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Characterizations of k-cutwidth critical trees

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Abstract The cutwidth problem for a graph G is to embed G into a path such that the maximum number of overlap edges (i.e., the congestion) is minimized. The investigations of critical graphs and their structures are meaningful in the study of a graph-theoretic parameters. We study the structures of k-cutwidth ($k > 1$) critical trees, and use them to characterize the set of all 4-cutwidth critical trees.

Keywords Graph labeling · Cutwidth · Critical tree

1 Introduction

Graphs in this paper are finite, simple and connected with undefined terminologies and notations following (Bondy and Murty 2008). The cutwidth problem for a graph G is to embed G into a path such that the maximum number of overlap edges (i.e., the congestion) is minimized. It is known that the problem for general graphs is NP-complete (Garey and Johnson 1979) while it is polynomially solvable for trees (Yannakakis 1985). The cutwidth problem has important applications to VLSI designs and communication networks (Diaz et al. 2002). It is closely related to other graph-

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theoretic parameters such as bandwidth, pathwidth and treewidth, see Chung and Seymour (1985), Diaz et al. (2002), Korach and Solel (1993), among others.

Let $n > 0$ be an integer, and $\overline{n} = \{1, 2, \ldots, n\}$. For a graph $G = (V, E)$ with $|V| = n$, a *labeling* of G is a bijection $f : V \to \overline{n}$, viewed as an embedding of G into the path P with vertices $\overline{n}$, where consecutive integers are the adjacent vertices. The cutwidth of G with respect to f is

$$c(G, f) = \max_{1 \le j < n} \left|\left\{uv \in E : f(u) \le j < f(v)\right\}\right|, \tag{1}$$

which represents the congestion of the embedding. The *cutwidth* of G is defined by

$$c(G) = \min_{f} c(G, f), \tag{2}$$

where the minimum is taken over all labelings f. If $k = c(G, f)$, then f, as well as the embedding induced by f, is called a *k-cutwidth embedding* of G. A labeling f attaining the minimum in (2) is called an *optimal labeling*. For each i with $1 \le i \le n$, let $u_i = f^{-1}(i)$, and let $S_j = \{u_1, u_2, \ldots, u_j\}$ be the set of the first j vertices. Define $\nabla(S_j) = \{u_i u_h \in E : i \le j < h\}$ is called the (edge) cut at $[j, j+1]$. By (1), we have

$$c(G, f) = \max_{1 \le j < n} |\nabla(S_j)|. \tag{3}$$

Any cut $\nabla(S_j)$ achieving the maximum in (3) is called an *f-max-cut* of G.

Let G be a graph and $i \ge 0$ be an integer. Let $D_i(G) = \{v \in V(G) : d_G(v) = i\}$, where any vertex in $D_1(G)$ is a *pendant vertex* of G. For each $v \in V(G)$, let $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. If G has a vertex $v \in D_2(G)$ with $N_G(v) = \{v_1, v_2\}$ and $v_1v_2 \notin E(G)$, then $G - v + v_1v_2$, the graph obtained from $G - v$ by adding a new edge v_1v_2, is call a series reduction of G. A graph G is *homeomorphically minimal* if G does not have any series reductions. Two graphs are said to be *homeomorphic* if they are isomorphic or can be reduced to isomorphic graphs by a sequence of series reductions.

Definition 1.1 Let $k \ge 1$ be an integer. A graph G is said to be *k-cutwidth critical* if $c(G) = k$ but for every proper subgraph G' of G, $c(G') < k$ and G is homeomorphically minimal, that is, G is not a subdivision of any simple graph.

The cutwidths of certain families of graphs, including complete graphs and complete bipartite graphs as well as cartesian products of cycles and paths, have been studied in (Lin et al. 2002; Lin 2003; Liu and Yuan 1995; Rolin et al. 1995). The relations between cutwidth and other graph-theoretic parameters were studied in various aspects (Chung and Seymour 1985; Korach and Solel 1993). The critical graphs with cutwidth at most three were studied in (Lin and Yang 2004).

Theorem 1.2 (Lin and Yang (2004)) *Let G be a graph. Each of the following holds.*

(i) *G is 1-cutwidth critical if and only if $G = K_2$.*
(ii) *G is 2-cutwidth critical if and only if $G \in \{K_3, K_{1,3}\}$.*

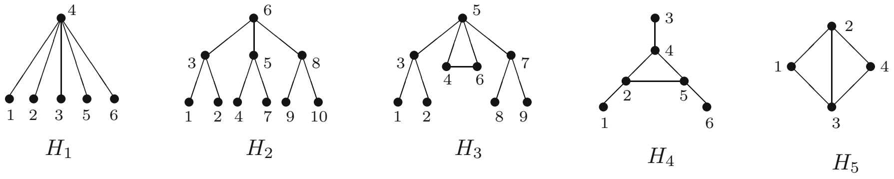


Fig. 1 The 3-cutwidth critical trees

(iii) *G is 3-cutwidth critical if and only if $G \in \{H_1, H_2, H_3, H_4, H_5\}$, where H_1, H_2, H_3, H_4, H_5 are the graphs depicted in Fig. 1.*

The investigation of 4-cutwidth critical trees was conducted in (Zhang and Lin 2012), where twelve 4-cutwidth critical trees $\tau_1, \tau_2, \ldots, \tau_{12}$ have been found, as depicted in Fig. 3 (see Appendix), together with their optimal labelings. It is natural to consider the problem of determining all 4-cutwidth critical trees. In this paper we will give a complete characterization of all 4-cutwidth critical trees. Let $\tau_{13}, \tau_{14}, \ldots, \tau_{18}$ be the trees depicted in Fig. 4 (see Appendix), and let $\mathcal{T}_4 = \{\tau_1, \tau_2, \ldots, \tau_{18}\}$. One of our main results in this paper is the following.

Theorem 1.3 *A tree G is 4-cutwidth critical if and only if $G \in \mathcal{T}_4$.*

A *caterpillar* is a tree G if $G - D_1(G)$ is a path. By Lin and Yang (2004), $c(G) = \lceil \frac{\Delta(G)}{2} \rceil$ for any caterpillar G, where $\Delta(G)$ denote the maximum degree of G. The tree family $\mathcal{T}_4$ can be used to characterize all trees with $c(T) = 3$ as follows.

Theorem 1.4 *A tree G has cutwidth ≤ 3 if and only if it is not homeomorphic to a caterpillar with $\Delta(G) \geq 7$ or it does not contain any subtree homeomorphic to a member in $\mathcal{T}_4$.*

Proof By Theorem 1.3, for a tree G, $c(G) \leq 3$ if and only if $G \notin \mathcal{T}_4$. Similarly, since $c(G) = \lceil \frac{\Delta(G)}{2} \rceil$ for any caterpillar G, $c(G) \leq 3$ if and only if $\Delta(G) \leq 6$, completing the proof. □

In Sect. 2, we investigate different ways of constructing k-critical trees. The proof of Theorem 1.3 is given in Sect. 3.

2 Constructions of k-cutwidth critical trees

In this section, we will present several constructions of k-cutwidth critical trees.

Definition 2.1 Let $T = (V, E)$ be a tree.

(i) For any $\{v, v_1, v_2, \ldots, v_r\} \subseteq V$, define $T(v; v_1, v_2, \ldots, v_r)$ as the largest subtree of T that contains v but does not contain any of $v_1, v_2, \ldots, v_r$.
(ii) Let $T_1, T_2, \ldots, T_t$ be trees and for $j \in \{1, 2, \ldots, t\}$, let $z_j \in D_1(T_j)$. Define $T = T(z_0; T_1, T_2, \ldots, T_t)$ to be a tree obtained from disjoint union of $T_1, T_2, \ldots, T_t$ by identifying $z_1, z_2, \ldots, z_t$ into a single vertex z_0 in T. As $z_0 = z_j$ in T_j, z_0 is viewed as the vertex z_j in T_j.

(iii) If $|V(T)| \geq 2$, then define $\mathcal{M}(T) = \{T - v : v \in D_1(T)\}$ to be the family of all proper maximal subtrees of T.

Lemma 2.2 *Let G be a graph and T be a tree. Then each of the following holds.*

(i) (Zhang and Lin (2012)) $c(G) \geq \lceil \Delta(G)/2 \rceil$.
(ii) (Chung et al. (1985)) $c(T) \leq k$ *if and only if every vertex v of degree at least 2 has neighbors v_1,v_2 such that* $c(T(v; v_1, v_2)) \leq k - 1$.
(iii) (Zhang and Lin (2012)) $c(T) \geq k$ *if and only if there exists a nonpendant vertex v such that $c(T(v; v_1, v_2)) \geq k - 1$ holds for any two neighbors v_1, v_2 of v.*
(iv) (Zhang and Lin (2012)) *Let $k \geq 2$ be an integer. If T_1, T_2 and T_3 be $(k-1)$-cutwidth trees, then $T = T(z_0; T_1, T_2, T_3)$ is a k-cutwidth tree.*

We first obtain a construction of k-cutwidth critical trees by extending Lemma 2.2(iv).

Theorem 2.3 *Let $k \geq 2$ be an integer, and let T_1, T_2 and T_3 be $(k-1)$-cutwidth critical trees (not necessarily distinct). If $T = T(z_0; T_1, T_2, T_3)$, then T is a k-cutwidth critical tree.*

Proof By Theorem 1.2, Theorem 2.3 holds with $k = 2$. Hence we assume that $k \geq 3$. We adopt the notation in Definition 2.1 in the proof. By Lemma 2.2(iv), $c(T) = k$. For $j \in \{1, 2, 3\}$, let x_j be the only vertex in T_j adjacent to z_0. Let $T' \in \mathcal{M}(T)$ be a maximal proper subtree of T. Then for some $v \in D_1(T)$, $T' = T - v$. Since $k \geq 3$, we have $v \notin N_T(z_0)$, and so we may assume that $w \in D_1(T_2)$. Since T_1, T_2, T_3 are $(k-1)$-cutwidth critical, $c(T_1 - z_0) \leq k-2$, $c(T_2 - w) \leq k-2$ and $c(T_3 - z_0) \leq k-2$. For $1 \leq i \leq 3$, let f_i be an optimal labeling of T_i such that $c(T_i, f_i) \leq k - 2$. Define $f : V(T) \to \overline{|V(T)|}$ as follows:

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in V(T_1 - z_0) \\ f_2(v) + |V(T_1)| - 1 & \text{if } v \in V(T_2 - w) \\ f_3(v) + |V(T_1)| + |V(T_2)| - 1 & \text{if } v \in V(T_3 - z_0) \end{cases}. \tag{4}$$

By (3), every f-max-cut of T must be an f_i-max-cut of T_i plus the newly added edge z_0x_1 or z_0x_3, and so $c(T') \leq k - 1$. Hence T is k-cutwidth critical. This completes the proof. □

The labeling f defined in (4) is called *the labeling by the order* (f_1, f_2, f_3), or *the labeling by the order* $(V(T_1 - z_0), V(T_2 - w), V(T_3 - z_0))$.

Definition 2.4 Let T be a tree, $v_0 \in V(T)$ with $i_0 = d_T(v_0)$ and $N_T(v_0) = \{v_i : 1 \leq i \leq i_0\}$. For each i with $1 \leq i \leq i_0$, let T_i be the component of $T - v_0$ with $v_i \in V(T_i)$, and define, for $j = 1, 2, 3$,

$$F_j = T_j \cup \left(\bigcup_{h=4}^{i_0} T_h \right) \cup \{v_0 v_h : 4 \leq h \leq i_0\} \cup \{v_0 v_j\}.$$

Lemma 2.5, Theorem 2.6, 2.7 and 2.8 below are on k-cutwidth critical trees with a construction stated in Definition 2.4. Hence notation in Definition 2.4 will be assumed in Lemma 2.5, as well as Theorems 2.6, 2.7 and 2.8.

Lemma 2.5 *Let $k \geq 3$ be an integer. With the notation in Definition 2.4, each of the following holds.*

(i) *If each of F_1, F_2 and F_3 is $(k-1)$-cutwidth critical, Then $c(T) = k$.*
(ii) *If each of F_1 and F_2 is $(k-1)$-cutwidth critical, and $F_3 \cong K_{1,2k-3}$, then $c(T) = k$.*
(iii) *If each of F_1 is $(k-1)$-cutwidth critical, and $F_2 \cong F_3 \cong K_{1,2k-3}$, then $c(T) = k$.*

Proof (i) Clearly, $i_0 < 2k-1$ because of $c(K_{2k-1}) = k$ by (Zhang and Lin 2012). Fix j with $1 \leq j \leq 3$. Since F_j is $(k-1)$-cutwidth critical, $c(T_j) \leq k-2$. Let f_1, f_2, f_3 be optimal labelings of T_1, F_2 and T_3, such that $c(T_1, f_1) \leq k-2$, $c(F_2, f_2) \leq k-1$ and $c(T_3, f_3) \leq k-2$, respectively. Define $f : V(T) \to \overline{|V(T)|}$ to be the labeling by the order of (f_1, f_2, f_3), as follows:

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in V(T_1) \\ f_2(v) + |V(T_1)| & \text{if } v \in V(F_2) \\ f_3(v) + |V(T_1)| + |V(F_2)| & \text{if } v \in V(T_3) \end{cases}.$$

By (3), we have $c(T) \leq c(T, f) = k$. On the other hand, it is routine to verify that, for any distinct $v_{i_1}, v_{i_2} \in N_T(v_0)$, $c(T(v_0; v_{i_1}, v_{i_2})) \geq k-1$, and so by Lemma 2.2(iii), we have $c(T) \geq k$. Thus $c(T) = k$.

The proofs for (ii) and (iii) are similar by utilizing the fact that $K_{1,2k-3}$ is a $(k-1)$-cutwidth critical tree, and so they will be omitted. □

Theorem 2.6 *If each of F_1, F_2 and F_3 is $(k-1)$-cutwidth critical, then T is k-cutwidth critical.*

Proof By Lemma 2.5, to show that T is k-cutwidth critical, it remains to show that, for any $T' \in \mathcal{M}(T)$, $c(T') \leq k-1$. Pick $T' = T - x \in \mathcal{M}(T)$ for some $x \in D_1(T)$. As $D_1(T) \subseteq \cup_{j=1}^{i_0} D_1(T_j)$, we have $x \in D_1(T_j)$ for some j with $1 \leq j \leq i_0$. We may assume that $x \in D_1(T_2)$ if $1 \leq j \leq 3$; and $x \in D_1(T_4)$ if $j \leq 4$. Thus we always have $x \in D_1(F_2)$. Since F_2 is $(k-1)$-cutwidth critical, each of $c(T_1)$, $c(F_2 - xy)$ and $c(T_3)$ is at most $k-2$. With an argument similar to the above, a labeling f of T' with $c(T', f) \leq k-1$ can be found, and so $c(T') \leq k-1$. This proves that T is k-cutwidth critical. □

Theorem 2.7 *Let $k \geq 3$ be an integer. If, with the notation in Definition 2.4, for some j with $1 \leq j \leq 3$, we have $T_j + v_0 v_j \cong K_{1,2k-3}$, and if each F_i, $i \in \{1, 2, 3\} - \{j\}$, is $(k-1)$-cutwidth critical, then T is k-cutwidth critical.*

Proof Without loss of generality, we assume that $T_1 + v_0 v_1 = K_{1,2k-3}$. By Lemma 2.5(ii), it suffices to show that for any $T' \in \mathcal{M}(T)$, $c(T') \leq k-1$. Pick $T' = T - x \in \mathcal{M}(T)$ for some $x \in D_1(T)$. If $x \in D_1(T_j)$ for some $j \geq 2$, then using the same arguments as in the proof of Theorem 2.6, we conclude that $c(T') \leq k-1$. Hence we assume that $x \in D_1(T_1)$, and so $T_1 + v_0 v_1 - x \cong K_{2k-4}$ with cutwidth $k-2$. Since

F_2 and F_3 are $(k-1)$-cutwidth critical, we have $c(T_2) \leq k-2$ and $c(T_3) \leq k-2$. Let f_1, f_2, f_3 be optimal labelings of T_1, F_2 and T_3, respectively. Define a labeling f of T to be the labeling by the order of $(V(T_1), V(F_2), V(T_3))$, as follows:

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in V(T_1) \\ f_2(v) + |V(T_1)| & \text{if } v \in V(F_2) \\ f_3(v) + |V(T_1)| + |V(F_2)| & \text{if } v \in V(T_3) \end{cases}.$$

As $c(T_2) \leq k-2$ and $c(T_3) \leq k-2$, we must have

$$\begin{aligned} \min\{f(v) : v \in V(T_2')\} &= \max\{f(v) : v \in V(T_1)\} + 1, \\ \min\{f(v) : v \in V(T_2)\} &= \max\{f(v) : v \in V(F_2 - V(T_2))\} - 1, \\ \max\{f(v) : v \in V(T_2)\} &= \max\{f(v) : v \in V(F_2)\} = \min\{f(v) : v \in V(T_3)\} - 1. \end{aligned} \tag{5}$$

To apply (3) to estimate $c(T, f)$, we present the embedding of $V(T)$ onto a path P_n with $n = |V(T)|$. Let $f(v_0) = q_0$ and denote $V(T_1) = \{u_1, \ldots, u_{i_1}\}$, $V_1(F_2 - V(T_2)) = \{v : v \in V(F_2 - V(T_2)), f(v) < q_0\} = \{u_{i_1+1}, \ldots, u_{i_1+q}\}$, $V_2(F_2 - V(T_2)) = \{v : v \in V(F_2 - V(T_2)), f(v) \geq q_0\} = \{u_{i_1+q+1}, \ldots, u_{i_2}\}$, $V(T_2) = \{u_{i_2+1}, \ldots, u_{i_3}\}$, $V(T_3) = \{u_{i_3+1}, \ldots, u_{i_4}\}$. Then, f can be viewed as an embedding ordering π of vertices of T on the path P_n:

$$u_1, u_2, \ldots, u_{i_1}, u_{i_1+1}, \ldots, u_{q_0-1}, u_{q_0}, \ldots, u_{i_2}, u_{i_2+1}, \ldots, u_{i_3}, u_{i_3+1}, \ldots, u_{i_4}.$$

By (3), for the restriction of π on $V(F_2 - V(T_2))$, we have $|\nabla(S_j)| \leq k-2$ for any $j < q_0$, and $|\nabla(S_j)| \leq k-3$ for any $j \geq q_0$. Now we consider the embedding ordering π' of vertices of $T' = T - x$, and let $x = u_l$ $(1 < l < i_1)$. Since $T_1 + v_0v_1 - x = K_{1,2k-4}$, we have $c(T_1 + v_0v_1 - x) = k-2$. On the basis of π, we arrange an embedding ordering π' of vertices of $T - x$ as follows:

$$\begin{aligned} &u_{i_4}, \ldots, u_{i_3+1}, u_1, u_2, \ldots, u_{l-1}, u_{l+1}, \ldots, u_{i_1}, u_{i_2}, \ldots, \\ &u_{q_0}, u_{q_0-1}, \ldots, u_{i_1}, u_{i_2}, u_{i_2+1}, \ldots, u_{i_3}, \end{aligned}$$

and define a labeling f' of T' according to π'. Since $c(T_1 - x) = k-2$, $c(T_2) \leq k-2$ and $c(T_3) \leq k-2$, the cardinality of any maximum cut is at most $k-1$ in π', and so by (3), $c(T - x, f') = k-1$. Thus $c(T - x) \leq k-1$,. This proves that T is k-cutwidth critical. □

The argument used in the proof of Theorem 2.7 can be further applied to prove the Theorem 2.8 below. Its detailed proof will be omitted.

Theorem 2.8 *Let $k \geq 3$ be an integer. If, with the notation in Definition 2.4, for some i, j with $1 \leq i < j \leq 3$, we have $T_j + v_0v_j \cong K_{1,2k-3}$, and if F_ℓ, $\ell \in \{1, 2, 3\} - \{i, j\}$, is $(k-1)$-cutwidth critical, then T is k-cutwidth critical.*

Following (Chung et al. 1985), let $\mathcal{T}_d(k)$ denote the set of all trees T such that $V(T) = D_1(T) \cup D_d(T)$ and $c(T) = k$. We shall use $T_d(k)$ to denote a member in $\mathcal{T}_d(k)$.

Theorem 2.9 (Chung et al. (1985)) *Each of the following holds.*

(i) $\mathcal{T}_3(1) = \{K_2\}$, $\mathcal{T}_3(2) = \{K_{1,3}\}$, *and* $\mathcal{T}_3(3) = \{H_2\}$.
(ii) *For* $k > 1$, *any* $T \in \mathcal{T}_3(k)$ *can be formed from the disjoint union of three (not necessarily distinct) trees* $T_1, T_2, T_3 \in \mathcal{T}_3(k-1)$ *by identifying a pendant vertex in each of* T_1, T_2 *and* T_3 *to form a degree 3 vertex* v_0 *in* T. *(The vertex* v_0 *is called the identified vertex of* T.*)*

Corollary 2.10 $T_3(k)$ *is* k*-cutwidth critical.*

Proof By definition, $c(T_3(k)) = k$. By Theorems 1.2 and 2.9(i), $T_3(k)$ is k-cutwidth critical for $1 \le k \le 3$. Now assume that $k > 3$ and for all $k' < k$, $T_3(k')$ is k'-cutwidth critical. By Theorem 2.9(ii) and 2.3, $T_3(k+1)$ is k-cutwidth critical. Thus the corollary is proved by induction. □

Definition 2.11 Let $k \ge 3$ be an integer.

(i) A graph G is *minimally homeomorphic* to a graph H if G is homeomorphic to H and G is homeomorphically minimal.
(ii) *(Definition of the family* $\mathcal{T}(k)$*)* Take a $T_3(k-1) \in \mathcal{T}_3(k-1)$. For $1 \le r \le 3$ let $\tilde{T}_r \cong T_3(k-1)$ and x_0^r be the identified vertex of $\tilde{T}_r$. For each $4 \le s \le 6$, let $\tilde{T}_s \cong T_3(k-1)$ and x_0^{3+s} be a pendent vertex of $\tilde{T}_s$. For $7 \le q \le 9$, let $\tilde{T}_q \cong K_{1,2k-3}$ and x_0^q be a pendent vertex of $\tilde{T}_q$. Define $S = \{x_0^i : 1 \le i \le 9\}$. Thus

$$\textit{For each } i \textit{ with } 1 \le i \le 9, \tilde{T}_i \textit{is } (k-1) - \textit{cutwidth critical}. \tag{6}$$

Let $L \cong K_{1,3}$ be a star $V(L) = \{u_0, u_1, u_2, u_3\}$, where $N_L(u_0) = \{u_1, u_2, u_3\}$. Let $\mathcal{T}(k)$ denote the family of trees such that $\tilde{T} \in \mathcal{T}(k)$ if and only if there exist some (i, j, p) with $1 \le i, j, p \le 9$ such that $\tilde{T}$ is minimally homeomorphic to the tree $T(i, j, p)$ obtained from the disjoint union of $\tilde{T}_i$, $\tilde{T}_j$, $\tilde{T}_p$ and L by identifying u_1 and x_0^i, u_2 and x_0^j, u_3 and x_0^p respectively. The vertices x_0^i, x_0^j, x_0^p are called *the identified vertices of* $\tilde{T}$.

Theorem 2.12 *For any* $\tilde{T} \in \mathcal{T}(k)$, $\tilde{T}$ *is a* k*-cutwidth critical tree.*

Proof We use the notation in Definition 2.11(ii). Let $T = T(i, j, p)$ and $\tilde{T} \in \mathcal{T}(k)$ with x_0^i, x_0^j, x_0^p being the identified vertices, for some $1 \le i \le j \le p \le 9$.

If $i \ge 4$, then x_0^i, x_0^j, x_0^p are in $D_2(T(i, j, p))$, and so by Definition 2.11, $\tilde{T} = T(z_0; \tilde{T}_i, \tilde{T}_j, \tilde{T}_p)$. It follows by Theorem 2.3 that $\tilde{T}$ is k-cutwidth critical.

Hence we assume that $i \le 3$. There are six cases to consider: (1) $x_0^i \in \{x_0^1, x_0^2, x_0^3\}$, $x_0^j \in \{x_0^4, x_0^5, x_0^6\}$, $x_0^p \in \{x_0^7, x_0^8, x_0^9\}$; (2) $x_0^i \in \{x_0^1, x_0^2, x_0^3\}$, $x_0^j, x_0^p \in \{x_0^4, x_0^5, x_0^6\}$; (3) $x_0^i \in \{x_0^1, x_0^2, x_0^3\}$, $x_0^j, x_0^p \in \{x_0^7, x_0^8, x_0^9\}$; (4) $x_0^i, x_0^j \in \{x_0^1, x_0^2, x_0^3\}$, $x_0^p \in \{x_0^4, x_0^5, x_0^6\}$; (5) $x_0^i, x_0^j \in \{x_0^1, x_0^2, x_0^3\}$, $x_0^p \in \{x_0^7, x_0^8, x_0^9\}$; (6) $x_0^i, x_0^j, x_0^p \in \{x_0^1, x_0^2, x_0^3\}$.

As the proof arguments are similar in each of these six cases, it suffices to show case (1). Without loss of generality, we assume that $i = 1$, $j = 4$ and $p = 7$. Let $x \in N_{\tilde{T}_4}(x_0^4)$ and $y \in N_{\tilde{T}_7}(x_0^7)$. Thus $\tilde{T} = T - \{u_0x_0^4, x_0^4x, u_0x_0^7, x_0^7y\} + \{u_0x, u_0y\}$. Since $\tilde{T}_1, \tilde{T}_4 \in \mathcal{T}_3(k-1)$ and $\tilde{T}_7 \cong K_{1,2k-3}$, both $\tilde{T}_1$ and $\tilde{T}_4$ are $(k-1)$-critical. It follows by Lemma 2.5 that $c(\tilde{T}) = k$.

It remains to show that, for any maximal proper subtree $\tilde{T}' \in \mathcal{M}(\tilde{T})$, $c(\tilde{T}') \leq k-1$. Let $\tilde{T}' = \tilde{T} - z$ for some $z \in D_1(\tilde{T})$. Then we have these possibilities. (1A) $z \in V(\tilde{T}_1)$; (1B) $z \in V(\tilde{T}_4 - u_0)$; (1C) $z \in V(\tilde{T}_7 - u_0)$.

As the arguments for each of (1A), (1B) and (1C) are similar, we only present the proof when (1A) holds. As $\tilde{T}_1 \in \mathcal{T}_3(k-1)$, by Theorem 2.9, $\tilde{T}_1$ is formed by identifying a pendent vertex in three copies $T_3^{(1)}(k-2), T_3^{(2)}(k-2), T_3^{(3)}(k-2) \in \mathcal{T}_3(k-2)$ with the identified vertex x_0^1. Let $N_{\tilde{T}_1}(x_0^1) = \{v_0', v_0'', v_0'''\}$ with $v_0' \in V(T_3^{(1)}(k-2))$, $v_0'' \in V(T_3^{(2)}(k-2))$, $v_0''' \in V(T_3^{(3)}(k-2))$. Without loss of generality, let $z \in D_1(T_3^{(2)}(k-2))$. As $c(\tilde{T}_1 - z) = k-2$, we have $c(T_3^{(2)}(k-2) - z) = k-3$. Using the $(k-2)$-cutwidth embeddings of $T_3'(k-2) - x_0^1$ and $T_3'''(k-2) - x_0^1$ and a $(k-3)$-cutwidth embedding of $T_3''(k-2) - z - x_0^1$, there exists a $(k-2)$-cutwidth embedding of $\tilde{T}_1$ defined as the labeling by the order $(V(T_3'(k-2)) - x_0^1, V(T_3''(k-2) - z - x_0^1), x_0^1, V(T_3'''(k-2)) - x_0^1)$.

Now using the $(k-1)$-cutwidth embeddings in $V(\tilde{T}_4 - u_0)$ and $V(\tilde{T}_7 - u_0)$, we obtain a labeling f of $\tilde{T} - \{u_0x_0^1, u_0x_0^4, u_0x_0^7\}$ by the order $(V(\tilde{T}_4 - u_0), V(T_3'(k-2)) - x_0^1, V(T_3''(k-2) - z - x_0^1), x_0^1, u_0, V(T_3'''(k-2)) - x_0^1, V(\tilde{T}_7 - u_0))$. Note that f is also a labeling of $\tilde{T}$. As one can put edges $x_0^1v_0', x_0^1v_0'', x_0^1v_0''', u_0x_0^1, u_0x_0^4, u_0x_0^7$ back. Obviously, the congestion is $k-1$ in the embedding ordering, which indicates $c(\tilde{T}') \leq k-1$. Consequently, $\tilde{T}$ is $k-$cutwidth critical. □

By Theorem 2.6, Lemma 2.2(iv) and Theorem 2.12 can be generalized to be a method to construct k-cutwidth trees.

Theorem 2.13 *Let T_1, T_2, T_3 be $(k-1)$-cutwidth trees, where at least one of them is critical, and $v_i \in V(T_i)$ $(i = 1, 2, 3)$, u_i $(i = 1, 2, 3)$ be a pendent vertex in $K_{1,3}$. If tree T is formed by identifying v_i and u_i $(i = 1, 2, 3)$ respectively, then T is a k-cutwidth tree.*

3 Proof of Theorem 1.3

Throughout this section, for two graphs G and H, we write $H \subseteq G$ to mean that H is a subgraph of G. Let $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_{18}\}$. To prove Theorem 1.3, we shall first show that every tree in $\mathcal{T}$ is 4-cutwidth critical. In (Zhang and Lin 2012), it is shown that

Lemma 3.1 *Each of the following holds.*

(i) (Zhang and Lin (2012)) *For $1 \leq i \leq 12$, every τ_i is 4-cutwidth critical.*
(ii) *For $13 \leq i \leq 18$, every τ_i is 4-cutwidth critical.*

Proof (ii) As it is similar to prove that for each i with $13 \le i \le 18$, τ_i is 4-cutwidth critical, we only present the proof for τ_{13}. Let H_2 be the graph depicted in Fig. 3 (see Appendix). The only vertex v_0 in H_2 that is of distance at most 2 to all vertices of H_2 is called the center of H_2 (in Fig. 1, v_0 is the vertex with label 6). By definition, $H_2 \cong T_3(3)$. Let $T_3^{(1)}(3), T_3^{(2)}(3), T_3^{(3)}(3)$ be three copies of $T_3(3)$ with centers $v_0^{(1)}$, $v_0^{(2)}$ and $v_0^{(3)}$ respectively. Let $T = K_{1,3}$ with $D_1(T) = \{u_1, u_2, u_3\}$. Obtain τ_{13} from the disjoint union of $T_3^{(1)}(3), T_3^{(2)}(3), T_3^{(3)}(3)$ and T by identifying u_i with $v_0^{(i)}$, for each $1 \le i \le 3$. By Theorem 2.12, τ_{13} is 4-cutwidth critical, and so the lemma follows. □

Lemma 3.2 *Let $T = T(z_0, T_1, T_2, \ldots, T_t)$ be a tree as defined in Definition 2.1.*

(i) *If $t = 3$ and if $c(T_1) < 3$, T_j and $T_j - z_0$ are k-cutwidth critical with $k \le 3$, for each $j \ge 2$, then $c(T) \le 3$.*
(ii) *If $t \ge 3$ and if $c(T_j) \ge 3$ for $j = 1, 2, 3$, then $c(T) \ge 4$.*

Proof (i) is a consequence of Theorem 2.3. To prove (ii), let $N_T(z_0) = \{v_0^i : v_0^i \in V(T_i), 1 \le i \le t\}$. Since $t \ge 3$, any $T(z_0; v_0^i, v_0^j)$ contains at least one subtree T_k $(k \ne i, j)$ for $1 \le i, j \le t$, resulting in $c(T(z_0; v_0^i, v_0^j)) \ge 3$ by the assumption. Thus by Lemma 2.2(iii), $c(T) \ge 4$, completing the proof. □

Proof of Theorem 1.3 By Lemma 3.1, it suffices to show that every 4-cutwidth critical tree must be in $\mathcal{T}$. Let T be a 4-cutwidth critical tree. By Definition 1.1, T is homeomorphically minimal, and so for any $v \in V(T) - D_1(T)$, $\Delta(T) \ge d_T(v) \ge 3$. If $\Delta(T) \ge 7$, then $\tau_1 \cong K_{1,7} \subseteq T$, and so $T = \tau_1$ by the minimality of T. By Lemma 2.2(i), we may assume that

$$3 \le \Delta(G) \le 6, \text{ and } T \text{ is not homeomorphic to a caterpillar.} \tag{7}$$

Let $\Delta = \Delta(T)$. Pick any $v_0 \in V(T) - D_1(T)$ with $\Delta = d_T(v_0) \ge 3$, and denote $N_T(v_0) = \{v_1, v_2, \ldots, v_\Delta\}$. Then by Lemma 2.2(iii) and since T is 4-cutwidth critical, we have

$$\text{For any } v', v'' \in N_T(v_0),\ c(T(v_0; v', v'')) = 3. \tag{8}$$

For each i with $1 \le i \le \Delta$, define T_i to be the largest subtree of T with $V(T_i) \cap (N_T(v_0) \cup \{v_0\}) = \{v_0, v_i\}$.

Case 1 $\Delta = 3$. For each T_i with $c(T_i) = 3$, the 3-cutwidth critical trees H_1 or H_2 (see Figure 1) must be contained in T_i. On the other hand, $\Delta(T) = d_T(v_0)$ must be 3 in this case, otherwise it is not hard to verify that $c(T) > 4$ by Lemma 3.2, a contradiction to $c(T) = 4$. Hence T must be one of $\{\tau_3, \ldots, \tau_6, \tau_{13}, \ldots, \tau_{18}\}$ by the minimality.

Case 2 $\Delta = 4$. Note that $c(T(v_0; v_i, v_j)) = 3$ for any two neighbors v_i, v_j of v_0 $(1 \le i < j \le 4)$, and the degree of v_0 is two in subtree $T(v_0; v_i, v_j)$. If one neighbor of v_0, say v_1, is a pendant vertex of T, then the other subtrees T_2, T_3, T_4 must have cutwidth 3, thus the subtree T_1 (namely the edge v_0v_1) can be deleted, which is reduced to be Case 1 leading to $T - v_1 \in \{\tau_3, \ldots, \tau_6, \tau_{13}, \ldots, \tau_{18}\}$, contradicting that T is 4-critical. So, we may assume that all neighbors v_1, v_2, v_3, v_4 of v_0 are not pendant. Due to that T is critical, among all subtrees $T(v_0; v_i, v_j)(1 \le i < j < 4)$, there must be one being minimal (if the degree two vertex v_0 is ignored, then it is critical). Therefore, at least one subtree $T(v_0; v_i, v_j)$ is an H_2 with v_0 as a subdivision vertex; and the subtree T_i and T_j in the remaining part may contain an H_1. By the minimality, T is one of $\{\tau_7, \tau_8, \tau_9\}$.

Case 3 $\Delta = 5$. If all neighbors v_i of v_0 have $d_T(v_i) \geq 3 (i = 1, 2, 3, 4, 5)$, then τ_2 is included in T and thus $T = \tau_2$ by the minimality. If only one neighbor of v_0, say v_1, is pendant, then by $c(T_1 \bigcup T_i \bigcup T_j) = 3$ $(2 \leq i < j \leq 5)$, the edge $v_0 v_1$ can be deleted without effect on $c(T) = 4$, i.e., $c(T - v_1) = 4$, which can be reduced to be Case 2, a contradiction to the assumptions. By $c(T(v_0; v_i, v_j)) = 3$, it is impossible that v_0 has three or more pendant neighbors. So, we may assume that there are two neighbors of v_0 being pendant. By the fact that $c(T(v_0; v_i, v_j)) = 3$ for any two neighbors v_i, v_j of $v_0 (1 \leq i < j \leq 5)$ and that T is critical, it can be seen that there must be a subtree $T(v_0; v_i, v_j)$ being an H_2 containing those two pendant neighbors of v_0. And the subtree T_i or T_j in the remaining parts may contain an H_1. Therefore, T is one of $\{\tau_{10}, \tau_{11}, \tau_{12}\}$ by the minimality.

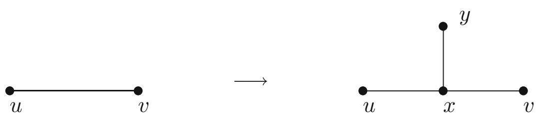
Case 4 $\Delta = 6$. By using the fact that $c(T(v_0; v_i, v_j)) = 3$ for any i and j $(1 \leq i < j \leq 6)$, it can be deduced that T must contain a subtree in Case 2 or Case 3, which contradicts that T is critical. This establishes the proof. □

4 Remarks

The paper first investigates combinatorial structures of k-cutwidth $(k > 1)$ critical trees, from which one can obtain some methods to construct k-cutwidth critical trees, and then characterizes the set of 4-cutwidth critical trees, which corrects the shortcomings of that of (Zhang and Lin 2012) by giving six new 4-cutwidth critical trees. As to more methods to construct the critical trees with cutwidth k, Zhang and Lin (2012) gave other two results: (1) star $K_{1,2k-1}$ is a critical tree with cutwidth k; (2) If tree T_1' is obtained from star $K_{1,2k-3}$ by replacing every edge uv of it with tree shown in Fig. 2, where $d_{K_{1,2k-3}}(u) = 2k - 3$, $d_{K_{1,2k-3}}(v) = 1$, x and y are new vertices and y is a new pendant vertex. Then tree T_1' is k-cutwidth critical.

From these, we think that we have found all ways of constructing k-cutwidth critical trees for any fixed integer k $(k > 1)$; In addition, for any k-cutwidth critical tree, there must exist an optimal embedding ordering π of vertices $v_1, v_2, \ldots, v_n$ arranged on path P_n such that there is a unique maximum cut $\nabla(S_j)$ in π (it is true for $T_3(k)$). These will be our emphases to study in the future works. Other further tasks are to characterize the set of 4-cutwidth nontree critical graphs which includes K_4 and all 5-critical graphs.

Fig. 2 Definition of T'



Appendix

See Figs. 3 and 4.

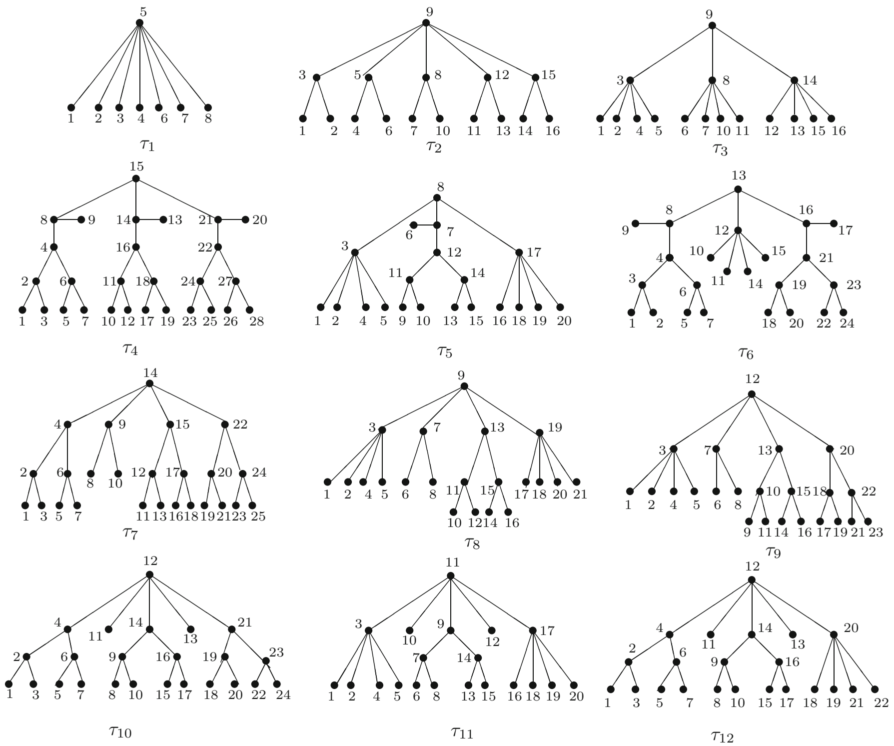


Fig. 3 The 4-cutwidth critical trees in Zhang and Lin (2012)

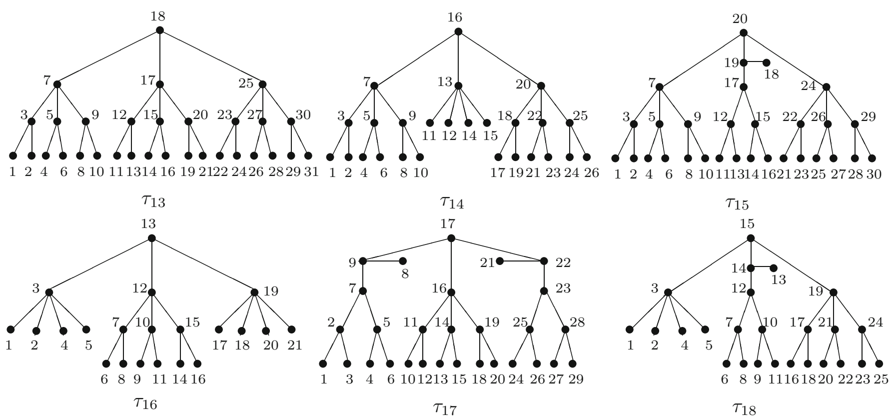


Fig. 4 New 4-cutwidth critical trees

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