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Supereulerian width of dense graphs

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ABSTRACT

For a graph G, the **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G has a spanning $(k; u, v)$-trail-system, for any integer k with $1 \leq k \leq s$, and for any $u, v \in V(G)$ with $u \neq v$. Thus $\mu'(G) \geq 2$ implies that G is supereulerian, and so graphs with higher supereulerian width are natural generalizations of supereulerian graphs. Settling an open problem of Bauer, Catlin (1988) proved that if a simple graph G on $n \geq 17$ vertices satisfy $\delta(G) \geq \frac{n}{4} - 1$, then $\mu'(G) \geq 2$. In this paper, we show that for any real numbers a, b with $0 < a < 1$ and any integer $s > 0$, there exists a finite graph family $\mathcal{F} = \mathcal{F}(a, b, s)$ such that for a simple graph G with $n = |V(G)|$, if for any $u, v \in V(G)$ with $uv \notin E(G)$, $\max\{d_G(u), d_G(v)\} \geq an + b$, then either $\mu'(G) \geq s + 1$ or G is contractible to a member in $\mathcal{F}$. When $a = \frac{1}{4}$, $b = -\frac{3}{2}$, we show that if n is sufficiently large, $K_{3,3}$ is the only obstacle for a 3-edge-connected graph G to satisfy $\mu'(G) \geq 3$.

1. Introduction

Graphs in this paper are finite and may have multiple edges but no loops. Terminology and notations not defined here are referred to [4]. In particular, for a graph G, $\delta(G)$, $\alpha(G)$, $\kappa(G)$ and $\kappa'(G)$ represent the minimum degree, the stability number (also called the independence number), the connectivity and the edge connectivity of the graph G, respectively. A trail with initial vertex u and terminal vertex v will be referred as a (u, v)-trail. We use $O(G)$ to denote the set of all odd degree vertices in G. A graph G is **Eulerian** if it is connected and $O(G) = \emptyset$, and is **supereulerian** if G has a Eulerian subgraph H with $V(H) = V(G)$. The study of supereulerian graphs was first raised by Boesch, Suffel and Tindel in [3]. Pulleyblank [15] showed that the problem to determine if a graph is supereulerian, even within planar graphs, is NP-complete.

Motivated by the Menger Theorem, a generalization of supereulerian graphs has been considered in the literature (see [12], for example). For a graph G and an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$-trail-system of G is a subgraph H consisting of s edge-disjoint (u, v)-trails. The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G has a spanning $(k; u, v)$-trail-system, for any integer k with $1 \leq k \leq s$. For any $u, v \in V(G)$ with $u \neq v$, Luo et al. in [14] defined graphs with $\mu'(G) \geq 1$ as **Eulerian-connected** graphs. They also investigated, for a given integer $r > 0$, the minimum value $\psi(r)$ such that if G is a $\psi(r)$-edge-connected graph, then for any $X \subseteq E(G)$ with $|X| \leq r$, $\mu'(G - X) \geq 2$. An open problem on $\psi(r)$ is raised in [14] and is settled in [17]. By definition, $\mu'(G) \geq 2$ implies that G is supereulerian. Supereulerian graphs have been intensively studied, as seen in surveys [5,8,10], among others.

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The concept of $\mu'(G)$ is formally introduced in [12], as a natural generalization of supereulerian graphs. Related studies can be found in [7] and [16]. One of the main problems in the study on the supereulerian width of graphs is to determine $\mu'(G)$ for a given graph G. As shown in [12], every collapsible graph (to be defined in Section 2) has supereulerian width at least 2. Settling an open problem of Bauer [1,2], Catlin prove Theorem 1.1(i) below, which was recently extended by Li et al. in [12].

Theorem 1.1. *Let G be a simple graph on n vertices.*

(i) *(Catlin, Theorem 9*(ii) *of [5]) If $n \geq 17$ and $\delta(G) \geq \frac{n}{4} - 1$, then $\mu'(G) \geq 2$.*

(ii) *(Li et al., Theorem 5.3*(i) *of [12]) For any positive integers p and s with $p \geq 2$, there exists an integer $N = N(s, p)$ and a finite family $\mathcal{F}_0$ of graphs with supereulerian width at most s such that if $\delta(G) \geq \frac{n}{p} - 1$, then either $\mu'(G) \geq s + 1$, or G is contractible to a member in $\mathcal{F}_0$.*

These motivate the current research. The main goal of this paper is to prove Theorems 1.2 and 1.3 below. When $a = \frac{1}{p}$ and $b = -1$, if $\delta(G) \geq \frac{n}{p} - 1$, then for any $u, v \in V(G)$ with $uv \notin E(G)$, $\max\{d_G(u), d_G(v)\} \geq \delta(G) \geq an + b$. Thus the hypothesis of Theorem 1.1(ii) implies a special case of the hypothesis of Theorem 1.2. Computationally, it takes the same order of computational complexity to examine finitely many graphs. In this sense, Theorem 1.2 extends Theorem 1.1(ii).

Theorem 1.2. *For any real numbers a, b with $0 < a < 1$ and any integer $s > 0$, there exists a finite family $\mathcal{F} = \mathcal{F}(a, b, s)$ such that for any simple graph G with $n = |V(G)|$, if for any pair of nonadjacent vertices u and v, $\max\{d_G(u), d_G(v)\} \geq an + b$, then $\mu'(G) \geq s + 1$ if and only if G is not contractible to a member in $\mathcal{F}$.*

Theorem 1.3. *For a simple graph G with $|V(G)| = n \geq 141$ and $\kappa'(G) \geq 3$, if for any pair of nonadjacent vertices u and v, $\max\{d_G(u), d_G(v)\} \geq \frac{n}{4} - \frac{3}{2}$, then $\mu'(G) \geq 3$ if and only if G is not contractible to $K_{3,3}$.*

The next section is devoted to a reduction method which will be employed in our study. Theorem 1.2 will be proved in the subsequent section. Section 4 is devoted to the proof of Theorem 1.3.

2. Reductions and s -collapsible graphs

Throughout this paper, we shall adopt the convention that any graph G is 0-edge-connected, and always assume that $s \geq 1$ is an integer. The maximum number of edge-disjoint spanning trees in a graph G is denoted by $\tau(G)$.

Definition 2.1. A graph G is s-**collapsible** if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning subgraph Γ_R such that
(i) both $O(\Gamma_R) = R$ and $\kappa'(\Gamma_R) \geq s - 1$, and
(ii) $G - E(\Gamma_R)$ is connected.

Catlin [5] first introduced collapsible graphs, which are exactly the 1-collapsible graphs defined here. A spanning subgraph Γ_R of G satisfying Definition 2.1(i) and (ii) is an (s, R)-**subgraph** of G. Let $\mathcal{C}_s$ denote the collection of all s-collapsible graphs. Then $\mathcal{C}_1$ is the collection of all collapsible graphs [5]. By definition, for $s \geq 1$, any $(s + 1, R)$-subgraph of G is also an (s, R)-subgraph of G. Thus $\mathcal{C}_{s+1} \subseteq \mathcal{C}_s$ for any positive integer s.

For a graph G, and for $X \subseteq E(G)$, the **contraction** G/X is obtained from G by identifying the two ends of each edge in X and then by deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$. If H is a connected induced subgraph of G and z is the vertex in G/X onto which H is contracted, then we call H the **(contraction) preimage** of z, and define $PI_G(z) = H$. A vertex $z \in V(G/X)$ with $PI_G(z) \cong K_1$ is often referred as a trivial vertex under the contraction. The following are known.

Proposition 2.2 (*Li [11], Corollary 2.4 of [12]*)**.** *Let $s \geq 1$ be an integer. Then $\mathcal{C}_s$ satisfies the following.*

(C1) $K_1 \in \mathcal{C}_s$

(C2) *If $G \in \mathcal{C}_s$ and if $e \in E(G)$, then $G/e \in \mathcal{C}_s$.*

(C3) *If H is a subgraph of G and if $H, G/H \in \mathcal{C}_s$, then $G \in \mathcal{C}_s$.*

Lemma 2.3 (*Li [11], Corollary 2.5 of [12]*)**.** *Let $s \geq 1$ be an integer. If a graph $G \in \mathcal{C}_s$, then $\mu'(G) \geq s + 1$.*

A graph is $\mathcal{C}_s$-**reduced** if it contains no nontrivial subgraph in $\mathcal{C}_s$. It is shown in [12] that every graph G has a unique collection of maximally s-collapsible subgraphs $H_1, H_2, \ldots, H_c$, and the graph $G'_s = G/(\cup_{i=1}^{c} E(H_i))$ is $\mathcal{C}_s$-reduced, which is called the $\mathcal{C}_s$-**reduction** of G. By the definition of $\mathcal{C}_s$-reduction and by Proposition 2.2, the $\mathcal{C}_s$-reduction of a graph is always $\mathcal{C}_s$-reduced.

Lemma 2.4 (*Li [11], Corollary 2.9 of [12]*)**.** *Let $s \geq 1$ be an integer, G be a graph and H be a subgraph of G such that $H \in \mathcal{C}_s$. Each of the following holds.*

(i) *$G \in \mathcal{C}_s$ if and only if $G/H \in \mathcal{C}_s$.*

(ii) *$\mu'(G) \geq s + 1$ if and only if $\mu'(G/H) \geq s + 1$. In particular, if G' is the $\mathcal{C}_s$-reduction of G, then $\mu'(G) \geq s + 1$ if and only if $\mu'(G') \geq s + 1$.*

Let $F(G, s)$ denote the minimum number of additional edges that must be added to G to result in a graph Γ with $\tau(\Gamma) \geq s$. The value of $F(G, s)$ has been determined in [13], whose matroidal versions are proved in [9] and [11].

Theorem 2.5. *Let G be a connected nontrivial graph, and $s \geq 1$ be an integer.*
(i) *(Li [11], Theorem 2.11 of [12]) If $F(G, s+1) \leq 1$, then $G \in \mathcal{C}_s$ if and only if $\kappa'(G) \geq s+1$.*
(ii) *(Catlin, Han and Lai, Theorem 1.3 of [6]) If $F(G, 2) \leq 2$, then G is 1-collapsible if and only if the $\mathcal{C}_1$-reduction of G is not in $\{K_2, K_{2,t} : t \geq 1\}$.*

Lemma 2.6 (*Li [11], Corollary 2.13 of [12]*)**.** *Let G be a connected nontrivial graph, and $s \geq 1$ be an integer.*
(i) *If $\tau(G) \geq s+1$, then $G \in \mathcal{C}_s$.*
(ii) *If G is $\mathcal{C}_s$-reduced, then for any nontrivial subgraph H of G, $\frac{|E(H)|}{|V(H)|-1} < s+1$.*
(iii) *If $\kappa'(G) \geq s+1$ and G is $\mathcal{C}_s$-reduced, then*

$$F(G, s+1) = (s+1)(|V(G)|-1) - |E(G)| \geq 2.$$

Let $\ell > 0$ be an integer and define ℓK_2 to be the graph with two vertices and ℓ edges connecting the two vertices. Catlin [5] showed that $\ell K_2 \in \mathcal{C}_1$ if and only if $\ell \geq 2$ and $K_n \in \mathcal{C}_1$ if and only if $n \geq 3$. Li et al. present the following characterization for larger values of s.

Lemma 2.7 (*Li et al., Corollary 3.1 and Theorem 3.3 of [12]*)**.** *Let ℓ, n be integers with $\ell \geq 1$ and $n \geq s \geq 2$. Each of the following holds.*
(i) *$\ell K_2 \in \mathcal{C}_s$ if and only if $\ell \geq s+1$.*
(ii) *$K_n \in \mathcal{C}_s$ if and only if $n \geq s+3$;*

Lemma 2.8 (*Li et al., Corollary 2.4 and 2.9 of [12]*)**.** *Let $s > 0$ be an integer. Each of the following holds.*
(i) *$\mu'(K_1) \geq s+1$.*
(ii) *If $e \in E(G)$, then $\mu'(G/e) \geq \mu'(G)$. In particular, if $\mu'(G) \geq s+1$ and $e \in E(G)$, then $\mu'(G/e) \geq s+1$.*

3. Proof of Theorem 1.2

Following [4], if $V' \subseteq V(G)$ (or $X \subseteq E(G)$, respectively), then $G[V']$ (or $G[X]$) is the subgraph of G induced by V' (by X, respectively). If $v \in V(G)$, let $N_G(v)$ be the vertices of G adjacent to v in G. If H is a graph and Z is a set of edges such that the end vertices of each edge in Z are in $V(H)$, then $H + Z$ denotes the graph with vertex set $V(H)$ and edge set $E(H) \bigcup Z$. For an integer $i \geq 0$, let $D_i(G)$ be the set of all vertices of degree i in G, and $d_i(G) = |D_i(G)|$. By Lemma 2.6(iii) and with algebraic manipulations, we obtain the following lemma.

Lemma 3.1. *If G' is $\mathcal{C}_s$-reduced and $\kappa'(G') \geq s+1$, then $|E(G')| \leq (s+1)(|V(G')|-1) - 2$ and*

$$\sum_{i=s+1}^{2s+1} (2s+2-i)d_i \geq \sum_{i \geq 2s+3} (i-2s-2)d_i + 2s+4. \tag{1}$$

Proof. By Lemma 2.6, for a $\mathcal{C}_s$-reduced graph H, we have

$$2(s+1)\sum_{i\geq 1} d_i(H) \geq \sum_{i \geq 1} i d_i(H) + 2s+4. \tag{2}$$

Hence

$$2(s+1)\sum_{i=1}^{s} d_i(H) + 2(s+1)\sum_{i=s+1}^{2s+1} d_i(H) + 2(s+1)d_{2s+2}(H) + 2(s+1)\sum_{i\geq 2s+3} d_i(H)$$
$$\geq \sum_{i=1}^{s} i d_i(H) + \sum_{i=s+1}^{2s+1} i d_i(H) + (2s+2)d_{2s+2}(H) + \sum_{i \geq 2s+3} d_i(H) + 2s+4.$$

Therefore

$$(2s+2-i)\sum_{i=1}^{s} d_i(H) + (2s+2-i)\sum_{i=s+1}^{2s+1} d_i(H) \geq \sum_{i\geq 2s+3}(i-2s-2)d_i(H) + 2s+4.$$

As $\kappa'(G') \geq s+1$, we have $\sum_{i=1}^{s} d_i(H) = 0$, and so we have $(2s+2-i)\sum_{i=s+1}^{2s+1} d_i(H) \geq \sum_{i\geq 2s+3}(i-2s-2)d_i(H) + 2s+4$. □

Throughout the rest of this section, we assume that a, b are given real numbers with $0 < a < 1$, s is a fixed positive integer, and G is a simple graph on n vertices. Define

$$W = W_{a,b}(G) = \{v \in V(G) | d_G(v) < an + b\}. \tag{3}$$

Let G' denote the $\mathcal{C}_s$-reduction of G and let $n' = |V(G')|$. Then by definition, G' is $\mathcal{C}_s$-reduced. Define

$$W' = \{z \in V(G') | PI_G(z) \cap W \neq \emptyset\}. \tag{4}$$

Let $c = 2s + 2$ be a real number. Define

$$X'_c = \{z \in V(G') | d_{G'}(z) < c\} \text{ and } X''_c = \{z \in X'_c | PI_G(z) \neq K_1\}. \tag{5}$$

Let $N_1 = 1 + \max\{s + 1, \frac{c-b}{a} + 1, \frac{-(a+2)(b+1)}{a^2}, (2s+3)(\lceil \frac{1}{a} \rceil + s + 2) - 2s - 4\}$ be an integer. Define $\mathcal{S}(a, b)$ to be the family of simple graphs such that a graph G is in $\mathcal{S}(a, b)$ if and only if G satisfies the following:

$$\text{for any } u, v \in V(G) \text{ with } uv \notin E(G),\ \max\{d_G(u), d_G(v)\} \geq an + b. \tag{6}$$

For a graph $G \in \mathcal{S}(a, b)$, recall that G' is the $\mathcal{C}_s$-reduction of G. As N_1 is completely determined by the values of a, b and s, N_1 is a finite number when a, b and s are given. Let $\mathcal{F} = \mathcal{F}(a, b, s) = \{G' | G \in \mathcal{S}(a, b), \mu'(G') \leq s \text{ and } |V(G')| \leq N_1\}$ to be the family of all $\mathcal{C}_s$-reductions with order at most N_1 of graphs in $\mathcal{S}(a, b)$ with supereulerian width at most s. Thus a graph $H \in \mathcal{F}$ if and only if H is the $\mathcal{C}_s$-reduction of a graph G in $\mathcal{S}(a, b)$, such that $\mu'(G') \leq s$ and $|V(G')| \leq N_1$. We have the following observations.

Proposition 3.2. *Each of the following holds.*
(i) If a graph G is contractible to a member in $\mathcal{F}$, then $\mu'(G) \leq s$.
(ii) $\mathcal{F}$ contains only finitely many graphs.
(iii) If $\kappa'(G) \leq s$, then G is contractible to a member in $\mathcal{F}$.

Proof. As (i) follows from Lemma 2.8 (ii), we start our proofs of (ii). By definition, every graph in $\mathcal{F}$ has at most N_1 vertices. By Lemma 2.7(i), if $G \in \mathcal{F}$, every edge of G has multiplicity at most s. Thus there are finitely many members in $\mathcal{F}$ and so (ii) holds.

By definition, $\kappa'(H) \geq \mu'(H)$ for any graph H. In particular, for any integer $\ell > 0$, $\mu'(\ell K_2) \leq \ell$ and so as $N_1 \geq s + 2$, we have $\ell K_2 \in \mathcal{F}$, for all $1 \leq \ell \leq s$. By definition, a connected graph G satisfies $\kappa'(G) = \ell > 0$ if and only if G can be contracted to a ℓK_2. Thus (iii) must hold. □

Necessity of Theorem 1.2. Let $G \in \mathcal{S}(a, b)$. If G is contractible to a member in $\mathcal{F}$, then by Proposition 3.2(i) and by the definition of $\mathcal{F}$, we have $\mu'(G) \leq s$.

Sufficiency of Theorem 1.2. We assume that $G \in \mathcal{S}(a, b)$ and that

$$G \text{ cannot be contracted to a member of } \mathcal{F}, \tag{7}$$

to show that $\mu'(G) \geq s + 1$. By (7) and by Proposition 3.2(iii), we in the rest of the proof will assume that

$$\kappa'(G) \geq s + 1 \text{ and } n = |V(G)| \geq N_1.. \tag{8}$$

Pick any $z \in X''_c - W'$ and let $H_z = PI_G(z)$. For each $v \in V(H_z)$, by (6), we have $|V(H_z)| \geq 1 + d_G(v) - d_{G'}(z) \geq an + b + 1 - c$. We claim that

$$\text{there must be a vertex } v' \in V(H_z) \text{ such that } N_G(v') \cap [V(G) - V(H_z)] = \emptyset \text{ for any } z \in X''_c - W'. \tag{9}$$

If (9) does not hold, then every vertex in H_z is adjacent to at least one vertex which is not in H_z. Let $|V(H_z)| = k$. Since $d_{G'}(z) < c$, we have $k \leq d_{G'}(z) \leq c - 1$. Since $n \geq N_1 \geq \frac{c-b}{a} + 1$, we have $an + b \geq c + 1$. This, together with the assumption that $z \in X''_c - W'$, implies $d_G(v) \geq c + 1$ for any $v \in H_z$.

For any $v \in H_z$, let m_v be the number of edges not in H_z but incident with v. If for any $v \in V(H_z)$, we have $m_v > \frac{c-1}{k}$, then $d_{G'}(z) = \sum_{v \in H_z} m_v > k \times \frac{c-1}{k} = c - 1$ which contradicts our assumption that $d_{G'}(z) \leq c - 1$. Hence there must be a vertex $v_0 \in H_z$ such that $m_{v_0} \leq \frac{c-1}{k}$, and so we have $k - 1 \geq |N_G(v_0) \cap V(H_z)| = d_G(v_0) - m_{v_0} \geq c + 1 - \frac{c-1}{k}$. Thus we have $k > c + 1$ which contradicts $k \leq c - 1$. Hence, it is impossible that every vertex in H_z is adjacent to a vertex which is not in H_z, implying that (9) must hold.

By (9), it follows that $|V(H_z)| \geq 1 + d_G(v')$, and so we have

$$\text{for any } z \in X''_c - W',\ |V(PI_G(z))| \geq an + b + 1. \tag{10}$$

Applying (10), we count the number of vertices contained in the preimages of vertices in $X''_c - W'$ to get $n \geq \sum_{z \in X''_c - W'} |V(PI_G(z))| \geq |X''_c - W'|(an + b + 1)$. It follows by $n \geq N_1 > \frac{-(a+2)(b+1)}{a^2}$ that $|X''_c - W'| \leq \frac{n}{an+b+1} < \frac{1}{a} + \frac{1}{2}$, and so

$$|X''_c - W'| \leq \lceil \frac{1}{a} \rceil. \tag{11}$$

By (3) and (6), $G[W]$ is a complete subgraph of G. By Lemma 2.7 and (4), we conclude that

$$|W'| \le s + 2. \tag{12}$$

By (5), we have $X_c' - X_c'' \subseteq W'$. Since $c = 2s + 2$, we have $|X_c'| \ge \sum_{i=s+1}^{2s+1} d_i$. It now follows from (1) that

$$\begin{aligned}(2s+2)|X_c'| \ge (2s+2)\sum_{i=s+1}^{2s+1} d_i &\ge \sum_{i=s+1}^{2s+1}(2s+2-i)d_i \ge \sum_{i\ge 2s+3}(i-2s-2)d_i + 2s+4 \\ &\ge \sum_{i\ge 2s+3} d_i + 2s+4 = |V(G')| - |X_c'| + 2s + 4.\end{aligned}$$

As $X_c' \subseteq W' \cup X_c'' = W' \cup (X_c'' - W')$, this, together with (11) and (12), implies

$$n' \le (2s+3)|X_c'| - 2s - 4 \le (2s+3)(|W'| + |X_c'' - W'|) - 2s - 4 \le (2s+3)(\lceil \frac{1}{a} \rceil + s + 2) - 2s - 4.$$

Hence $|V(G')| \le N_1$. By (7), we must have $\mu'(G') \ge s + 1$. It follows by Lemma 2.4 that $\mu'(G) \ge s + 1$. This completes the proof of the sufficiency of Theorem 1.2(i). $\square$

4. The proof of Theorem 1.3

Throughout this section, we assume $a = \frac{1}{4}$, $b = -\frac{3}{2}$ and $s = 2$ in our discussion. We will use the notation in the previous section, set $c = 2s + 2 = 6$ andfollow (3)–(5) to define W, W', X_c' and W_c'', respectively. The main goal of this section is to prove Theorem 1.3. We will need the following lemmas in this section.

Lemma 4.1 (*Theorem 4.4 of [12]*). *Let G be a 3-edge-connected graph on $n \le 6$ vertices. Then $\mu'(G) \ge 3$ if and only if $G \ncong K_{3,3}$.*

Lemma 4.2 (*Catlin [5], Corollary 1*). *Let G be a connected graph. If for any $e \in E(G)$, G has a collapsible subgraph H_e with $e \in E(H_e)$, then G is collapsible.*

Lemma 4.3. *Let G be a graph with $|V(G)| = n \ge 138$ and $\kappa'(G) \ge 3$ and let G' be the $\mathcal{C}_2$-reduced graph of G. If for any $u, v \in V(G)$ with $uv \notin E(G)$,*

$$\max\{d_G(u), d_G(v)\} \ge \frac{n}{4} - \frac{3}{2} \tag{13}$$

then $|V(G')| \le 7$.

Proof. By (13), the subgraph of G' induced by vertices in W' must be a complete graph. Thus by Lemma 2.7 with $s = 2$, we conclude that $|W'| \le 4$.

Let $k = |V(G') - W'|$. We shall show that $k \le 4$. Firstly, we claim that

$$\text{for any } z \in V(G') - W', \text{ there exists } v \in V(PI_G(z)) \text{ such that } N_G(v) \subseteq V(PI_G(z)). \tag{14}$$

Fix a $z_0 \in V(G') - W'$ which violates (14). Then for every $v \in V(PI_G(z_0))$, we have $N_G(v) - V(PI_G(z_0)) \ne \emptyset$. By (9), $z_0 \notin X_6'' - W'$. By (4), (5) and (13), for any $z \in X_6'' - W'$, we have $|V(PI_G(z))| \ge \frac{n}{4} - \frac{1}{2}$. It follows that $n \ge (\frac{n}{4} - \frac{1}{2})|X_6'' - W'|$, implying $|X_6'' - W'| \le 4$.

Since G' is $\mathcal{C}_2$-reduced, by Lemma 2.6 (iii) with $s = 2$, we have $|E(G')| \le 3(|V(G')| - 1) - 2 \le 3(k + 4 - 1) - 2 = 3k + 7$. Denote $|V(PI_G(z_0))| = m$. As $|N_G(v) - V(PI_G(z))| \ge 1$ for each $v \in V(PI_G(z))$,

$$\begin{aligned}0 < m \le d_{G'}(z_0) &= \sum_{z\in V(G')} d_{G'}(z) - \sum_{z\in V(G'-z_0)} d_{G'}(z) \le 2(3k+7) - \sum_{z\ne z_0} d_{G'}(z) \\ &\le 2(3k+7) - 3|X_6'' - W'| - \sum_{z\ne z_0, z\notin X_6''-W'} d_{G'}(z) \le 2(3k+7) - 3\times 4 - (k-5)6 = 33.\end{aligned}$$

It follows that there exists $v \in V(PI_G(z_0))$ such that $N_G(v) - V(PI_G(z_0)) \le \frac{33}{m}$. As $n \ge 138$, we have $\frac{n}{4} - \frac{1}{2} \ge 34$. Thus $34 \le d_G(v) \le \frac{33}{m} + m - 1$, forcing either $m \le 0$ or $m \ge 34$, contrary to (15). This justifies (14).

By (14), for each $z \in V(G') - W'$, there exists $v \in V(PI_G(z))$ such that $N_G(v) \subseteq V(PI_G(z))$. It follows by (13) that,

$$|V(PI_G(z))| \ge |N_G(v) \cup \{v\}| = d_G(v) + 1 \ge \frac{n}{4} - \frac{1}{2}. \tag{15}$$

As $n = |V(G)| \ge \bigcup_{z\in V(G')-W'} |V(PI_G(z))| \ge |V(G') - W'| \left(\frac{n}{4} - \frac{1}{2}\right)$, and our choice of n, we conclude that $|V(G') - W'| \le 4$.

Thus $|V(G')| = |W'| + |V(G') - W'| \leq 8$, where equality holds if and only if $|W'| = |V(G') - W'| = 4$. If $|V(G')| = 8$, then by the choice of n and by (15), we have $8 = |V(G')| \leq 4 + \left[n - 4\left(\frac{n}{4} - \frac{1}{2}\right)\right] = 6$, a contradiction. Therefore we must have $|V(G')| \leq 7$. □

Lemma 4.4. *Let J be a graph $\kappa'(J) \geq 3$ and $|V(J)| \leq 4$. For any edge subset $X \subseteq E(J)$ with $1 \leq |X| \leq 2$ such that $J[X]$ is a path, each of the following holds.*

(i) *$J - X$ is 1-collapsible if and only if $\kappa'(J - X) \geq 2$.*

(ii) *If $J \in \{K_4, K_4 - e\}$ where $e \in E(K_4)$, then J is 1-collapsible.*

Proof. Since every edge of K_4 and $K_4 - e$ lies in a triangle, by Lemma 4.2, we have (ii) holds. By Lemma 2.3, if $J - X$ is 1-collapsible, then $\kappa'(J - X) \geq \mu'(J - X) \geq 2$. It remains to show the sufficiency of (i).

Suppose $\kappa'(J - X) \geq 2$ and assume that $J - X$ is not collapsible. Let $(J - X)'$ be the 1-reduction of $J - X$. Then by Proposition 2.2 (C3), we have $2 \leq |V[(J - X)']| \leq 4$. As it is known (Page 38 of [5]) that every 2-edge-connected graph with at most 3 vertices are 1-collapsible, we must have that $|V[(J - X)']| = 4$. Since $\kappa'[(J - X)'] = \kappa'(J - X) \geq 2$, we have $|E(J')| \geq \frac{1}{2}\sum_{v \in V(J')} d_{J'}(v) \geq \frac{4\times 2}{2} = 4$. It follows by Lemma 2.6 (iii) that $F((J - X)', 2) \leq 2$. By Theorem 2.5 (ii), we have $(J - X)' \cong K_{2,2}$. This implies that $(J - X)' = J - X$. Since $\kappa'(J) \geq 3$, and since $J - X = K_{2,2}$, X must be a matching of size 2 in J, contrary to the assumption that $J[X]$ is a path in J. This completes the proof. □

Lemma 4.5. *Let H be a graph with $\kappa'(H) \geq 3$ and $|V(H)| = 7$. If H contains a subgraph $L \cong K_4$, then for any distinct $u, v \in H$ there exists a (u, v)-path P in H such that $H - E(P)$ is 1-collapsible.*

Proof. For integers $\ell > 0$ and $t > 0$, let ℓP_t denote the graph obtained from a path $P_t = z_1 z_2 \ldots z_t$ by replacing each edge of P_t by a set of ℓ parallel edges. We will use this notation in the proof. Note that if H/L is spanned by a $3P_4$, then by Lemma 4.2, Lemma 4.5 holds trivially. Hence we assume that

$$H/L \text{ is not spanned by a } 3P_4. \tag{16}$$

Fix the vertices $u, v \in V(H)$. If $uv \in E(G)$, then let $P = H[\{uv\}]$, $L_1 = L$ if $uv \notin E(L)$ and $L' = L - uv$ if $uv \in E(L)$. Then $J = H/L' \cong H/L$ is a 3-edge-connected graph with $|V(J)| \leq 4$. It follows by Lemma 4.4(i) that $(H - E(P))/L'$ is 1-collapsible. By Lemma 4.4(ii) and Proposition 2.2 (C3), $H - E(P)$ is 1-collapsible. Hence in the following arguments, we assume that $uv \notin E(H)$, and so $|\{u, v\} \cap V(L)| \leq 1$. Let $J = H/L$ and v_L be the vertex in J onto which L is contracted. We further assume that if $|\{u, v\} \cap V(L)| = 1$, then $v \in V(L)$, in which case we adopt the convention to denote $v = v_L$.

By the assumption of the lemma, $|V(J)| = 4$ with $\kappa'(J) \geq 3$. By (16), it is a routine inspection to conclude that J always has a (u, v)-path P' with $|E(P')| \leq 2$ and $\kappa'(J - E(P')) \geq 2$. It follows by Lemma 4.4(i) that $J - E(P')$ is 1-collapsible.

If $v_L \notin V(P')$, then $v \notin V(L)$ and so P' is a path in H, in this case we define $P = P'$. If $v_L \in V(L)$ such that P' is a (u, v_L)-path in J, then $v \in V(L)$. In this case, let $v' \in V(L)$ be the vertex in L such that $H[E(P')]$ is an (u, v')-path; and define $P = P'$ if $v = v'$, and $P = P' + v'v$ if $v \neq v'$. If $v_L \in V(P')$ is an internal vertex of P', then $v \notin V(L)$ and there exist distinct vertices $v', v'' \in V(L)$ such that $H[E(P')]$ consists of a (u, v')-path T' and a (v'', v)-path T''. In this case we define $P = T' + v_i v_i' + T''$. In any case, P is a (u, v)-path in H satisfying $|E(P) \cap E(L)| \leq 1$. By Lemma 4.4, $L - (E(P) \cap E(L))$ is 1-collapsible. By the definition of contraction,

$$(H - E(P))/(L - (E(P) \cap E(L))) = (H - E(P'))/L = J - E(P'),$$

is 1-collapsible. We conclude that $H - E(P)$ is 1-collapsible by applying Proposition 2.2(C3). □

With these Lemmas, now we are ready to present the proof of Theorem 1.3.

Proof of Theorem 1.3. Necessity; Let G be a graph which is contractible to $K_{3,3}$, then by Lemma 4.1 and Lemma 2.8 (ii), $\mu'(G) \leq 2$.

Sufficiency: Let G be a graph which is not contractible to $K_{3,3}$. Let G' be the $\mathcal{C}_2$-reduction of G. Then by Lemma 4.3, $|V(G')| \leq 7$. If $|V(G')| \leq 6$, then since G is not contractible to $K_{3,3}$ and by Lemma 4.1 we have $\mu'(G') \geq 3$. If $|V(G')| = 7$, then by Lemma 4.5 we have $\mu'(G') \geq 3$. Finally, since $\mu'(G') \geq 3$, by Lemma 2.8 we know that $\mu'(G) \geq 3$. □

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