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3-dynamic coloring and list 3-dynamic coloring of $K_{1,3}$-free graphs

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ABSTRACT

For positive integers k and r, a (k, r)-coloring of a graph G is a proper coloring of the vertices with k colors such that every vertex of degree i will be adjacent to vertices with at least $\min\{i, r\}$ different colors. The r-dynamic chromatic number of G, denoted by $\chi_r(G)$, is the smallest integer k for which G has a (k, r)-coloring. For a k-list assignment L to vertices of G, an (L, r)-coloring of G is a coloring c such that for every vertex v of degree i, $c(v) \in L(v)$ and v is adjacent to vertices with at least $\min\{i, r\}$ different colors. The list r-dynamic chromatic number of G, denoted by $\chi_{L,r}(G)$, is the smallest integer k such that for every k-list L, G has an (L, r)-coloring.

In this paper, the behavior and bounds of 3-dynamic coloring and list 3-dynamic coloring of $K_{1,3}$-free graphs are investigated. We show that if G is $K_{1,3}$-free, then $\chi_{L,3}(G) \le \max\{\chi_L(G)+3, 7\}$ and $\chi_3(G) \le \max\{\chi(G)+3, 7\}$. The results are best possible as 7 cannot be reduced.

1. Introduction

Graphs in this paper are simple and finite. Undefined terminologies and notations are referred to [3]. For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of vertices adjacent to v in G, and let $d_G(v) = |N_G(v)|$ denote the degree of v. Denote $N_G[v] = N_G(v) \cup \{v\}$ to be the closed neighborhood of v. When G is understood from the context, we often use $N(v)$, $d(v)$ and $N[v]$ for $N_G(v)$, $d_G(v)$ and $N_G[v]$, respectively. For an integer $i \ge 0$, let $D_i(G)$ denote the set of all vertices of degree i in G; vertices in $D_i(G)$ are called i-vertices of G. For a graph G, $\delta(G)$, $\Delta(G)$, and $\chi(G)$ denote the minimum degree, the maximum degree and the chromatic number of G, respectively.

For a vertex set $V_0 \subseteq V(G)$, denote $G[V_0]$ to be the induced subgraph of V_0 in G. For a vertex $v \in V(G)$, we abbreviate $G[V(G) - \{v\}]$ to $G - v$. We call a vertex v is *locally connected* if $G[N_G(v)]$ is connected. The *clique number* of G, denoted by $\omega(G)$, is the maximum number k such that G contains a subgraph isomorphic to K_k.

For an integer $k > 0$, we define $\bar{k} = \{1, 2, \ldots, k\}$. If $c : V(G) \mapsto \bar{k}$ is a mapping and $S \subseteq V(G)$, then $c(S) = \{c(u) \mid u \in S\}$. For positive integers k and r, a (k, r)-*coloring* of a graph G is a mapping $c : V(G) \to \bar{k}$ such that both of the following hold:
(C1) if $u, v \in V(G)$ are adjacent vertices in G, then $c(u) \ne c(v)$; and
(C2) for every $v \in V(G)$, $|c(N_G(v))| \ge \min\{|N_G(v)|, r\}$.
When G has a (k, r)-coloring, we say that G is (k, r)-colorful. The r-*dynamic chromatic number* of G, denoted by $\chi_r(G)$, is the smallest k such that G is (k, r)-colorful. By the definition, we observe that

$$\chi(G) = \chi_1(G) \le \chi_2(G) \le \cdots \le \chi_{\Delta(G)}(G) = \chi_{\Delta(G)+i}(G) \le |V(G)|.$$

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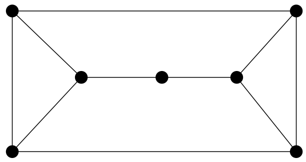

Fig. 1. A 3-colorable graph with 3-dynamic chromatic number 7.

Let L be an assignment such that assigns to every $v \in V(G)$ a list $L(v)$ of colors available at v. For an integer $k > 0$, an L is a *k-list* if for any $v \in V(G)$, $|L(v)| = k$. An *L-coloring* is a proper coloring c such that $c(v) \in L(v)$, for every $v \in V(G)$. The *list chromatic number* of G, denoted by $\chi_L(G)$, is the minimum number k such that for every k-list L, G admits an L-coloring. Given an assignment L and a positive integer r, an *(L, r)-coloring* c is an L-coloring such that $|c(N_G(v))| \geq \min\{|N_G(v)|, r\}$ for every $v \in V(G)$. The *list r-dynamic chromatic number* of G, denoted by $\chi_{L,r}(G)$, is the minimum number k such that for every k-list L, G has an (L, r)-coloring. By its definition, we observe that $\chi_r(G) \leq \chi_{L,r}(G)$, and

$$\chi_L(G) = \chi_{L,1}(G) \leq \chi_{L,2}(G) \leq \cdots \leq \chi_{L,\Delta(G)}(G) = \chi_{L,\Delta(G)+i}(G) \leq |V(G)|.$$

The study of r-dynamic colorings started in [10,12]. The r-dynamic chromatic number of certain graph families has been determined, as seen in [4,5,7,9,11], among others. It has been indicated in [10,12] that $\chi_2(G) - \chi(G)$ can be arbitrarily large. In [9], it is initiated the study of finding graph families $\mathcal{F}$ such that $\chi_2(G) - \chi(G)$ is bounded by a constant for all graphs in $\mathcal{F}$. For positive integers r and s, a graph G is *(r, s)-normal* if $\chi_r(G) - \chi(G) \leq s$. A $(2, 0)$-normal graph is also called a normal graph in [9]. When $r = 2$, it is conjectured in [12] that every regular graph is $(2, 2)$-normal. Let H be a graph. A graph G is *H-free* if G does not have an induced subgraph isomorphic to H. In particular, a $K_{1,3}$-free graph is also called a claw-free graph. The following is proved in [9].

Theorem 1.1 (*Theorem 4.2 of [9]*)**.** *Let G be a claw free graph. Each of the following holds.*

(i) *G is $(2, 2)$-normal.*

(ii) *If G is connected, then $\chi_2(G) = \chi(G) + 2$ if and only if G is a cycle of length 5 or an even cycle of length not a multiple of 3.*

More results to investigate the difference $\chi_2 - \chi$ can be found in [1,2,6,9], among others. Motivated by Theorem 1.1, in this paper, we obtain the following results.

Theorem 1.2. *Let G be a claw-free graph. Then $\chi_{L,3}(G) \leq \max\{\chi_L(G) + 3, 7\}$.*

Theorem 1.3. *Let G be a claw-free graph. Then $\chi_3(G) \leq \max\{\chi(G) + 3, 7\}$.*

Both Theorems 1.2 and 1.3 are best possible in some sense. Let G be the graph depicted in Fig. 1. It is routine to verify that $\chi_L(G) = \chi(G) = 3$ while $\chi_{L,3}(G) = \chi_3(G) = 7$.

Necessary preliminaries are presented in Section 2. Our main results are proved in Section 3.

2. Preliminaries

In this section, we present lemmas and observations that will be needed in the arguments to prove the main results.

Lemma 2.1 (*[1] and [9]*)**.** *For positive integer $r \geq 2$, each of the following holds:*

$$\chi_{L,r}(C_n) = \chi_r(C_n) = \begin{cases} 5, & \text{if } n = 5; \\ 3, & \text{if } n \equiv 0 \,(\text{mod } 3); \\ 4, & \text{otherwise.} \end{cases}$$

We shall adopt the idea of partial colorings from [13]. Throughout this section, if L is a list of G and H is a subgraph of G, we shall adopt the convention of using L to denote the restriction of L to $V(H)$. If $V' \subseteq V(G)$, we define a mapping c on V' to be a *partial (L, r)-coloring* of G if c is an (L, r)-coloring of $G[V']$. The *support* of c, also denoted by $S(c)$, equals V'. If $L(v) = \overline{k}$ for every $v \in V(G)$, then a *partial (L, r)-coloring* of G is also called a *partial (k, r)-coloring* of G. If c_1 and c_2 are two partial (L, r)-colorings of G such that $S(c_1) \subseteq S(c_2)$ and such that for any $v \in S(c_1)$, $c_1(v) = c_2(v)$, then c_2 is an *extension* of c_1. Given a partial (L, r)-coloring c with $S(c) = V'$, for each $v \in V - V'$, define $\{c(v)\} = \emptyset$; and for every vertex $v \in V$, we extend the definition of $c(N_G(v)) = \cup_{z \in N_G(v)}\{c(z)\}$, and define

$$c[v] = \begin{cases} \{c(v)\}, & \text{if } |c(N_G(v))| \geq r; \\ \{c(v)\} \cup c(N_G(v)), & \text{otherwise.} \end{cases}$$

By definition, $|c[v]| \leq r$ for any $v \in V$. We have the following observation.

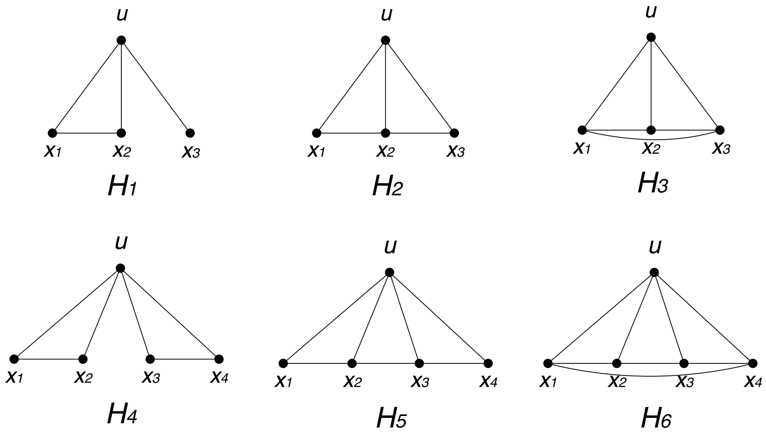

Fig. 2. Local structures in claw-free graphs.

Observation 2.2. *Let c be a partial (L, r)-coloring of G with support $S(c)$. For any $u \notin S(c)$, and for any $v \in N_G(u)$, by the definition of $c[v]$, we have $|c[v]| \leq \min\{d(v), r\}$ and $c[v]$ represents the colors that cannot be used as $c(u)$ if one wants to extend the support of c to include u. In other words, the colors in $L(u) - \cup_{v\in N_G(u)} c[v]$ are available colors to define $c(u)$ in extending the support of c from $S(c)$ to $S(c) \cup \{u\}$.*

Lemma 2.3. *Let G be a simple graph with $u \in D_1(G)$. Let k be an integer such that $k \geq 4$. Then both of the following hold:*
(i) *If $\chi_{L,3}(G - u) \leq k$, then $\chi_{L,3}(G) \leq k$;*
(ii) *If $\chi_3(G - u) \leq k$, then $\chi_3(G) \leq k$.*

Proof. The proof for Part (ii) is similar to that for Part (i). Thus we only prove Part (i) and omit the proof for Part (ii).
Assume that $N_G(u) = \{v\}$. Let L be a k-list of G. Since $\chi_{L,3}(G - u) \leq k$, let c be a partial $(L, 3)$-coloring of G with $S(c) = V(G) - u$. By Observation 2.2, $|c[v]| \leq 3$, and c could be extended to an $(L, 3)$-coloring of G by coloring u with a color in $L(u) - c[v]$. □

Lemma 2.4. *Let G be a claw-free graph with $\delta(G) = 2$. Let $u \in D_2(G)$ with $N_G(u) = \{v_1, v_2\}$. Denote $G' = G - u + v_1v_2$ if $v_1v_2 \notin E(G)$, or $G' = G - u$ if $v_1v_2 \in E(G)$. Let k be an integer such that $k \geq 7$. Then both of the following hold:*
(i) *If $\chi_{L,3}(G') \leq k$, then $\chi_{L,3}(G) \leq k$;*
(ii) *If $\chi_3(G') \leq k$, then $\chi_3(G) \leq k$.*

Proof. As the proof for Part (ii) is similar to that for Part (i), we only present the proof for art (i) and omit that for Part (ii).
Let L be a k-list of G. Let L' be a k-list of G' where $L'(v) = L(v)$ for any $v \in V(G')$. Since $\chi_{L,3}(G') \leq k$, let c be an $(L', 3)$-coloring of G'. Define a coloring c_0 on G in the following way: let $c_0(v) = c(v)$, for every $v \in V(G) - u$; and choose $c_0(u) \in L(u) - S_1 - S_2$ where

$$S_i = \begin{cases} \{c_0(v_i)\}, & \text{if } |c_0(N_G(v_i) - \{u\})| \geq 3; \\ \{c_0(v_i)\} \cup c_0(N_G(v_i) - \{u\}), & \text{otherwise} \end{cases}$$

for $i = 1, 2$. Then c_0 is an $(L, 3)$-coloring of G. □

Lemma 2.5 (*Lemma 3.4 of [8]*). *Let G be a claw-free graph with $\delta(G) \geq 3$ and $v \in V(G)$ be a locally connected vertex. Then $G[N_G(v)]$ has a Hamilton path.*

Lemma 2.6. *Define $H_1, H_2, \ldots, H_6$ to be the graphs depicted in Fig. 2. Let G be a claw-free graph with $\delta(G) \geq 3$. Then each of the following holds:*
(i) *For every $u \in D_3(G)$, $G[N_G[u]]$ is isomorphic to one graph in $\{H_1, H_2, H_3\}$;*
(ii) *For every $u \in D_4(G)$, either $G[N_G(u)]$ has K_3 as a subgraph or $G[N_G[u]]$ is isomorphic to one graph in $\{H_4, H_5, H_6\}$;*
(iii) *For every u with degree at least 5, $\chi(G[N_G(u)]) \geq 3$.*

Proof. By the definition of claw-free graphs and Lemma 2.5, we have (i) and (ii). For every u with degree at least 5, it is sufficient to show that $G[N_G(u)]$ contains a K_3 or a C_5. If u is locally connected, then by Lemma 2.5, $G[N_G(u)]$ has a Hamilton path. Since G is claw-free, $G[N_G(u)]$ contains a K_3 or a C_5. If u is not locally connected, then by the definition of claw-free graphs, $G[N_G(u)]$ has exactly two components, both of which are complete graphs. As $|N_G(u)| \geq 5$, at least one component of $G[N_G(u)]$ contains a K_3. $\square$

Lemma 2.7. *Let G be a claw-free graph with $\delta(G) = 3$. Let $u \in D_3(G)$ with $N_G(u) = \{x_1, x_2, x_3\}$, $x_1x_2 \in E(G)$ and $x_1x_3 \notin E(G)$. Then each of the following holds:*

(i) *Let $k = \max\{\chi_L(G) + 3, 7\}$. If $\chi_{L,3}(G - u) \leq k$, then for any k-list L of G, there is a partial $(L, 3)$-coloring c of G with $S(c) = V(G) - u$, such that $|c(\{x_1, x_2, x_3\})| = 3$;*

(ii) *Let $k' = \max\{\chi(G) + 3, 7\}$. If $\chi_3(G - u) \leq k'$, then there is a partial $(k', 3)$-coloring c' of G with $S(c') = V(G) - u$, such that $|c'(\{x_1, x_2, x_3\})| = 3$.*

Proof. The proof for Part (ii) is similar to that for Part (i). Thus we only prove Part (i) and omit the proof for Part (ii).

Denote $H = G[N_G[x_3] - \{u, x_2\}]$. Since G is claw-free, H is a complete graph. Since $\delta(G) \geq 3$ and $x_1x_3 \notin E(G)$, $|V(H)| \geq 2$. Note that $k \geq k' \geq \chi(G) + 3 \geq |V(H)| + 3$.

Let L be an assignment on $V(G)$. Since $\chi_{L,3}(G - u) \leq k$, G has a partial $(L, 3)$-coloring c with $S(c) = V(G) - u$. If $c(x_3) \notin \{c(x_1), c(x_2)\}$, then $|c(\{x_1, x_2, x_3\})| = 3$. Hence we assume that $c(x_3) \in \{c(x_1), c(x_2)\}$.

If $|V(H)| \geq 4$, then we could obtain a desired partial $(L, 3)$-coloring from c by recoloring x_3 with a color in $L(x_3) - c(V(H) \cup \{x_1, x_2\})$.

If $|V(H)| = 3$, then we assume $V(H) = \{x_3, y_1, y_2\}$. Since $d_G(y_1) = d_{G-u}(y_1) \geq 3$, then there exists $z_1 \in N_G(y_1)$ such that $|c(\{x_3, y_2, z_1\})| = 3$. Similarly, there exists $z_2 \in N_G(y_2)$ such that $|c(\{x_3, y_1, z_2\})| = 3$. Define an L-coloring c_0 of $G - u$ in the following way:

$$c_0(v) = c(v) \text{ if } v \neq x_3, \text{ and } c_0(x_3) = a \text{ where } a \in L(x_3) - c(\{x_1, x_2, y_1, y_2, z_1, z_2\}).$$

For $v \in V(G) - \{u, x_2, x_3, y_1, y_2\}$, $|c_0(\{N_{G-u}(v)\})| = |c(\{N_{G-u}(v)\})| \geq \min\{d_{G-u}(v), 3\}$. For y_i $(i = 1, 2)$, $|c_0(\{N_{G-u}(y_i)\})| \geq |\{c(z_i), c(y_{3-i}), a\}| = 3$. For x_2 and x_3, if $x_2x_3 \notin E(G)$, then $|c_0(\{N_{G-u}(x_2)\})| = |c(\{N_{G-u}(x_2)\})| \geq \min\{d_{G-u}(x_2), 3\}$ and $|c_0(\{N_{G-u}(x_3)\})| = |\{c(y_1), c(y_2)\}| = d_{G-u}(x_3) = 2$. If $x_2x_3 \in E(G)$, then $c(x_3) = c(x_1)$ and $c(\{N_{G-u}(x_2)\}) \subseteq c_0(\{N_{G-u}(x_2)\})$. Thus $|c_0(\{N_{G-u}(x_2)\})| \geq |c(\{N_{G-u}(x_2)\})| \geq \min\{d_{G-u}(x_2), 3\}$, and $|c_0(\{N_{G-u}(x_3)\})| = |\{c(x_2), c(y_1), c(y_2)\}| = 3$. Therefore, c_0 is a desired partial $(L, 3)$-coloring.

If $|V(H)| = 2$, then $d_G(x_3) = 3$ and $x_2x_3 \in E(G)$ as $\delta(G) = 3$. Since $c(x_2) \neq c(x_3)$, $c(x_3) = c(x_1)$. We assume that $N_G(x_3) = \{u, x_2, y_1\}$. Since $d_G(y_1) = d_{G-u}(y_1) \geq 3$, then there exists $z_1, z_2 \in N_G(y_1)$ such that $|c(\{x_3, z_1, z_2\})| = 3$. Therefore, we could obtain a required partial $(L, 3)$-coloring from c by recoloring x_3 with a color in $L(x_3) - c(\{x_1, x_2, y_1, z_1, z_2\})$. $\square$

Lemma 2.8. *Let G be the bull graph, where $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_5\}$. For any assignment L to $V(G)$ satisfying $|L(v_i)| = 4$ for $i = 1, 2, 3$ and $|L(v_j)| = 3$ for $j = 4, 5$, there exists an $(L, 3)$-coloring of G.*

Proof. If $L(v_4) \cap L(v_5) \neq \emptyset$, then choose $a \in L(v_4) \cap L(v_5)$. We could find an $(L, 3)$-coloring c of G in the following way: $c(v_1) \in L(v_1) - \{a\}$; $c(v_2) \in L(v_2) - \{c(v_1), a\}$; $c(v_3) \in L(v_3) - \{c(v_1), c(v_2), a\}$; $c(v_4) = c(v_5) = a$. Thus we assume that $L(v_4) \cap L(v_5) = \emptyset$. without loss of generality, we assume that $L(v_4) = \{1, 2, 3\}$ and $L(v_5) = \{4, 5, 6\}$.

If $1 \notin L(v_1)$, we could find an $(L, 3)$-coloring c of G in the following way: $c(v_4) = 1$; $c(v_5) = 6$; $c(v_2) = L(v_2) - \{1, 6\}$; $c(v_3) = L(v_3) - \{1, 6, c(v_2)\}$; $c(v_1) \in L(v_1) - \{6, c(v_2), c(v_3)\}$. If $1 \notin L(v_2)$, we could find an $(L, 3)$-coloring c of G in the following way: $c(v_4) = 1$; $c(v_5) = 6$; $c(v_3) = L(v_3) - \{1, 6\}$; $c(v_1) = L(v_1) - \{1, 6, c(v_3)\}$; $c(v_2) \in L(v_2) - \{6, c(v_1), c(v_3)\}$. If $1 \notin L(v_3)$, we could find an $(L, 3)$-coloring c of G in the following way: $c(v_4) = 1$; $c(v_5) = 6$; $c(v_1) = L(v_1) - \{1, 6\}$; $c(v_2) = L(v_2) - \{1, 6, c(v_1)\}$; $c(v_3) \in L(v_3) - \{6, c(v_1), c(v_2)\}$. Thus, we could assume that $1 \in L(v_1) \cap L(v_2) \cap L(v_3)$. By a similar argument, we could assume that $2, 3, 4 \in L(v_1) \cap L(v_2) \cap L(v_3)$. Therefore, $L(v_i) = \{1, 2, 3, 4\}$ for $i = 1, 2, 3$. Hence we could find an $(L, 3)$-coloring c of G in the following way: $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 4$, $c(v_4) = 1$, $c(v_5) = 5$. $\square$

3. Proof of main results

Proof of Theorem 1.2. Let $k = \max\{\chi_L(G) + 3, 7\}$. By contradiction, we choose a counterexample G to Theorem 1.2 such that

$$\chi_{L,3}(G) > k \text{ with } |V(G)| \text{ minimized.} \tag{1}$$

As $k \geq 7$, Theorem 1.2 holds trivially for all graphs with at most 7 vertices, and so we assume $|V(G)| \geq 8$. By Lemma 2.1, G is not a 2-regular graph. By (1),

$$\text{there exists a } k\text{-list } L \text{ such that } G \text{ does not have an } (L, 3)\text{-coloring.} \tag{2}$$

Claim 1. $\delta(G) \geq 3$.

Suppose $u_1 \in D_1(G)$ and denote $G' = G - u_1$. By (1), $\chi_{L,3}(G') \leq \max\{\chi_L(G') + 3, 7\} \leq k$. By Lemma 2.3(i), $\chi_{L,3}(G) \leq k$, contrary to (1). Therefore, $D_1(G) = \emptyset$.

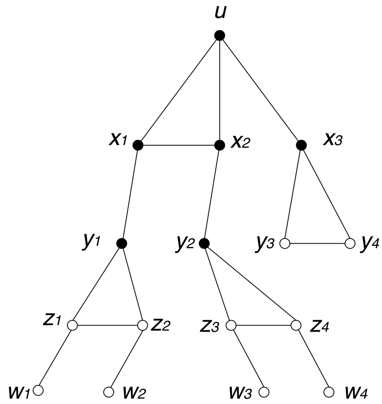

Fig. 3. A vertex is represented by a solid point if all of its incident edges are drawn, otherwise it is represented by a hollow point.

Suppose $u_2 \in D_2(G)$ with $N_G(u_2) = \{v_1, v_2\}$. Denote $G' = G - u + v_1v_2$ if $v_1v_2 \notin E(G)$, or $G' = G - u$ if $v_1v_2 \in E(G)$. By (1), $\chi_{L,3}(G'') \leq \max\{\chi_L(G'') + 3, 7\} \leq k$. By Lemma 2.4(i), $\chi_{L,3}(G) \leq k$, contrary to (1). Thus $D_2(G) = \emptyset$. This completes the proof of Claim 1.

Case 1 $\delta(G) = 3$.

Pick $u \in D_3(G)$ and denote $N_G(u) = \{x_1, x_2, x_3\}$. By (1), $\chi_{L,3}(G-u) \leq \max\{\chi_L(G-u)+3, 7\} \leq k$. By Lemma 2.6(i), $G[N_G[u]]$ is isomorphic to one of H_1, H_2 and H_3.

If $G[N_G[u]] \cong H_3$, then there exists a partial $(L, 3)$-coloring c_1 of G with $S(c_1) = V(G) - u$. The partial coloring c_1 could be extended to an $(L, 3)$-coloring of G by coloring u with a color in $L(u) - c_1\{N_G(u)\}$, contrary to (2).

If $G[N_G[u]] \cong H_2$, then by Lemma 2.7(i), there exists a partial $(L, 3)$-coloring c_2 of G with $S(c_2) = V(G) - u$ and $|c_2(\{x_1, x_2, x_3\})| = 3$. Since $\delta(G) = 3$, $d_{G-u}(x_1) \geq 2$. Choose $y_1 \in N_G(x_1) - u$ such that $c_2(y_1) \neq c_2(x_2)$. Similarly, choose $y_2 \in N_G(x_3) - u$ such that $c_2(y_2) \neq c_2(x_2)$. Therefore, the partial coloring c_2 could be extended to an $(L, 3)$-coloring of G by coloring u with a color in $L(u) - c_2(\{x_1, x_2, x_3, y_1, y_2\})$, contrary to (2).

If $G[N_G[u]] \cong H_1$, then by Lemma 2.7(i), there exists a partial $(L, 3)$-coloring c_3 of G with $S(c_3) = V(G) - u$ and $|c_3(\{x_1, x_2, x_3\})| = 3$. Since $\delta(G) = 3$, choose $y_1 \in N(x_1)$, $y_2 \in N(x_2)$ and $y_3, y_4 \in N(x_3)$ such that $|c_3\{y_1, x_2\}| = 2$, $|c_3\{x_1, y_2\}| = 2$ and $|c_3\{y_3, y_4\}| = 2$. If $L(u) - c_3(\{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}) \neq \emptyset$, then the partial coloring c_3 could be extended to an $(L, 3)$-coloring of G by coloring u with a color in $L(u) - c_3(\{x_1, x_2, x_3, y_1, y_2, y_3, y_4\})$, contrary to (2). Therefore, we assume that $L(u) = c_3(\{x_1, x_2, x_3, y_1, y_2, y_3, y_4\})$ and $k = 7$.

If $d_G(x_1) \geq 4$, then there exists $y_1' \in N(x_1)$ such that $|c_3(\{y_1, x_2, y_1'\})| = 3$. Then coloring u by $c_3(y_1)$ extends c_3 to an $(L, 3)$-coloring of G, contrary to (2). It follows that $d_G(x_1) = 3$ and $N_G(x_1) = \{u, x_2, y_1\}$. Similarly, $d_G(x_2) = 3$ and $N_G(x_2) = \{u, x_1, y_2\}$. If $d_G(x_3) \geq 4$, then there exists $y_3' \in N(x_3)$ such that $|c_3(\{y_3, y_4, y_3'\})| = 3$. Then coloring u by $c_3(y_3)$ extends c_3 to an $(L, 3)$-coloring of G, contrary to (2). Thus $d_G(x_3) = 3$ and $N_G(x_3) = \{u, y_3, y_4\}$. (See Fig. 3.)

Since G is claw-free, $G[N_G[y_1] - x_1]$ is a complete graph with $d_G(y_1)$ vertices. If $d_G(y_1) \geq 4$, then an $(L, 3)$-coloring of G could be obtained from c_3 by coloring u with $c_3(x_1)$ and recoloring x_1 with a color in $L(x_1) - c_3(\{x_1, x_2, x_3, y_1, y_2\})$, leading to a contradiction to (2). Thus, $d_G(y_1) = 3$ and we assume that $N_G(y_1) = \{x_1, z_1, z_2\}$. If $L(x_1) - c_3(\{x_1, x_2, x_3, y_1, y_2, z_1, z_2\}) \neq \emptyset$, then we could obtain an $(L, 3)$-coloring of G from c_3 by coloring u with $c_3(x_1)$ and recoloring x_1 with a color in $L(x_1) - c_3(\{x_1, x_2, x_3, y_1, y_2, z_1, z_2\})$, a contradiction to (2). Thus $L(x_1) = c_3(\{x_1, x_2, x_3, y_1, y_2, z_1, z_2\})$. By symmetry, $d_G(y_2) = 3$. Denote $N_G(y_2) = \{x_2, z_3, z_4\}$. With a similar argument, we conclude that $L(x_2) = c_3(\{x_1, x_2, x_3, y_1, y_2, z_3, z_4\})$.

Since $d_G(z_i) \geq 3$ $(i = 1, 2)$, there exists $w_i \in N_G(z_i)$ such that $|c_3(\{y_1, z_{3-i}, w_i\})| = 3$. Since $d_G(z_i) \geq 3$ $(i = 3, 4)$, there exists $w_i \in N_G(z_i)$ such that $|c_3(\{y_2, z_{7-i}, w_i\})| = 3$. If $L(y_1) - c_3(\{x_1, x_2, y_1, z_1, z_2, w_1, w_2\}) \neq \emptyset$, then we could obtain an $(L, 3)$-coloring of G from c_3 by coloring u with $c_3(y_1)$ and recoloring y_1 with a color in $L(y_1) - c_3(\{x_1, x_2, y_1, z_1, z_2, w_1, w_2\})$, a contradiction to (2). Thus $L(y_1) = c_3(\{x_1, x_2, y_1, z_1, z_2, w_1, w_2\})$. Similarly, $L(y_2) = c_3(\{x_1, x_2, y_2, z_3, z_4, w_3, w_4\})$.

Denote $G_1 = G[\{u, x_1, x_2, y_1, y_2\}]$. Define an assignment L' on $V(G_1)$ in the following way: $L'(u) = L(u) - c_3(\{x_3, y_3, y_4\})$; $L'(x_1) = L(x_1) - c_3(\{x_3, z_1, z_2\})$; $L'(x_2) = L(x_2) - c_3(\{x_3, z_3, z_4\})$; $L'(y_1) = L(y_1) - c_3(\{z_1, z_2, w_1, w_2\})$; $L'(y_2) = L(y_2) - c_3(\{z_3, z_4, w_3, w_4\})$. By Lemma 2.8, there exists an $(L', 3)$-coloring c_0 of G_1. Then we could define an $(L, 3)$-coloring c_4 of G in the following way:

$$c_4(v) = \begin{cases} c_0(v), & \text{if } v \in \{u, x_1, x_2, y_1, y_2\}; \\ c_3(v), & \text{otherwise}, \end{cases}$$

a contradiction to (2).

Case 2 $\delta(G) = 4$.

Choose $u \in D_4(G)$. Assume that $N_G(u) = \{x_1, x_2, x_3, x_4\}$. By (1), $\chi_{L,3}(G-u) \leq \max\{\chi_L(G-u)+3, 7\} \leq k$. Let c' be a partial $(L, 3)$-coloring of G with $S(c') = V(G) - u$. If $|c'(\{x_1, x_2, x_3, x_4\})| \geq 3$, then c' could be extended to an $(L, 3)$-coloring of G by coloring u with a color in $L(u)-c'(\{x_1,x_2,x_3,x_4\})$, contrary to (2). Thus $|c'(\{x_1,x_2,x_3,x_4\})|=2$. Therefore, $G[N_G(u)]$ does not contain K_3 as a subgraph. By Lemma 2.6(ii), $G[N_G[u]]$ is isomorphic to one of H_4, H_5 and H_6.

By the structure of H_4, H_5 and H_6, $x_1x_2, x_3x_4 \in E(G)$ and $x_1x_3, x_2x_4 \notin E(G)$. As $|c'(\{x_1,x_2,x_3,x_4\})|=2$, without loss of generality, we assume that $c'(x_1)=c'(x_3)$ and $c'(x_2)=c'(x_4)$. Denote $G_2=G[N_G[x_1]-\{u,x_2,x_4\}]$. Since G is claw-free, G_2 is a complete graph. Since $d_{G-u}(x_1)\geq 3$, $|c'(N_{G-u}(x_1))|\geq 3$. As $c'(x_2)=c'(x_4)$, there exists $y_1,y_1'\in N_G(x_1)-N_G[u]$ such that $|c'(\{x_2,y_1,y_1'\})|=3$. Similarly, for x_2, there exists $y_2,y_2'\in N_G(x_2)-N_G[u]$ such that $|c'(\{x_1,y_2,y_2'\})|=3$; and for x_i $(i=3,4)$, there exists $y_i,y_i'\in N_G(x_i)-N_G[u]$ such that $|c'(\{x_{7-i},y_i,y_i'\})|=3$. Note that $\{x_1,y_1,y_1'\}\subseteq V(G_2)$ and $k\geq \chi_L(G)+3\geq \omega(G)+3\geq |V(G_2)|+3$.

If $|V(G_2)|\geq 4$, then an $(L,3)$-coloring c_0' of G could be obtained in the following way:

$$c_0'(v)=\begin{cases} a & \text{if } v=x_1, \text{ and } a\in L(x_1)-c'(V(G_2))-\{c'(x_2)\};\\ b & \text{if } v=u, \text{ and } b\in L(u)-\{c'(x_1),c'(x_2),c'(y_2),c'(y_2'),a\};\\ c'(v) & \text{otherwise.}\end{cases}$$

This is a contradiction to (2).

So $V(G_2)=\{x_1,y_1,y_2\}$. Choose colors $a_1,a_2,a_3\in L(x_1)-c'(\{x_1,x_2,y_1,y_1'\})$. Define three L-colorings of G in the following way: for $i=1,2,3$,

$$c_i'(v)=\begin{cases} a_i & \text{if } v=x_1;\\ b_i & \text{if } v=u, \text{ and } b_i\in L(u)-\{c'(x_1),c'(x_2),c'(y_2),c'(y_2'),a_i\};\\ c'(v) & \text{otherwise.}\end{cases}$$

Define $BV(c_i')=\{v\in V(G): |c_i'(N_G(v))|\leq 2\}$ for $i=1,2,3$. Note that $|c_i'(N_G(u))|=3$, $|c_i'(N_G(x_1))|\geq |c'(\{x_2,y_1,y_1'\})|=3$, $|c_i'(N_G(x_2))|\geq |\{c'(y_2),c'(y_2'),b_i\}|=3$ and $|c_i'(N_G(x_j))|\geq |c'(\{x_{7-j},y_j,y_j'\})|=3$ for $j=3,4$. For any $v\in V(G)-N_G[u]-\{y_1,y_1'\}$, $|c_i'(N_G(v))|=|c'(N_G(v))|\geq 3$. Thus $BV(c_i')\subseteq\{y_1,y_1'\}$ for $i=1,2,3$.

By (2), c_1' and c_2' are not an $(L,3)$-coloring of G. Then $BV(c_1')\neq\emptyset$ and $BV(c_2')\neq\emptyset$. Without loss of generality, we assume that $|c_1'(N_G(y_1))|=2=|\{a_1,c'(y_1')\}|$. Since $|c'(N_G(y_1))|\geq 3$, there exists $z_1\in N_G(y_1)$ such that $c'(z_1)=a_1$. Thus $|c_2'(N_G(y_1))|\geq |\{a_2,c'(z_1),c'(y_1')\}|=3$ and $BV(c_2')=\{y_1'\}$. As $|c_2'(N_G(y_1'))|=2=|\{a_2,c'(y_1)\}|$ and $|c'(N_G(y_1'))|\geq 3$, there exists $z_1'\in N_G(y_1')$ such that $c'(z_1')=a_2$. Therefore, $|c_3'(N_G(y_1))|\geq |\{a_3,c'(z_1),c'(y_1')\}|=3$ and $|c_3'(N_G(y_1'))|\geq |\{a_3,c'(z_1'),c'(y_1)\}|=3$. Hence, $BV(c_3')=\emptyset$ and c_3' is an $(L,3)$-coloring of G, a contradiction to (2).

Case 3 $\delta(G)\geq 5$.

Since $k\geq\chi_L(G)$, there is an L-coloring c'' of G. By Lemma 2.6(iii), $|c''(N_G(v))|\geq 3$ for any $v\in V(G)$. Therefore c'' is also an $(L,3)$-coloring of G, a contradiction to (2). □

Although Theorem 1.3 is not a corollary of Theorem 1.2, the proof of Theorem 1.3 is quite similar to the proof of Theorem 1.2. So it is omitted here.

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