k'' + t) - k'' = 2k + h - k''$. Since $|(y, Y)_D \cup (Y, y)_D| \leq 2h$, and since $d_{D[N^+(v)]}(y) \geq 2h + k'' + 1$, we have $|(y, X)_D \cup (X, y)_D| \geq k'' + 1$. By Claim 1(ii), $|(x, Y)_D \cup (Y, x)_D| \leq h$. As $d_{D[N^+(v)]}(x) \geq 2k + h - k''$, we have $|(x, X)_D \cup (X, x)_D| \geq 2k - k''$. This implies that there are at least $k - k''$ vertices $x' \in X$ such that $(x, x'), (x', x) \in A(D)$. As $|(y, X)_D \cup (X, y)_D| \geq k'' + 1$, there must be an $x' \in X$ with $(x, x'), (x', x) \in A(D)$ such that $|\{(y, x'), (x', y)\} \cap A(D)| \geq 1$, contrary to Claim 1(i). Hence Case 3 does not hold.

Case 4. $s = k - 1$ and $m = k + h = 3k - 2$.

By (8), $h + k = m = 3s + t = 3k - 2$, and so $h = 2k - 2$. By (5), for each $z \in N^+(v)$ $d_{D[N^+(v)]}(z) \geq 4s + t + \frac{t+1}{3} = 4k - \frac{7}{3}$. Since $d_{D[N^+(v)]}(z)$ is an integer, we have $d_{D[N^+(v)]}(z) \geq 4k - 2$. It follows that $d_{D[N^+(v)]}(x) \geq 2k + (2k - 2) = 2k + h$ and $d_{D[N^+(v)]}(y) \geq 2(2k - 2) + 2 = 2h + 2$, contrary to Claim 2. Thus Case 4 also leads to a contradiction.

As every case leads to a contradiction, this establishes the theorem. □

Proof of Corollary 1.9. Let D be a locally $(\frac{2}{3}, 0)^+$-arc-connected strong simple digraph. Then by definition of locally $(\alpha, \beta)^+$-arc-connected strong simple digraph Corollary 1.9 means that $\forall v \in V(D)$, $\lambda(D[N^+(v)]) \geq \frac{2}{3}|N^+(v)| + 0$ which by (1) implies that $\forall v \in V(D)$, $d_{[N^+(v)]}(u) \geq 2(\frac{2}{3})|N^+(v)| = \frac{4}{3}|N^+(v)| \geq \frac{4}{3}|N^+(v)| - \frac{7}{3}$, $\forall u \in N^+(v)$. It follows that D is a locally $(\frac{4}{3}, \frac{-7}{3})^+$-dense strong simple digraph. By Theorem 1.8, D is supereulerian. □

With similar arguments to those in the proof of Theorem 1.8, the following theorem can also be justified.

Theorem 3.1. *Every locally $(\frac{4}{3}, \frac{-7}{3})$-dense strong simple digraph is supereulerian.*

Sketch of Proof. Assume that

$$D \text{ is a locally } \left(\frac{4}{3}, \frac{-7}{3}\right)\text{-dense strong simple digraph.} \tag{9}$$

Since D is strong, D must have an eulerian subdigraph. Let S be an eulerian subdigraph of D such that among all eulerian subdigraphs of D

$$|V(S)| \text{ being maximized.} \tag{10}$$

If $|V(S)| = |V(D)|$, then S is a spanning eulerian subdigraph of D and we are done. Assume by contradiction that $|V(D)| > |V(S)| > 1$. Hence $V(D) - V(S) \neq \emptyset$. Since D is strong and S is a maximum eulerian subdigraph, there exists an (S, S)-path Q on at least three vertices. Let Q be chosen so that:

$$\text{the length of a shortest dipath } P \text{ in } S \text{ between the endpoints of } Q \text{ is minimized.} \tag{11}$$

Assume that $V(Q) \cap V(S) = \{v, z\}$ with $(v, x) \in A(Q)$. If $(v, z) \in A(S)$, then $S - (v, z) + A(Q)$ is an eulerian subdigraph with at least one more vertex than S, contrary to (10). Therefore, there must be a vertex $y \in V(P) - \{v, z\}$ such that $(v, y) \in A(P)$. Let $X = X^+ \cup X^-$ where $X^+ = N^+(v) - (\{x, y\} \cup V(S))$ and $X^- = N^-(v) - V(S)$. Also let $Y = Y^+ \cup Y^-$ where $Y^+ = N^+(v) \cap (V(S) - \{y\})$ and $Y^- = N^-(v) \cap V(S)$. As in the proof of Theorem 1.8, denote such that

$$|X| = k,\ |Y| = h,\ |X| + |Y| = m = k + h,\quad |N(v)| = n_v. \tag{12}$$

The rest of the proof will be identical to that of Theorem 1.8, using $N(v)$ instead of $N^+(v)$. Therefore, these similar arguments are omitted. □

The final corollary follows from Theorem 3.1.

Corollary 3.2. *Every locally $(\frac{2}{3}, 0)$-arc-connected strong simple digraph is supereulerian.*

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