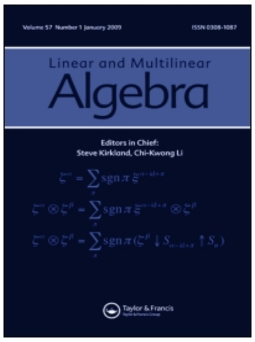

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On the permanental nullity and matching number of graphs

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ABSTRACT

For a graph G with n vertices, let $\nu(G)$ and $A(G)$ denote the matching number and adjacency matrix of G, respectively. The permanental polynomial of G is defined as $\pi(G,x) = \mathrm{per}(Ix - A(G))$. The permanental nullity of G, denoted by $\eta_{per}(G)$, is the multiplicity of the zero root of $\pi(G,x)$. In this paper, we use the Gallai–Edmonds structure theorem to derive a concise formula which reveals the relationship between the permanental nullity and the matching number of a graph. Furthermore, we prove a necessary and sufficient condition for a graph G to have $\eta_{per}(G) = 0$. As applications, we show that every unicyclic graph G on n vertices satisfies $n - 2\nu(G) - 1 \le \eta_{per}(G) \le n - 2\nu(G)$, that the permanental nullity of the line graph of a graph is either zero or one and that the permanental nullity of a factor critical graph is always zero.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with n vertices and m edges. The *neighbourhood* of vertex $v \in V(G)$ in a graph G, denoted by $N_G(v)$ (or just $N(v)$, when G is understood from the context), is the set of vertices adjacent to v. Suppose that T is a nonempty subset of $V(G)$. The *induced subgraph* of G is denoted by $G[T]$, whose vertex set is T and whose edge set is the set of those edges of G that have both ends in T. Similarly, suppose that $E'(G)$ is a nonempty subset of $E(G)$. The *edge-induced subgraph* of G whose vertex set is the set of ends of edges in $E'(G)$ and whose edge set is $E'(G)$ is denoted by $G[E'(G)]$. The *line graph* $L(G)$ of G is the graph whose vertex set is $E(G)$, where two vertices of $L(G)$ are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G.

A set M of edges in G is *a matching* if every vertex of G is incident with at most one edge in M. For two matchings M and N, the *symmetric difference* of M and N is defined to be $M \triangle N = (M \cup N) - (M \cap N)$. A vertex v is said to be *covered* (or *saturated*) by M if some edge of M is incident with v. A *maximum matching* is one which covers as many vertices as possible. In particular, a maximum matching covering all vertices of G is called a *perfect matching*. A *near-perfect* matching in a graph G is one covering all but exactly one vertex of G. The size of a maximum matching in G is called the *matching number* of G and is denoted by $\nu(G)$. A graph G is said to be *factor-critical* if $G - v$ has a perfect matching for every $v \in V(G)$.

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The *permanent* of a matrix $A = (a_{ij})_{n\times n}$ is defined as

$$\mathrm{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the sum is taken over all permutations σ of $\{1, 2, \ldots, n\}$. Valiant [1] showed that computing the permanent of a matrix is #P-complete even when restricted to (0, 1)-matrices.

For an n by n matrix A, define $\mathrm{per}(xI - A)$ to be the permanental polynomial of A. If G is a graph and $A(G)$ is the *adjacency matrix* of G, then *permanental polynomial* of G is defined to be $\pi(G, x) = \mathrm{per}(xI - A(G))$. That is, the permanental polynomial of $A(G)$. The *permanental spectrum* of G, denoted by $ps(G)$, is the collection of all roots (together with their multiplicities) of $\pi(G, x)$. The multiplicity of the zero root of $\pi(G, x)$, denoted by $\eta_{per}(G)$ is called the *permanental nullity* (per-nullity for short) of G.

It seems that the permanental polynomials of graphs were first considered by Turner [2]. Subsequently, Merris et al. [3] and Kasum et al. [4] systematically introduced permanental polynomial and its potential applications in mathematical and chemical studies, respectively. Since then, very few research papers on the permanental polynomial were published for a period of time (see [5]). This may be due to the difficulty of computing the permanent $\mathrm{per}(xI - A(G))$. However, permanental polynomials and their applications have received a lot of attention from researchers in recent years, as shown in [6–16] and the references therein.

The *spectrum* of a graph (i.e. the roots of the characteristic polynomial of a graph with their multiplicities. See [17]) encodes useful combinatorial information of the graph. The relationship between the structural properties of a graph and its spectrum has been studied extensively over the years. Nevertheless, only a few results on the permanental spectrum have been published. Brenner and Brualdi [18] proved the following: if A is an n by n matrix with nonnegative entries and spectral radius ρ, then every root of the permanental polynomial of A must be in $\{z : |z| \le \rho\}$. Merris [19] observed that if A is hermitian with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, then each real permanental root of A is in the interval $[\lambda_n, \lambda_1]$. Borowiecki [20] proved that G has $ps(G) = \{i\lambda_1, \ldots, i\lambda_n\}$ if and only if G is bipartite without cycles of length $4k$ ($k = 1, 2, \ldots$), where i is imaginary unit and $\{\lambda_1, \ldots, \lambda_n\}$ is the adjacency spectrum of G. Zhang et al. [21] proved that every graph does not have a negative real permanental root. In particular, they showed that a bipartite graph has no real permanental roots except possibly zero. Additionally, several papers have been published on graphs uniquely determined by their permanental spectra, see [22–25], among others.

In [26], Wu and Zhang introduced the per-nullity of a graph, and presented some elementary properties of per-nullity. Furthermore, they characterized the extremal graphs of order n whose per-nullities are $n-2$, $n-3$, $n-4$ and $n-5$, respectively. It is natural to consider the problem of computing the per-nullity of graphs. In this paper, we investigate the problem of computing the per-nullity of graphs, and find a relationship between per-nullity and matching number of a graph. The rest of this paper is organized as follows. In Section 2, we demonstrate some preliminaries on per-nullity of graphs. In Section 3, using the Gallai–Edmonds structure theorem, we obtain a relationship between the per-nullity and the matching number of a graph. In Section 4, we determine all graphs with zero per-nullity. In the last section, we apply our main results to several classes of graphs, including unicyclic graphs, line graphs and factor critical graphs.

2. Preliminaries

A *Sachs graph* is a simple graph such that each component is regular and has degree 1 or 2. In other words, the components are single edges and cycles.

Lemma 2.1 (Merris et al. [3]): *Let G be a graph with $\pi(G,x) = \sum_{k=0}^{n} b_k(G)x^{n-k}$. Then,*

$$b_k(G) = (-1)^k \sum_{H} 2^{c(H)}, 1 \leq k \leq n,$$

where the sum is taken over all Sachs subgraphs H of G on k vertices, and $c(H)$ is the number of cycles in H.

Let $S(G)$ be a maximum Sachs subgraph of G (i.e. $S(G)$ has the maximum number of vertices among all Sachs subgraph of G). By the definition of a Sachs graph, $S(G)$ has three possible structures: a maximum matching, union of disjoint cycles or union of some disjoint single edges and cycles.

In [26], two elementary properties of per-nullity of graphs are introduced as follows.

Lemma 2.2 (Wu and Zhang [26]): *Let G be a simple graph with n vertices.*

(i) *$\eta_{per}(G) = n$ if and only if G is an empty graph.*
(ii) *If $G_1, G_2, \ldots, G_t$ are the connected components of G, then $\eta_{per}(G) = \sum_{i=1}^{t} \eta_{per}(G_i)$.*

Lemma 2.3 (Wu and Zhang [26]): *Let G be a graph with n vertices and $S(G)$ be a maximum Sachs subgraph of G. Then, $\eta_{per}(G) = n - |V(S(G))|$.*

In the following, we present the famous Gallai–Edmonds structure theorem on matchings of graphs. Definition 2.4(i) comes from [27,28]. The notation of Definition 2.4 (i) and (ii) will be used throughout this paper.

Definition 2.4: Let G be a graph.

(i) Let $D(G)$ be the set of all vertices in G which are not saturated by at least one maximum matching of G. Define $B(G) = \{v \in (V(G) - D(G))$: there exists a $u \in B(G)$ with $uv \in E(G)\}$ and $C(G) = V(G) - (D(G) \cup B(G))$.
(ii) Let $D_0'(G)$ be the set of all isolated vertices in $G[D(G)]$ and $\mathcal{F}(G)$ be the set of components in $G[D(G)]$ each of which has order at least 3.

By definition 2.4, it can be known that $D(G)$, $B(G)$ and $C(G)$ is a vertex partition of $V(G)$. With this partition, the Gallai–Edmonds structure theorem is stated as follows.

Theorem 2.5 (Gallai–Edmonds Structure Theorem [27,28]): *Let G be a graph and let $B(G)$, $C(G)$ and $D(G)$ be the vertex partition defined above. Each of the following holds:*

(i) *The components of the subgraph induced by $D(G)$ are factor critical.*
(ii) *The subgraph induced by $C(G)$ has a perfect matching.*
(iii) *Any maximum matching M of G contains a near-perfect matching of each component of $G[D(G)]$ and a perfect matching of each component of $G[C(G)]$, and M matches all vertices of $B(G)$ with vertices in distinct components of $G[D(G)]$.*

(iv) *The size of maximum matching is* $\frac{1}{2}(|V(G)| - c(D(G)) + |B(G)|)$, *where* $c(D(G))$ *denotes the number of components of the graph induced by* $D(G)$.

By Theorem 2.5, we obtain the following lemma, which will be used later in our arguments.

Lemma 2.6: *Let* G *be a graph with* $\mathcal{F}(G) \neq \emptyset$ *and without a perfect matching. If a maximum matching* M *of* G *covers as many isolated vertices in* $G[D(G)]$ *as possible, then there must exist at least one component of* $G[D(G)]$ *in* $\mathcal{F}(G)$ *not covered by* M.

***Proof*:** Since G does not have a perfect matching, it follows from Theorem 2.5(iv) that $c(G[D(G)]) > |B(G)|$. By Theorem 2.5(iii), M contains a subset M_{BD} which matches $B(G)$ with vertices in distinct components of $D(G)$. Let $W \subseteq D(G)$ be the set of vertices covered by M_{BD}. Then, W consists of some isolate vertices in $G[D(G)]$ and some vertices of components each of which has at least 3 vertices in $G[D(G)]$. If all vertices in W are taken isolate vertices in $G[D(G)]$, or if $G[D(G)]$ does not have any isolated vertices, then the conclusion follows from the fact that $c(G[D(G)]) > |B(G)|$ and the assumption that $\mathcal{F}(G) \neq \emptyset$. Let $D_0''(G)$ be a subset of $D_0'(G)$. Define $B'(G) = \{u \in B(G) \ : \text{for some } w \in D(G), uw \in M_{BD}\}$, $B_1'(G) = \{v \in B'(G) \ : \text{for some } w \in D_0''(G) \subseteq D_0'(G), vw \in M_{BD}\}$ and $B_2'(G) = B'(G) - B_1'(G)$. Since the choice of M maximizes $|B_1'(G)|$, for every vertex $v \in B_2'(G)$, $N_G(v) \cap (D_0'(G) - D_0''(G)) = \emptyset$. It follows from the definition of $D(G)$ that there must be a vertex in $B'(G)$ adjacent to at least two components in $G[D(G)]$ that are in $\mathcal{F}(G)$. The conclusion of the lemma now follows. □

Theorem 2.7 (Chartrand et al. [29]): *Let* G *be a nontrivial connected graph. Then, the line graph* $L(G)$ *contains a perfect matching if and only if* $|E(G)| \equiv 0$ *(mod 2).*

Corollary 2.8 (Chartrand et al. [29]): *The line graph* $L(G)$ *of a nontrivial graph* G *has a perfect matching if and only if every component of* $L(G)$ *has even order.*

Let G be a connected graph with at least 3 edges and $|E(G)| \equiv 1 \pmod 2$. Then, as G has a spanning tree, G has an edge e such that $G - e$ is either connected or has two components with one being an isolated vertex. If G is 2-edge-connected, then for any $e \in E(G)$, $G - e$ is connected and has an even number of edges. With these observations, we have the following consequence of Theorem 2.7.

Theorem 2.9: *Let* G *be a connected graph with at least 3 edges and with* $|E(G)| \equiv 1$ *(mod 2). Each of the following holds.*

(i) *The line graph* $L(G)$ *contains a near-perfect matching.*
(ii) *If, in addition,* G *is 2-edge-connected, then* $L(G)$ *is factor critical.*

3. A relationship between the per-nullity and the matching number of graphs

By Lemma 2.2 and by working componentwise, it suffices to discuss connected graphs in this sections. We start with a lemma.

Lemma 3.1: *Let* G *be a factor-critical graph with* $n \geq 3$ *vertices. Each of the following holds.*

(i) *Every vertex* $v \in V(G)$ *lies in an odd cycle of* G.
(ii) *There exists an odd cycle* C *and a maximum matching* M *of* G *such that* G *is covered by* $E(C) \cup M$ *and such that the maximum degree of edge-induced subgraph* $G[E(C) \cup$

$(M - E(G[V(C)]))]$ of G is 2. (Thus, $G[E(C) \cup M]$ is a maximum Sachs subgraph of G.)

***Proof*:** Since G is a factor-critical graph, $|V(G)|$ is odd, and G is connected and not a bipartite graph. If G is an odd cycle, then Lemma 3.1 is obvious. Thus, suppose that G is not a cycle below.

Let uv be an edge of G. Since G is factor critical, $G - v$ has a perfect matching M_v. Similarly, $G - u$ has a perfect matching M_u. It follows that the symmetric difference $M_u \triangle M_v$ contains exactly one path P of even length joining u and v. By the choices of M_u and M_v, the edge uv is not in P, and so $C = P + uv$ is odd containing v. Since M_v covers all vertices of $G - v$, $E(C) \cup (M_v - E(G[V(C)]))$ is a cover of G. By the definition of M_v, no edge in $M_v - E(G[V(C)])$ is incident with an edge in C, and so the maximum degree of $G[E(C) \cup (M - E(G[V(C)]))]$ is 2. This completes the proof of the lemma. □

Lemma 3.2: *Let G be a connected graph with n vertices and with the size of a maximum matching being $\nu(G)$. The following are equivalent.*

(i) $\eta_{per}(G) = n - 2\nu(G)$.
(ii) *Either G has a perfect matching or $E(G[D(G)]) = \emptyset$.*

***Proof*:** Assume (i) to prove (ii). By Lemma 2.3, the equality $\eta_{per}(G) = n - 2\nu(G)$ implies that a maximum matching of G is a maximum Sachs subgraph. By the definition of $D(G)$, we observe that if $D(G) = \emptyset$, then G has a perfect matching, and so (ii) holds. Hence, we assume that $D(G) \neq \emptyset$. Suppose that there exists at least one component of $G[D(G)]$ having at least 3 vertices. By Lemma 2.6, there must be at least one component of $G[D(G)]$ in $\mathcal{F}(G)$ not covered by M. Let $\mathcal{F}_M(G)$ denote all such components. By Lemma 3.1, for each $L \in \mathcal{F}_M(G)$, there exists an odd cycle C_L and a subset $M_L \subset M \cap E(L)$, such that $E(C_L) \cup M_L$ covers $V(L)$ and such that the maximum degree of $L[E(C_L) \cup M_L]$ is at most 2. Thus, $H = G[\cup_{L \in \mathcal{F}_M(G)}(E(C_L) \cup M_L)]$ is a Sachs graph H such that $|V(H)|$ is more than the number of vertices in G covered by the maximum matching M. This implies that any maximum matching of G is not a maximum Sachs subgraph, contrary to the fact that any maximum matching must also be a maximum Sachs subgraph. Therefore, $\mathcal{F}(G) = \emptyset$ and so $E(G[D(G)]) = \emptyset$.

We now assume (ii) to prove (i). If G has a perfect matching, then the perfect matching is a maximum Sachs subgraph of G. By Lemma 2.3, $\eta_{per}(G) = n - 2\nu(G)$. Suppose that G does not have a perfect matching and $E(G[D(G)]) = \emptyset$. Since every maximum matching of G is a maximum Sachs subgraph of G, it follows by Lemma 2.3 that $\eta_{per}(G) = n - 2\nu(G)$. □

Definition 3.3: For a maximum matching M of G such that

$$\text{the number of isolated vertices in } G[D(G)] \text{ covered by } M \text{ is maximized,} \tag{1}$$

define $M(G)$ to be the number of components of order at least 3 in $G[D(G)]$ each of which has just a vertex not covered by M.

By Theorem 2.5(i), every graph in $\mathcal{F}(G)$ is factor critical. By Lemma 3.2, if $\mathcal{F}(G) \neq \emptyset$, then $\eta_{per}(G) < n - 2\nu(G)$. The next lemma describes the per-nullity of the graphs with $\mathcal{F}(G) \neq \emptyset$.

Lemma 3.4: *Let G be a connected graph with n vertices and without a perfect matching. If $\mathcal{F}(G) \neq \emptyset$, then*

$$\eta_{per}(G) = n - 2\nu(G) - M(G).$$

***Proof*:** Let M be a maximum matching of G satisfying (1). Since G does not have a perfect matching, by Theorem 2.5, $c(D(G)) > |B(G)|$. Then, there exists at least one $H \in \mathcal{F}(G)$ such that H has just a vertex not covered by M. By Lemma 3.1 every $H \in \mathcal{F}(G)$ has an odd cycle C_H such that $E(E(C_H)) \cup (M - E(G[V(C_H)]))]$ is a cover of H. It follows that G has a maximum Sachs subgraph $S(G)$ consisting of disjoint odd cycles $\{C_H : H \in \mathcal{F}(G)\}$, and a subset of M. It is routine to verify that $|S(G)| = 2\nu(G) + M(G)$, and so by Lemma 2.3, we have $\eta_{per}(G) = n - 2\nu(G) - M(G)$. □

By applying Lemmas 3.2 and 3.4, we obtain the main result of this section. Recall that $D(G)$ and $M(G)$ are defined in Definitions 2.4 and 3.3, for a given maximum matching M of G.

Theorem 3.5: *Let G be a connected graph with n vertices, and let M be a maximum matching of G satisfying (1). Then,*

$$\eta_{per}(G) = \begin{cases} n - 2\nu(G) & \text{if } G \text{ has a perfect matching or } \mathcal{F}(G) = \emptyset, \\ n - 2\nu(G) - M(G) & \text{otherwise.} \end{cases}$$

4. The graphs with zero per-nullity

For a simple graph G on n vertices, it is known that $0 \leq \eta_{per}(G) \leq n - 2$. In this section, we will characterize the graphs with zero per-nullity. Note that by Lemma 2.3,

$$\eta_{per}(G) = 0 \text{ if and only if } G \text{ has a spanning Sachs subgraph.} \tag{2}$$

Theorem 4.1: *Let $n \geq 2$ be an integer, G be a connected graph on n vertices. Then, $\eta_{per}(G) = 0$ if and only if one of the following holds:*

(i) *G has a perfect matching, or*
(ii) *$G[D(G)]$ has no isolated vertices, or*
(iii) *$G[D(G)]$ has isolated vertices and G has a maximum matching covering every isolated vertices of $G[D(G)]$.*

***Proof*:** Assume first that G satisfies one of (i), (ii) and (iii). We are to show that $\eta_{per}(G) = 0$.

If (i) holds, then by Theorem 3.5, $\eta_{per}(G) = 0$. Hence, we may assume that G has no perfect matchings, and so $|D(G)| > 0$.

Suppose (ii) holds. Then, $|\mathcal{F}(G)| = c(D(G))$. By Lemma 3.1, each $H \in \mathcal{F}(G)$ has a Sachs subgraph $S(H)$ to cover $V(H)$. Let M be a perfect matching of $G[C(G)]$. Then, $S(G) = M \cup (\cup_{H \in \mathcal{F}(G)} E(S(H)))$ is a spanning Sachs subgraph of G, and so by (2), $\eta_{per}(G) = n - |V(S(G))| = 0$.

Finally, we assume that G has a maximum matching M covering every isolated vertices of $G[D(G)]$. By Lemma 3.1, a Sachs subgraph exists to cover every graph in $\mathcal{F}(G)$. Edges in these Sachs subgraphs in the graphs of $\mathcal{F}(G)$ together with a subset of M induce a spanning Sachs subgraph of G. By (2), $\eta_{per}(G) = n - |V(S(G))| = 0$.

Conversely, we assume that $\eta_{per}(G) = 0$ to show (i) or (ii) or (iii) must occur. Choose a maximum matching M satisfying (1). By Theorem 3.5, if $M(G) = 0$, then the assumption $\eta_{per}(G) = 0$ leads to $|V(G)| = 2\nu(G)$. In this case, G has a perfect matching, and so (i) follows. Hence, we assume that $M(G) > 0$. By Lemma 3.4, $|V(G)| = 2\nu(G) + M(G)$. This implies that either all components of order 1 in $G[D(G)]$ are covered by M, whence (ii) holds; or every component of $G[D(G)]$ is in $\mathcal{F}(G)$, whence (iii) follows. This completes the proof of the theorem. □

5. Some applications

In this section, we determine the per-nullity of some classes of graphs as applications of Theorems 3.5 and 4.1.

An *unicyclic* graph is a connected graph with equal number of vertices and edges. The theorem below determines the per-nullity of unicyclic graphs.

Theorem 5.1: *Let G be an unicyclic graph with n vertices and the unique cycle in G is denoted by C_ℓ. Then,*

$$\eta_{per}(G) = \begin{cases} n - 2\nu(G) - 1 & \text{if } \ell \text{ is odd and } \nu(G) = \frac{\ell-1}{2} + \nu(G - C_\ell), \\ n - 2\nu(G) & \text{otherwise.} \end{cases}$$

Proof: Since C_ℓ is the unique cycle in G, it is routine to see that only $C_\ell \in \mathcal{F}(G)$ is a factor-critical component. By Theorem 3.5, we have $n - 2\nu(G) - 1 \le \eta_{per}(G) \le n - 2\nu(G)$.

If $\eta_{per}(G) = n - 2\nu(G) - 1$, then by Theorem 3.5, there exists a maximum matching M of G satisfying (1), and $|\mathcal{F}(G)| = 1$. Since G is an unicyclic graph, the factor-critical component of G must be C_ℓ. This implies that C_ℓ is odd. By (*iii*) of Theorem 2.5, we have $\nu(G) = \frac{\ell-1}{2} + \nu(G - C_\ell)$.

Assume that C_ℓ is odd and $\nu(G) = \frac{\ell-1}{2} + \nu(G - C_\ell)$. Then, C_ℓ is factor critical, and there exists a maximum matching covering all vertices of C_ℓ excepting a vertex. It follows from Theorem 3.5 that $\eta_{per}(G) = n - 2\nu(G) - 1$.

By Theorem 3.5, it is routine to verify that in this case, $\eta_{per}(G) = n - 2\nu(G)$ if and only if G has a perfect matching or if $\mathcal{F}(G) = \emptyset$, The theorem now follows. □

Theorem 5.2: *Let G be an unicyclic graph with a unique cycle C. Then, $\eta_{per}(G) = 0$ if and only if G is an odd cycle, G has a perfect matching or $G - V(C)$ has a perfect matching.*

Proof: By (i) and (ii) of Theorem 4.1, it is routine to verify that if G is an odd cycle or G has a perfect matching, then $\eta_{per}(G) = 0$. Thus, we assume that G is not an odd cycle and G does not have a perfect matching, and that $G - V(C)$ has a perfect matching M_C. Then, $E(C) \cup M_C$ is a spanning Schas subgraph of G, and so by (2), $\eta_{per}(G) = 0$.

Conversely, assume that $\eta_{per}(G) = 0$, and that G is not an odd cycle and G does not have a perfect matching. We are to show that $G - V(C)$ has a perfect matching. By contradiction, suppose that $G - V(C)$ does not have a perfect matching. By Theorem 4.1, (ii) or (iii) of Theorem 4.1 must hold. Since G is an unicyclic graph, the cycle C of G must be the only factor-critical component order at least 3 in $G[D(G)]$. Hence, $|V(C)|$ is odd. If Theorem 4.1 (ii) holds, then G must be an odd cycle, contrary to the assumption that G is not an odd cycle. Hence, Theorem 4.1 (iii) must hold. By Theorem 2.5 (iii), $G - V(C)$ has a perfect matching. This completes the proof of the theorem. □

By Theorems 2.7, 2.9 and 3.5, we obtain the following results.

Theorem 5.3: *Let $L(G)$ be the line graph of G. Then, the per-nullity of $L(G)$ equals zero or one.*

Theorem 5.4: *Let G be a factor-critical graph on $n(n \geq 2)$ vertices. Then, $\eta_{per}(G) = 0$.*

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