Discrete Mathematics 341 (2018) 400–404

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

A property on reinforcing edge-disjoint spanning hypertrees in uniform hypergraphs

Xiaofeng Gu [a,*], Hong-Jian Lai [b]

[a] Department of Mathematics, University of West Georgia, Carrollton, GA 30118, USA
[b] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

ARTICLE INFO

Article history:
Received 26 August 2015
Received in revised form 1 September 2017
Accepted 5 September 2017
Available online 4 October 2017

Keywords:
Spanning hypertrees
Partition connectivity
Uniformly dense

ABSTRACT

Suppose that H is a simple uniform hypergraph satisfying $|E(H)| = k(|V(H)| - 1)$. A k-partition $\pi = (X_1, X_2, \ldots, X_k)$ of $E(H)$ such that $|X_i| = |V(H)| - 1$ for $1 \le i \le k$ is a uniform k-partition. Let $P_k(H)$ be the collection of all uniform k-partitions of $E(H)$ and define $\varepsilon(\pi) = \sum_{i=1}^{k} c(H(X_i)) - k$, where $c(H)$ denotes the number of maximal partition-connected sub-hypergraphs of H. Let $\varepsilon(H) = \min_{\pi \in P_k(H)} \varepsilon(\pi)$. Then $\varepsilon(H) \ge 0$ with equality holds if and only if H is a union of k edge-disjoint spanning hypertrees. The parameter $\varepsilon(H)$ is used to measure how close H is being from a union of k edge-disjoint spanning hypertrees.

We prove that if H is a simple uniform hypergraph with $|E(H)| = k(|V(H)| - 1)$ and $\varepsilon(H) > 0$, then there exist $e \in E(H)$ and $e' \in E(H^c)$ such that $\varepsilon(H - e + e') < \varepsilon(H)$. This generalizes a former result, which settles a conjecture of Payan. The result iteratively defines a finite ε-decreasing sequence of uniform hypergraphs $H_0, H_1, H_2, \ldots, H_m$ such that $H_0 = H$, H_m is the union of k edge-disjoint spanning hypertrees, and such that two consecutive hypergraphs in the sequence differ by exactly one hyperedge.

1. Introduction

We consider finite graphs and finite hypergraphs. Definitions will be introduced in Section 2. Throughout the paper, let $k \ge 1$ be an integer, H denotes a hypergraph, $c(H)$ denotes the number of maximal partition-connected sub-hypergraphs of H, and $\omega(H)$ denotes the number of connected components of H. By definition, for a graph G, partition-connectedness is equivalent to connectedness, and so $c(G) = \omega(G)$. For a hypergraph H, as mentioned in [2], partition-connectedness is a stronger property than connectedness and so $\omega(H)$ and $c(H)$ are different in general. For $X \subseteq E(H)$, $H(X)$ denotes the spanning sub-hypergraph of H with edge set X, whereas $H[X]$ denotes the sub-hypergraph of H induced by X. If $H = (V, E)$ is an r-uniform hypergraph, then the **complement** of H, denoted by H^c, is an r-uniform hypergraph with $V(H^c) = V(H)$ and $E(H^c) = V^{[r]} - E(H)$.

In [7], Payan considered the following problem. Let G be a connected simple graph on $n \ge 2$ vertices and $k(n-1)$ edges. Payan introduced an integral function $\varepsilon(G)$ to measure how the graph G is closed to having k edge-disjoint spanning trees in such a way that G has k edge-disjoint spanning tree if and only if $\varepsilon(G) = 0$. Payan asked the question whether it is always possible to make a finite number of edge exchanges between edges in G and edges not in G so that the corresponding values of ε will be strictly decreasing until it becomes zero. Payan [7] conjectured that the problem has an affirmative answer (confirmed in [4]). In this paper, we study the corresponding problem in hypergraphs.

* Corresponding author.
E-mail addresses: xgu@westga.edu (X. Gu), hjlai@math.wvu.edu (H.-J. Lai).

http://dx.doi.org/10.1016/j.disc.2017.09.007

Suppose that H is a simple uniform hypergraph satisfying $|E(H)| = k(|V(H)| - 1)$. A k-partition $\pi = (X_1, X_2, \ldots, X_k)$ of $E(H)$ such that $|X_i| = |V(H)| - 1$ for $1 \le i \le k$ is called a **uniform k-partition**. Let $P_k(H)$ be the collection of all uniform k-partitions of $E(H)$. We define

$$\varepsilon(\pi) = \sum_{i=1}^{k} c(H(X_i)) - k,$$

and

$$\varepsilon(H) = \min_{\pi \in P_k(H)} \varepsilon(\pi).$$

By definition, $\varepsilon(H) \ge 0$. By Corollary 2.6 of [2] or Theorem 2.2(i), $\varepsilon(H) = 0$ if and only if for every $1 \le i \le k$, $H(X_i)$ is a spanning hypertree of H. Thus $\varepsilon(H) = 0$ if and only if H has k edge-disjoint spanning hypertrees.

The following result was conjectured by Payan [7] and proved in [4].

Theorem 1.1 ([4]). *If G is a simple graph with $|E(G)| = k(|V(G)| - 1)$ and $\varepsilon(G) > 0$, then there exist $e \in E(G)$ and $e' \in E(G^c)$ such that $\varepsilon(G - e + e') < \varepsilon(G)$.*

Note that a simple graph is a 2-uniform hypergraph. The main purpose of this note is to extend Theorem 1.1 to all uniform hypergraphs.

Theorem 1.2. *If H is a simple uniform hypergraph with $|E(H)| = k(|V(H)| - 1)$ and $\varepsilon(H) > 0$, then there exist $e \in E(H)$ and $e' \in E(H^c)$ such that $\varepsilon(H - e + e') < \varepsilon(H)$.*

Remark. (1) The parameter $\varepsilon(H)$, first introduced by Payan in [7] for graphs, can be considered as a measurement that how close H is from being an edge-disjoint union of k spanning hypertrees. Theorem 1.2 iteratively defines a finite ε-decreasing sequence of uniform hypergraphs $H_0, H_1, H_2, \ldots, H_m$ such that $H_0 = H$, H_m is the union of k edge-disjoint spanning hypertrees, and such that any two consecutive hypergraphs in the sequence differ by exactly one hyperedge.
(2) This problem is related to connectivity augmentation problems for a network (modeled as a graph or hypergraph). The traditional connectivity augmentation problem is, adding some edges to increase the connectivity (or edge connectivity, partition connectivity, etc.) of a network. Here a kind of "dynamic augmentation" is considered, i.e.,

- The number of edges in the network does not change.
- In each stage, one edge is deleted and another edge is added from outside, where the two edges are called an edge pair.
- In each stage, partition connectivity augmentation happens, which is so-called "dynamic augmentation".

In this paper, the existence of such edge pairs to augment partition connectivity of a uniform hypergraph is confirmed. It is still interesting to design algorithms to locate those edge pairs.

2. Preliminaries

A **hypergraph** H is a pair (V, E) where V is the vertex set of H and E is a collection of not necessarily distinct nonempty subsets of V, called **hyperedges** or simply **edges** of H. A **loop** is a hyperedge that consists of a single vertex. A hypergraph H is **nontrivial** if $E(H) \neq \emptyset$. A hypergraph H is **simple** if for any $e_1, e_2 \in E(H)$, $e_1 \not\subseteq e_2$. For an integer $r > 0$, and a set V, let $V^{[r]}$ denote the family of all r-subsets of V. A simple hypergraph $H = (V, E)$ is r-uniform if $E \subseteq V^{[r]}$. If $H = (V, E)$ is an r-uniform hypergraph, then the **complement** of H, denoted by H^c, is an r-uniform hypergraph with $V(H^c) = V(H)$ and $E(H^c) = V^{[r]} - E(H)$.

If $W \subseteq V(H)$, the hypergraph (W, E_W), where $E_W = \{e \in E(H) : e \subseteq W\}$ is a **sub-hypergraph induced by the vertex subset** W, and is denoted by $H[W]$. If $X \subseteq E(H)$ and $V_X = \cup_{e \in X} e$, then (V_X, X) is defined as **the sub-hypergraph induced by the edge subset** X and is denoted by $H[X]$.

A hypergraph H is **connected** if there is a hyperedge intersecting both W and $V - W$ for every non-empty proper subset W of $V(H)$. A **connected component** of a hypergraph H is a maximal connected sub-hypergraph of H. A hypergraph H is k-**partition-connected** if $\|P\| \ge k(|P| - 1)$ for every partition P of $V(H)$, where $|P|$ denotes the number of classes in P and $\|P\|$ denotes the number of edges intersecting at least two classes of P. Equivalently, H is k-partition-connected if, for any subset $X \subseteq E(H)$, $|X| \ge k(\omega(H - X) - 1)$. A 1-partition-connected hypergraph is also referred as a **partition-connected** hypergraph. It follows from definition that a graph is partition-connected if and only if it is connected. In general, partition-connected hypergraphs must be connected, but a connected hypergraph may not be partition-connected. The **partition connectivity** of H is the maximum k such that H is k-partition-connected.

A hypergraph H is a **hyperforest** if for every nonempty subset $U \subseteq V(H)$, $|E(H[U])| \le |U| - 1$. A hyperforest T is called a **hypertree** if $|E(T)| = |V(T)| - 1$. By a **hypercircuit**, we mean a hypergraph C with $|E(C)| = |V(C)|$ but $|X| < |V(C[X])|$ for any proper subset $X \subset E(C)$. For a hypergraph H, let $\tau(H)$ be the maximum number of edge-disjoint spanning hypertrees in H and $a(H)$ be the minimum number of edge-disjoint hyperforests whose union is $E(H)$. For a graph G, $\tau(G)$ is the spanning tree packing number of G and $a(G)$ is the arboricity of G.

The following theorem of Nash-Williams and Tutte shows that the k-partition-connectedness of a graph G is equivalent to the property that G has k edge-disjoint spanning trees. Nash-Williams then published a dual theorem, characterizing graphs that can be decomposed to at most k forests.

Theorem 2.1. *Let G be a graph.*
(i) (Nash-Williams [5], Tutte [8]). *$\tau(G) \geq k$ if and only if for any $X \subseteq E(G)$, $|X| \geq k(\omega(G - X) - 1)$.*
(ii) (Nash-Williams [6]). *$a(G) \leq k$ if and only if for any subgraph S, $|E(S)| \leq k(|V(S)| - 1)$.*

Frank, Király and Kriesell [2] extended both results to hypergraphs.

Theorem 2.2 (*Frank, Király and Kriesell [2]*). *Let H be a hypergraph.*
(i) *$\tau(H) \geq k$ if and only if for every $X \subseteq E(H)$, $|X| \geq k(\omega(H - X) - 1)$ (or, equivalently, H is k-partition-connected).*
(ii) *$a(H) \leq k$ if and only if for any sub-hypergraph S, $|E(S)| \leq k(|V(S)| - 1)$.*

By Theorem 2.2(i), $\tau(H)$ is the partition connectivity of H and a hypertree is a minimal partition-connected hypergraph.

Let e be a hyperedge in a hypergraph H (notice that e is also a subset of $V(H)$). By H/e we denote the hypergraph obtained from H by **contracting** the hyperedge e into a new vertex v_0 and by removing resulting loops if there are any. That is, $V(H/e) = (V(H) - e) \cup \{v_0\}$ and a hyperedge $e' \in E(H/e)$ if and only if either $e' = e''$ for some $e'' \in E(H)$ with $e'' \cap e = \emptyset$ or $e' = (e'' - e) \cup \{v_0\}$ for some $e'' \in E(H) \setminus \{e\}$ with $e'' \cap e \neq \emptyset$. If $X \subseteq E(H)$, then H/X is a hypergraph obtained from H by contracting all hyperedges in X. If S is a sub-hypergraph of H, then H/S denotes $H/E(S)$.

For any nonempty subset $X \subseteq E(H)$, the **density** of X is defined to be

$$d_H(X) = \frac{|X|}{|V(H[X])| - \omega(H[X])}.$$

We often use $d(H)$ for $d_H(E(H))$. Following [1,3], the **strength** $\eta(H)$ and the **fractional arboricity** $\gamma(H)$ of a nontrivial hypergraph H are defined, respectively, as

$$\eta(H) = \min\left\{\frac{|X|}{\omega(H - X) - \omega(H)}\right\} \text{ and } \gamma(H) = \max\left\{d(H[X])\right\},$$

where the minimum and maximum are taken over all edge subsets $X \subseteq E(H)$ so that the denominators are nonzero. It is mentioned in [3] that, Theorem 2.2 shows that for a connected hypergraph H, $\tau(H) \geq k$ if and only if $\eta(H) \geq k$; and $a(H) \leq k$ if and only if $\gamma(H) \leq k$, which gives

$$\tau(H) = \lfloor \eta(H) \rfloor \text{ and } \alpha(H) = \lceil \gamma(H) \rceil \text{ for a connected hypergraph } H. \tag{1}$$

Remark. There is also an equivalent definition for $\eta(H)$, as below

$$\eta(H) = \min\left\{\frac{|E(H) - X|}{|V(H/X)| - \omega(H)}\right\}. \tag{2}$$

Proof of the Remark. Let $X' = E(H) - X$. Each connected component of $H - X'$ corresponds to a vertex of H/X, and thus $\omega(H - X') = |V(H/X)|$. Hence

$$\min\left\{\frac{|E(H) - X|}{|V(H/X)| - \omega(H)}\right\} = \min\left\{\frac{|X'|}{\omega(H - X') - \omega(H)}\right\},$$

where the minimums are taken over all edge subsets $X \subseteq E(H)$ (or equivalently $X' \subseteq E(H)$) so that the denominators are nonzero, which finishes the proof of the remark.

It follows by definitions that for any nontrivial hypergraph H, $\eta(H) \leq d(H) \leq \gamma(H)$. A hypergraph H is **uniformly dense** if $d(H) = \gamma(H)$. We have the following property.

Theorem 2.3 (*[3]*). *For a hypergraph H, the following are equivalent.*
(i) *$\eta(H) = \gamma(H)$.*
(ii) *$\eta(H) = d(H)$.*
(iii) *$d(H) = \gamma(H)$.*

Theorem 2.3 generalizes the corresponding results in [1] from graphs to hypergraphs. Let $\mathcal{T}_k$ be the family of all k-partition-connected hypergraphs. By Theorem 2.2(i), $\mathcal{T}_k$ is the family of all hypergraphs each of which contains k edge-disjoint spanning hypertrees.

Proposition 2.4 (*[3]*). *Each of the following statements holds.*
(C1) *$\mathcal{T}_k \neq \emptyset$.*
(C2) *If $e \in E(H)$ and $H \in \mathcal{T}_k$, then $H/e \in \mathcal{T}_k$.*
(C3) *If for some $S \subset E(H)$, both $S, H/S \in \mathcal{T}_k$, then $H \in \mathcal{T}_k$.*

Lemma 2.5. *Let H be a hypergraph with $d(H) \geq k > \eta(H)$. Then H has a connected sub-hypergraph S such that $\eta(S) > k$. In particular, $d(S) > k$ and $\tau(S) \geq k$.*

Proof. As $d(H) \geq k > \eta(H)$, by Theorem 2.3, $\gamma(H) > k$. By the definition of $\gamma(H)$, there exists a connected sub-hypergraph S such that $d(S) = \gamma(H) > k$. (We can always choose S to be connected, for otherwise if S contains s connected components S_i, $1 \leq i \leq s$, we claim $d(S_i) = \gamma(H)$. First, by definition of $\gamma(H)$, $d(S_i) \leq d(S) = \gamma(H)$. Thus $\frac{|E(S_i)|}{|V(S_i)|-1} \leq d(S)$, which implies $|E(S_i)| \leq d(S)(|V(S_i)| - 1)$. If one of $d(S_i)$ is strictly less than $d(S)$, then $\sum_{1\leq i\leq s}|E(S_i)| < \sum_{1\leq i\leq s}d(S)(|V(S_i)| - 1) = d(S)(\sum_{1\leq i\leq s}|V(S_i)| - s)$. Thus $d(S) = \frac{\sum|E(S_i)|}{\sum|V(S_i)|-s} < d(S)$, a contradiction. Thus $d(S_i) = d(S) = \gamma(H)$. In this case, we can choose S to be any connected component.)

Thus $d(S) \leq \gamma(S) \leq \gamma(H) = d(S)$, which implies that $d(S) = \gamma(S)$. By Theorem 2.3, $\eta(S) = d(S) = \gamma(S) > k$. In particular, $\tau(S) \geq k$. □

Lemma 2.6. *Let C be a hypercircuit and $e \in E(C)$. Then $C - e$ is a hypertree (and thus partition-connected).*

Proof. By the definition of a hypercircuit, for every nonempty subset $U \subseteq V(C) = V(C - e)$, $|E(C[U])| \leq |U| - 1$. Since $|E(C)| = |V(C)|$, we have $|E(C - e)| = |V(C)| - 1$. By definition, $C - e$ is a hypertree. □

3. The proof of Theorem 1.2

Let $P'_k(H)$ be the collection of all k-partitions of $E(H)$, and define $\varepsilon'(H) = \min_{\pi\in P'_k(H)}\varepsilon(\pi)$.

Lemma 3.1. *For any uniform hypergraph H with $|E(H)| = k(|V(H)| - 1)$, $\varepsilon(H) = \varepsilon'(H)$.*

Proof. Since $P_k(H) \subseteq P'_k(H)$, it follows from definition that $\varepsilon(H) \geq \varepsilon'(H)$. Thus it suffices to show that $\varepsilon(H) \leq \varepsilon'(H)$.

Let $n = |V(H)|$. For each $\pi = (X_1, X_2, \ldots, X_k) \in P'_k(H)$, define

$$\varphi(\pi) = \sum_{i=1}^{k} \max\{|X_i| - n + 1, 0\}.$$

Thus $\varphi(\pi) \geq 0$, and $\varphi(\pi) = 0$ if and only if $\pi \in P_k(H)$.

Choose a $\pi = (X_1, X_2, \ldots, X_k) \in P'_k(H)$ such that $\varepsilon(\pi) = \varepsilon'(H)$ and such that $\varphi(\pi)$ is minimized. We claim that $\pi \in P_k(H)$. Assume that $\varphi(\pi) > 0$, and without loss of generality, we may assume that $|X_1| > n - 1$ and $|X_2| < n - 1$. Thus $H(X_1)$ must contain a hypercircuit C, and let $e \in E(C)$. Define $\pi' = (X'_1, X'_2, \ldots, X'_k)$ as below.

$$X'_i := \begin{cases} X_1 - \{e\}, & \text{if } i = 1, \\ X_2 \cup \{e\}, & \text{if } i = 2, \\ X_i, & \text{if } i > 2. \end{cases}$$

Then $\pi' \in P'_k(H)$. By Lemma 2.6, the removal of an edge in a hypercircuit does not affect the partition-connectedness, and so we have $c(H(X_1)) = c(H(X'_1))$. We also have $c(H(X_2)) \geq c(H(X'_2))$. Thus $\varepsilon(\pi') \leq \varepsilon(\pi) = \varepsilon'(H)$, but $\varphi(\pi') \leq \varphi(\pi) - 1$, contrary to the choice of π. Thus $\varphi(\pi) = 0$, and so $\pi \in P_k(H)$. Hence $\varepsilon(H) \leq \varepsilon(\pi) = \varepsilon'(H)$, which completes the proof. □

In the rest of the paper, we prove Theorem 1.2. Lemma 3.1 suggests that by using $\varepsilon'(H)$, we do not have to restrict our discussion to uniform k-partitions, and so in the proof arguments below, all k-partitions may not be uniform.

Proof of Theorem 1.2. Throughout the proof, we assume that H is an r-uniform hypergraph for some integer $r \geq 2$. Since $\varepsilon(H) > 0$, H does not have k edge-disjoint spanning hypertrees, and so by (1), $\eta(H) < k$. Since $|E(H)| = k(|V(H)| - 1)$, we have $d(H) = k$. By Lemma 2.5, there exists a maximal connected sub-hypergraph S with $d(S) > k$ and $\tau(S) \geq k$. In other words, $|E(S)| > k(|V(S)| - 1)$ and S has k edge-disjoint spanning hypertrees.

Claim 1. *For any $v \in V(H) - V(S)$, there exist $w_1, \ldots, w_{r-1} \in V(S)$ such that $\{w_1, \ldots, w_{r-1}, v\} \notin E(H)$.*

Proof of Claim 1. If not, then v is adjacent to every $(r - 1)$-subset of $V(S)$ in H. Let $|V(S)| = s$. Then there are at least $\binom{s}{r-1}$ hyperedges joining S and v. Since S is a simple r-uniform hypergraph, $\binom{s}{r} \geq |E(S)| > k(s - 1)$. Then $\binom{s}{r-1} = \binom{s}{r} \cdot \frac{r}{s-r+1} > k \cdot \frac{r(s-1)}{s-r+1} > k$. Thus $|E(H[V(S) \cup \{v\}])| > |E(S)| + k$, and so

$$d(H[V(S) \cup \{v\}]) = \frac{|E(H[V(S) \cup \{v\}])|}{|V(S) \cup \{v\}| - 1} > \frac{|E(S)| + k}{|V(S)|} > \frac{k(|V(S)| - 1) + k}{|V(S)|} = k.$$

Since $H[V(S) \cup \{v\}]/S$ is a multigraph with two vertices and at least k edges, $H[V(S) \cup \{v\}]/S$ is k-partition-connected. As S is k-partition-connected, it follows by Proposition 2.4(C3) that $H[V(S) \cup \{v\}]$ is k-partition-connected. Thus $\tau(H[V(S) \cup \{v\}]) \geq k$, contrary to the maximality of S. This proves the claim.

Since S has k edge-disjoint spanning hypertrees, $E(S)$ has a k-partition $(Y_1, Y_2, \ldots, Y_k)$ such that each $S(Y_i)$ is a spanning partition-connected sub-hypergraph of S for $1 \le i \le k$. As $|E(S)| > k(|V(S)| - 1)$, one of these spanning partition-connected sub-hypergraphs, (say $S(Y_1)$), must contain a hypercircuit C. Let $e \in E(C)$.

Choose $\pi = (X_1, X_2, \ldots, X_k) \in P'_k(H)$ such that $\varepsilon(\pi) = \varepsilon'(H)$. Define $X'_i = (X_i - E(S)) \cup Y_i$ for $1 \le i \le k$ and $\pi' = (X'_1, X'_2, \ldots, X'_k)$.

Claim 2. $c(H(X'_i)) \le c(H(X_i))$ *for* $1 \le i \le k$.

Proof of Claim 2. It suffices to show that for any maximal partition-connected sub-hypergraph T of $H(X_i)$, the sub-hypergraph T' induced by $(E(T)-E(S)) \cup Y_i$ is also partition-connected. Since T is partition-connected, by Proposition 2.4(C2), T/S is partition-connected. Thus $T'/Y_i = T/S$ is partition-connected. As $S(Y_i)$ is partition-connected, by Proposition 2.4(C3), T' is partition-connected. This proves the claim.

By Claim 2, $\varepsilon'(H) \le \varepsilon(\pi') \le \varepsilon(\pi) = \varepsilon'(H)$. Thus $\varepsilon(\pi') = \varepsilon(\pi) = \varepsilon'(H)$. By Lemma 3.1, $\varepsilon'(H) = \varepsilon(H) > 0$. Thus we may assume that $c(H(X'_j)) \ge 2$ for some j. Since $H(X'_j)$ has a partition-connected sub-hypergraph $S(Y_j)$, $H(X'_j)$ has a maximal partition-connected sub-hypergraph R containing $S(Y_j)$. Furthermore, $H(X'_j)$ must have a vertex $v \in V(H) - V(S)$ such that v is not in R since $c(H(X'_j)) \ge 2$. By Claim 1, there are vertices $w_1, w_2, \ldots, w_{r-1} \in V(S)$ such that $e' = \{w_1, w_2, \ldots, w_{r-1}, v\} \notin E(H)$. Define the hyperedge subset

$$X''_i := \begin{cases} (X'_i - \{e\}) \cup \{e'\}, & \text{if } i = j, \\ X'_i - \{e\}, & \text{if } i \ne j. \end{cases}$$

Note that $j = 1$ is possible and the hyperedge e is in X'_1.

Let $F = H - e + e'$. Then $\pi'' = (X''_1, X''_2, \ldots, X''_k) \in P'_k(F)$.

Claim 3. $\varepsilon(\pi'') < \varepsilon(\pi')$.

Proof of Claim 3. When $i \ne j$, since $e \in E(C) \subseteq X'_1$, $c(F(X''_i)) = c(H(X'_i))$. When $i = j$, let R' be the maximal partition-connected sub-hypergraph of $H(X'_j)$ that contains v. As $(R+e')/R = K_2$ (a graph with two vertices and one edge) is partition-connected, and R is partition-connected, by Proposition 2.4(C3), $R+e'$ is partition-connected. Also, $((R+e') \cup R')/(R+e') = R'$ is partition-connected. Again by Proposition 2.4(C3), $(R+e') \cup R'$ is partition-connected in $F(X''_i)$. As R and R' are two maximal partition-connected sub-hypergraphs in $H(X'_i)$, it follows that $c(F(X''_i)) < c(H(X'_i))$. By definition, $\varepsilon(\pi'') < \varepsilon(\pi')$, completing the proof of the claim.

By Lemma 3.1 and Claim 3, $\varepsilon(F) = \varepsilon'(F) \le \varepsilon(\pi'') < \varepsilon(\pi') = \varepsilon'(H) = \varepsilon(H)$. This proves the theorem. □

Acknowledgments

The authors would like to thank anonymous referees for their valuable comments and suggestions to improve the presentation of the article. The first author is partially supported by a grant from the Simons Foundation (No. 522728, XG).

References

[1] P.A. Catlin, J.W. Grossman, A.M. Hobbs, H.-J. Lai, Fractional arboricity, strength and principal partitions in graphs and matroids, Discrete Appl. Math. 40 (1992) 285–302.
[2] A. Frank, T. Király, M. Kriesell, On decomposing a hypergraph into k connected sub-hypergraphs, Discrete Appl. Math. 131 (2003) 373–383.
[3] X. Gu, H.-J. Lai, Augmenting and preserving partition connectivity of a hypergraph, J. Combin. 5 (2014) 271–289.
[4] H.-J. Lai, H. Lai, C. Payan, A property on edge-disjoint spanning trees, European J. Combin. 17 (1996) 447–450.
[5] C.St.J.A. Nash-williams, Edge-disjoint spanning trees of finite graphs, J. Lond. Math. Soc. 36 (1961) 445–450.
[6] C.St.J.A. Nash-Williams, Decompositions of finite graphs into forests, J. Lond. Math. Soc. 39 (1964) 12.
[7] C. Payan, Graphes équilibrés et arboricité rationnelle, European J. Combin. 7 (1986) 263–270.
[8] W.T. Tutte, On the problem of decomposing a graph into n factors, J. Lond. Math. Soc. 36 (1961) 221–230.