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Panconnected index of graphs

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ABSTRACT

For a connected graph G not isomorphic to a path, a cycle or a $K_{1,3}$, let $\mathrm{pc}(G)$ denote the smallest integer n such that the nth iterated line graph $L^n(G)$ is panconnected. A path P is a divalent path of G if the internal vertices of P are of degree 2 in G. If every edge of P is a cut edge of G, then P is a **bridge divalent path** of G; if the two ends of P are of degree s and t, respectively, then P is called a **divalent** (s, t)**-path**. Let $\ell(G) = \max\{m : G$ has a divalent path of length m that is not both of length 2 and in a $K_3\}$. We prove the following.

(i) If G is a connected triangular graph, then $L(G)$ is panconnected if and only if G is essentially 3-edge-connected.

(ii) $\mathrm{pc}(G) \le \ell(G) + 2$. Furthermore, if $\ell(G) \ge 2$, then $\mathrm{pc}(G) = \ell(G) + 2$ if and only if for some integer $t \ge 3$, G has a bridge divalent $(3, t)$-path of length $\ell(G)$.

1. The problem

We consider finite and loopless graphs which allow multiple edges and follow [1] for notations and terminology undefined in this paper. Let G be a graph. The **line graph** of G, denoted $L(G)$, has vertex set $E(G)$, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges share at least one common vertex in G. For a connected graph G, the n**th iterated line graph** $L^n(G)$ is defined recursively by $L^0(G) = G$ and $L^n(G) = L(L^{n-1}(G))$. Since the iterated line graph of a path will eventually diminish, and since the line graph of a cycle remains unchanged, in the discussions of iterated line graph problems, it is generally assumed that graphs under considerations are connected but not isomorphic to paths, cycles or $K_{1,3}$. For this reason, we let $\mathcal{G}$ denote the family of all connected graphs that are neither a path or a cycle, nor isomorphic to $K_{1,3}$.

The hamiltonian index (to be defined below) of a graph was first introduced in [3] by Chartrand. Other hamiltonian like indices were given by Clark and Wormald in [6]. More generally, we have the following definition.

Definition 1.1 (*[11]*)**.** For a property $\mathcal{P}$ and a connected nonempty graph $G \in \mathcal{G}$, the $\mathcal{P}$-**index** of G, denoted $\mathcal{P}(G)$, is defined by

$$\mathcal{P}(G) = \begin{cases} \min\{k : L^k(G) \text{ has property } \mathcal{P}\} & \text{if at least one such integer } k \text{ exists} \\ \infty & \text{otherwise.} \end{cases}$$

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When $\mathcal{P}$ represents the properties of being hamiltonian, edge-hamiltonian, pancyclic, vertex-pancyclic, edge-pancyclic, hamiltonian-connected, the corresponding indices are denoted (as in [6]) by $h(G)$, $\mathrm{eh}(G)$, $p(G)$, $\mathrm{vp}(G)$, $\mathrm{ep}(G)$, $\mathrm{hc}(G)$, respectively. In particular, $h(G)$ is called the **hamiltonian index** of G. Clark and Wormald [6] showed that if $G \in \mathcal{G}$, then the indices $h(G)$, $\mathrm{eh}(G)$, $p(G)$, $\mathrm{vp}(G)$, $\mathrm{ep}(G)$, $\mathrm{hc}(G)$ exist as finite numbers. In [4] and [11], it is shown that if G has any one of these properties mentioned above, then $L(G)$ also has the same property. In [14], Ryjáček, Woeginger and Xiong proved that determining the value of $h(G)$ is a difficult problem.

There have been many studies to investigate upper bounds of the hamiltonian index, hamiltonian-connected index and (vertex) pancyclic index. Interested readers may refer to [2,4,5,7–10,15–21] for further details. A path P of G is a **divalent path** of G if every internal vertex of P has degree 2 in G. Define

$$\ell(G) = \max\{m : G \text{ has a divalent path of length } m \text{ that is not both of length 2 and in a } K_3\}. \tag{1}$$

Let P be a divalent path of G. If every edge of P is a cut edge of G, then P is a **bridge divalent path** of G; Moreover, if the two ends of P are of degree s and t, respectively, then P is called a **divalent** (s, t)**-path**. Sharp upper bounds of the hamiltonian index, hamiltonian-connected index, s-hamiltonian index and pan-cyclic index have been obtained in terms of $\ell(G)$, see [5,8,10,16,17,20,21], among others. A graph G on $n \geq 3$ vertices is **panconnected** if for every pair of vertices u and v in G and for each s with $d(u, v) \leq s \leq n-1$, G always has a (u, v)-path of length s. Let $\mathcal{P}$ denote the property of being panconnected and following [6], let $\mathrm{pc}(G)$ denote the panconnected index of a graph $G \in \mathcal{G}$. There has been little study on $\mathrm{pc}(G)$. This observation motivates the current study. An edge cut X of G is an **essential edge cut** if at least two components of $G - X$ have at least one edge respectively. A graph G is **essentially** k**-edge-connected** if G does not have an essential edge cut with less than k edges. The main result of this paper is the following.

Theorem 1.2. *Let G be a graph in $\mathcal{G}$. Then $pc(G) \leq \ell(G) + 2$. Furthermore, if $\ell(G) \geq 2$, then $pc(G) = \ell(G) + 2$ if and only if for some integer $t \geq 3$, G has a bridge divalent $(3, t)$-path of length $\ell(G)$.*

The proof of Theorem 1.2 needs the assistance of an associate result stated below.

Theorem 1.3. *Let G be a graph in $\mathcal{G}$. If every edge of G lies in a cycle of length at most 3, in G, then $L(G)$ is panconnected if and only if G is essentially 3-edge-connected.*

In the next section, we present some of the preliminaries as preparations of the arguments. In Section 3, we prove the associate result which will play an important role to justify Theorem 1.2. We prove Theorem 1.2 in the last section.

2. Preliminaries

Throughout this section, we always assume that graphs are in $\mathcal{G}$. Let G be a graph. For a vertex $v \in V(G)$, define $N_G(v)$ to be the set of all vertices in G adjacent to v, and $E_G(v) = \{e \in E(G) | \ e \text{ is incident with } v \text{ in } G\}$. Following [1], we denote a trail $T = v_0e_1v_1 \cdots v_{t-1}e_tv_t$ such that each edge $e_i = v_{i-1}v_i$, for every i with $1 \leq i \leq t$, and such that all edges are distinct. For convenience, we sometimes view that T is associated with a natural orientation in which every edge e_i in the trail is oriented from v_{i-1} to v_i. If $v_0 = v_t$, then T is a closed trail. To emphasize the terminal vertices, T is called a (v_0, v_t)-trail. As the terminal edges of this trail T are e_1 and e_t, we also refer to T as an (e_1, e_t)-trail. The set of internal vertices of T is defined to be $T^o = \{v_1, v_2, \ldots, v_{t-1}\}$. If T is a trail of G, define

$$\partial_G(T) = \{e \in E(G) : \ e \text{ is incident with a vertex in } T^o\}. \tag{2}$$

As in [5,12,13], an (e, e')-trail T in G is a dominating trail if $\partial(T) = E(G)$, and is a spanning trail if T is dominating with $V(T) = V(G)$. The theorem below is well known.

Theorem 2.1. *Let G be a graph with $|E(G)| \geq 3$.*

(i) *(Harary and Nash-Williams) $L(G)$ is hamiltonian if and only if G has a dominating closed trail.*
(ii) *(Proposition 2.2 of [13]) $L(G)$ is hamiltonian-connected if and only if for every pair of distinct edges e, e' in $E(G)$, G has a dominating (e, e')-trail.*

By the definition of line graphs, we have the following lemma.

Lemma 2.2. *Let $s > 0$ be an integer, and $e, e' \in E(G)$. Each of the following holds.*

(i) *There is an (e, e')-path of length s in $L(G)$ if and only if G has an (e, e')-trail T with $|E(T)| \leq s + 1$ and $|\partial_G(T)| \geq s + 1$.*
(ii) *The distance between e and e' in $L(G)$ is s if and only if G has a shortest (e, e')-path of length $s + 1$.*

Proof. By the definition of line graphs, (ii) follows from (i) and so it suffices to prove Part (i) only. Suppose G has an (e, e')-trail $T = v_0e_1v_1e_2 \cdots v_{m-1}e_mv_m$ with $e = e_1$ and $e' = e_m$, satisfying $m = |E(T)| \leq s + 1$ and $|\partial_G(T)| \geq s + 1$. Then $L(T)$ is an (e, e')-path of length $m - 1$ in $L(G)$. For each i with $0 < i < m$, let $X_i' = E_G(v_i) - E(T)$, $X_1 = X_1'$ and for $2 \leq j < m$, let

$X_j = X'_j - (\cup_{1\le i<j} X'_i)$. Then $X_1, X_2, \ldots, X_m$ are pairwise disjoint and $\partial(T) = E(T) \cup (\cup_{i=1}^{m-1} X_i)$. Since $m = |E(T)| \le s+1$ and $|\partial_G(T)| \ge s+1$, we have $\sum_{i=1}^{m-1} |X_i| \ge (s+1) - m$. Hence there must be an integer m' with $1 \le m' \le m-1$ and a subset $X' \subseteq X_{m'}$ such that $|X_1 \cup X_2 \cup \cdots \cup X_{m'-1} \cup X'| = s - m$. Since every $E_G(v_i)$ induces a complete subgraph of $L(G)$, for each i with $1 \le i \le m'-1$, $L(G)[E_G(v_i)]$ has an (e_i, e_{i+1})-path P_i using exactly the vertices in $X_i \cup \{e_i, e_{i+1}\}$ in $L(G)$, and $L(G)[E_G(v_{m'})]$ has an $(e_{m'}, e_{m'+1})$-path $P_{m'}$ using exactly the vertices in $X' \cup \{e_{m'}, e_{m'+1}\}$ in $L(G)$. Let $P_{m'+1}$ be the subpath $e_{m'+1}e_{m'+2}\cdots e_m$ of $L(T)$. It follows that $L(G)$ has an (e, e')-path of length s obtained by putting all the paths $P_1, P_2, \ldots, P_{m'}, P_{m'+1}$ together.

Conversely, assume that $L(G)$ has an (e, e')-path P of length s. Then $V(P) \subseteq E(G)$. Since P is an (e, e')-path, the edge induced subgraph $G[V(P)]$ of G is connected and contains e and e'. Thus $G[V(P)]$ has a longest (e, e')-trail T. Since T is longest, and since $L(G[V(P)]) = P$, it follows that $E(T) \subseteq V(P) \subseteq \partial(T)$, and so $|E(T)| \le |V(P)| = s+1$ and $|\partial_G(T)| \ge |V(P)| = s+1$. □

3. An associate result on triangular graphs

Theorem 1.3 is an associate result which will play a useful role in our study of the panconnected index of connected graphs in $\mathcal{G}$. We introduce the terms used in this section first. If H is a subgraph of a graph G, the **vertex of attachment of** H **in** G, is

$$A_G(H) = \{v \in V(H) \ : \ v \text{ is adjacent to a vertex in } V(G) - V(H)\}. \tag{3}$$

If $X \subseteq E(H)$ and $Y \subseteq E(G) - E(H)$, then we define $H - X + Y = G[(E(H) - X) \cup Y]$. A graph G is **triangular** if G is connected with $E(G) \neq \emptyset$ such that every edge in $E(G)$ lies in a cycle of length at most 3 in G. For sets X and Y, the symmetric difference of X and Y is defined as

$$X \Delta Y = (X \cup Y) - (X \cap Y).$$

We are to investigate conditions for a line graph to be panconnected. It is well known that

$$\text{every panconnected graph is 3-connected.} \tag{4}$$

The main result of this section is given below, which, together with (4), implies Theorem 1.3.

Theorem 3.1. *If G is an essentially 3-edge-connected triangular graph, then $L(G)$ is panconnected.*

Proof. Since G is triangular, throughout the rest of the proof of this theorem, for each edge $f \in E(G)$, we define C_f to be a shortest cycle containing f. Thus $|E(C_f)| \le 3$ for any $f \in E(G)$. We argue by contradiction and assume that Theorem 3.1 has a counterexample G with $e, e' \in E(G)$ and a positive integer $s < |E(G)| - 1$ such that

$$L(G) \text{ has an } (e, e')\text{-path of every length at most } s \text{ but no } (e, e')\text{-paths of length } s+1. \tag{5}$$

By Lemma 2.2, G has an (e, e')-trail

$$T = v_0e_1v_1e_2\cdots v_{m-1}e_mv_m \text{ with } e = e_1 \text{ and } e' = e_m \tag{6}$$

with $|E(T)| \le s+1$ and $|\partial(T)| \ge s+1$. Assume that the choice of G satisfies (5), and that, subject to $|E(T)| \le s+1$ and $|\partial(T)| \ge s+1$,

$$|E(T)| \text{ is maximized.} \tag{7}$$

If $|\partial(T)| \ge s+2$, then by Lemma 2.2, $L(G)$ has an (e, e')-path of length $s+1$, contradicting (5). Hence we must have $|\partial(T)| = s+1 < |E(G)|$.

Claim 1. *For any edge $f = uv \in \partial(T) - E(T)$, we have $u, v \in V(T)$.*

By contradiction, assume that there exists an edge $f = uv \in \partial(T) - E(T)$ violating the claim. By (2), we may assume that $v \in T^o$ and $u \notin V(T)$. Since G is triangular, there exists a cycle C_f of length at most 3 in G containing f. If $C_f = \{f, f'\}$ is a cycle of length 2, then the trail $G[E(T) \cup C_f]$ violates (7). Assume that C_f is a 3-cycle with $V(C_f) = \{u, v, w\}$. As $uv, uw \notin E(T)$, it follows that $G[E(T) \Delta E(C_f)]$ is an (e, e')-trail in G violating (7). This justifies Claim 1.

Claim 2. *With the notation in (6), each of the following holds.*

(i) *There exists an edge $f \in E(G) - \partial(T)$ such that f shares at least one vertex with an edge in $\partial(T)$.*
(ii) *For any edge $f \in E(G) - \partial(T)$, if f shears at least one vertex with an edge in $\partial(T)$, then f must be incident with either v_0 or v_m.*
(iii) *$A_G(G[\partial(T)]) \subseteq \{v_0, v_m\}$.*
(iv) *Let $H = G[\partial(T)]$. For any $z \in \{v_0, v_m\}$, let $x \in \{e, e'\}$ be the corresponding edge incident with z. If there exists an edge $f \in E_G(z) - \partial(T)$, then x is not in any cycle of length 2 and*

$$N_H(z) - V(C_x) = \emptyset. \tag{8}$$

Claim 2(i) follows from the assumptions that G is connected and that $|\partial(T)| < |E(G)|$. To justify Claim 2(ii), suppose that an edge $f \in E(G) - \partial(T)$ shares at least one vertex with an edge y in $\partial(T)$. If $y \notin \{e, e'\}$, then by Claim 1, both ends are in T^o, and so f is incident with a vertex in T^o, leading to the contradiction that $f \in \partial(T)$. Hence we may assume that $y = e$. As $v_1 \in T^o$ and as $f \in E(G) - \partial(T)$, f must be incident with v_0. Similarly, if $y = e'$, then f must be incident with v_m. This shows Claim 2(ii).

Claim 2(iii) follows from (ii) and (3). We now show (iv) and assume, by symmetry, that $z = v_0$ and $x = e$, and there exists an edge $f \in E_G(v_0) - \partial(T)$. By (2), $v_0 \notin T^o$. If $\{e, e''\}$ is a cycle of G, then by (6), $T' = v_1 e v_0 e'' v_1 e_2 \cdots v_{m-1} e' v_m$ is also an (e, e')-trail in G with $E(T') \subseteq \partial(T)$ and with $|E(T')| > |E(T)|$, a contradiction to (7). Thus e is not in any cycle of length 2. Assume now that there exists a vertex $z' \in N_H(v_0) - V(C_e)$. Since G is triangular, G has a cycle $C_{v_0 z'}$ of length 2 or 3. By Claim 1, $E(C_{v_0 z'}) \subseteq \partial(T)$. As $v_0 \notin T^o$, $E(C_{v_0 z'}) \cap E(T) = \emptyset$ and so $G[E(T) \cup E(C_{v_0 z'})]$ is an (e, e')-trail violating (7). Hence we have (8).

Claim 3. *Suppose that $v_0 \neq v_m$ and that there exists an edge $f \in E_G(v_0) - \partial(T)$. Then each of the following holds.*

(i) *$V(C_e) = \{v_0, v_1, w\}$ is a cycle of length 3 and $v_1 \neq v_m$.*
(ii) *If $v_m \notin V(C_e)$, then G has an (e, e')-trail T' such that $|E(T')| \le |\partial(T)|$ and $|\partial(T')| \ge |\partial(T)| + 1$.*
(iii) *If C'_e is a cycle of length 3 in G containing e, then $v_0 v_1, v_1 w \in E(C'_e)$.*
(iv) *$w = v_m$.*
(v) *If $v_1 \neq v_{m-1}$, then either $v_1 v_m \notin E(T)$ or $v_1 v_m$ is not a cut edge of T.*
(vi) *The vertex v_1 is a cut vertex of G.*

Throughout the justification of Claim 3, we let $H = G[V(T)]$ and assume that $v_0 \neq v_m$, and there exists an edge $f \in E_G(v_0) - \partial(T)$. If $E(C_e) \cap E(T) = \{e\}$, then $G[E(C_e) \cup E(T)]$ is an (e, e')-trail violating (7). Hence we may assume that $V(C_e) = \{v_0, v_1, w\}$ and $v_0 v_1, v_1 w \in E(T)$. By contradiction, assume that $v_1 = v_m$. If $w = v_{m-1}$, then let $L = T[E(T) - \{e, e'\}]$ be a (v_1, v_{m-1})-subtrail of T. View L as a trail oriented by its direction from v_1 to v_{m-1}. Define L^{-1} to be the (v_{m-1}, v_1)-trail obtained from L by reversing the orientations. Then L^{-1} together with the oriented edges $v_1 v_0$ and $v_m v_{m-1}$ is an (e, e')-trail T_2 with $E(T_2) = E(T)$ and $\partial(T) \cup \{f\} \subseteq \partial(T_2)$. It follows that $|\partial(T_2)| \ge |\partial(T)| + 1 = s + 2$. By Lemma 2.2, $L(G)$ has an (e, e')-path of length $s + 1$, which contradicts (5).

Hence we assume that $w \neq v_{m-1}$. If $v_1 w$ is not a cut edge of $T - \{e, e'\}$, then $T - \{e, e', v_1 w\} + \{v_0 w\}$ is a (v_0, v_{m-1})-trail. Thus $T_1 = T - \{v_1 w\} + \{v_0 w\}$ is an (e, e')-trail with $\partial(T) \subseteq \partial(T_1)$ and $v_0 \in T_1^o$. It follows that $f \in \partial(T_1)$, and so by Lemma 2.2, we obtain a contradiction to (5).

Thus $v_1 w$ is a cut edge of $T - \{e, e'\}$. Let J_1 and J_2 be the two components of $T - \{e, e', v_1 w\}$ with $v_1 = v_m \in V(J_1)$ and $w \in V(J_2)$. Since $v_1 w$ is a cut edge of $T - \{e, e'\}$, we have $v_{m-1} \in V(J_2)$. Thus $T - \{e, e'\}$ is a (v_1, v_{m-1})-trail. If $v_0 \in V(C_{e'})$, then $V(C_{e'}) = \{v_0, v_1, v_{m-1}\}$, and so by $v_1 = v_m$, $T - \{e, e'\} + \{v_0 v_{m-1}\}$ is a (v_0, v_m)-trail. It follows that $T_2 = T + \{v_0 v_{m-1}\}$ is a (e, e')-trail with $\partial(T) \subseteq \partial(T_2)$ and $v_0 \in T_2^o$, and so a contradiction to (5) is obtained. Hence $v_0 \notin V(C_{e'})$. By (7), $|E(C_{e'})| = 3$ and so $V(C_{e'}) = \{z, v_1, v_{m-1}\}$ for some $z \neq v_0$. It follows that $T_3 = G[E(T) \Delta E(C_{e'}) \Delta (E(C_e) + \{e, e'\})]$ is an (e, e')-trail with $\partial(T) \subseteq \partial(T_3)$ and $v_0 \in T_3^o$, once again a contradiction to (5) is obtained. This shows that $v_1 \neq v_m$, and justifies (i).

Assume that $v_m \notin V(C_e)$. If $T - v_1 w$ is connected, then $T_4 = G[E(T) \Delta (E(C_e) - \{e\})]$ is an (e, e')-trail with $v_0 \in T_4^o$, and so $E(T_4) \subseteq \partial(T)$ and $\partial(T) \cup \{f\} \subseteq \partial(T_4)$, implying (ii). Hence we may assume that $T - v_1 w$ has two components L_1 and L_2 such that $e \in E(L_1)$ and $w \in V(L_2)$. Since T is an (e, e')-trail and $w \neq v_m$, $e' \neq v_1 w$ and so $e' \in E(L_2)$. Since G is essentially 3-edge-connected, $\{v_0 v_1, v_1 w\}$ is not an essential edge cut. By Claim 2(ii), there must be an edge $e'' = z_1 z_2 \in \partial(T) - E(T)$ with $z_1 \in V(L_1)$ and $z_2 \in V(L_2)$. Since G is triangular, there exists a cycle $C_{e''}$ of length 2 or 3 containing e''. Since any cycle intersects any edge cut with an even number of edges, $C_{e''}$ has two edges incident with both $V(T_1)$ and $V(T_2)$. It follows that $T_5 = G[E(T) \Delta E(C_{e''})]$ is also an (e, e')-trail of G with $E(T_5) \subseteq \partial(T)$ and $\partial(T) = \partial(T_5)$. Since $C_{e''}$ has two edges incident with both $V(L_1)$ and $V(L_2)$, the edge $v_1 w$ is not a cut edge of T'. Hence $T_6 = G[E(T_5) \Delta E(C_e - e)]$ is an (e, e')-trail with the first edge $v_1 v_0$ and with $E(T_6) \subseteq \partial(T) \subseteq \partial(T_6)$. However, $v_0 \in (T_6)^o$, and so $f \in \partial(T_6) - \partial(T)$, which is a contradiction to (5). This proves Claim 3(ii).

For (iii), suppose next that C'_e is a cycle of length 3 in G containing e. By contradiction, assume that $V(C'_e) = \{v_0, v_1, w'\}$ for some $w' \neq w$. By Claim 3(i), $v_1 \neq v_m$. Hence we may assume that $w \neq v_m$. It follows that by Claim 3(ii), G has an (e, e')-trail T' such that $|E(T')| \le |\partial(T)| = s + 1$ and $|\partial(T')| \ge |\partial(T)| + 1 = s + 2$. By Lemma 2.2, we have a contradiction to (5). This justifies (iii). Claim 3(iv) now follows from Claim 3(ii) and (iii).

Now suppose that $v_1 \neq v_{m-1}$. Since $v_0 \notin T^o$, we have $v_1 v_m \notin E(T)$. By contradiction, assume that $v_1 v_m \in E(T)$ and $v_1 v_m$ is a cut edge of T. To avoid introducing too many new notations, we again assume that $T - v_1 v_m$ has two components L_1 and L_2 such that $e \in E(L_1)$ and $v_m \in V(L_2)$. Since T is an (e, e')-trail and $v_{m-1} \neq v_1$, $e' \neq v_1 w$ and so $e' \in E(L_2)$. Since G is essentially 3-edge-connected, $\{v_0 v_1, v_1 v_m\}$ is not an essential edge cut. By Claim 2(ii), there must be an edge $e'' = z_1 z_2 \in \partial(T) - E(T)$ with $z_1 \in V(L_1)$ and $z_2 \in V(L_2)$. Since G is triangular, there exists a cycle $C_{e''}$ of length 2 or 3 containing e''. Since any cycle intersects any edge cut with an even number of edges, $C_{e''}$ has two edges incident with both $V(T_1)$ and $V(T_2)$. It follows that $T_5 = G[E(T) \Delta E(C_{e''})]$ is also an (e, e')-trail of G with $E(T_5) \subseteq \partial(T)$ and $\partial(T) = \partial(T_5)$. Since $C_{e''}$ has two edges incident with both $V(L_1)$ and $V(L_2)$, the edge $v_1 w$ is not a cut edge of T'. Hence $T_6 = G[E(T_5) \Delta E(C_e - e)]$ is an (e, e')-trail with the first edge $v_1 v_0$ and with $E(T_6) \subseteq \partial(T) \subseteq \partial(T_6)$. However, $v_0 \in (T_6)^o$, and so $f \in \partial(T_6) - \partial(T)$, which contradicts (5). Therefore, if $v_1 v_m \in E(T)$, then $v_1 v_m$ is not a cut edge of T. This justifies (v).

We argue by contradiction to prove (vi) and assume that v_1 is not a cut vertex of G. By Claim 2(ii), $\{v_0, v_m\}$ is a vertex cut of G such that if J is a component of $G - \{v_0, v_m\}$ containing v_1, then $G[V(J) \cup \{v_0, v_m\}] = H$.

Suppose first that $v_1 \neq v_{m-1}$ or $v_m \in T^o$. If v_1v_m is not a cut edge of $T - \{e, e'\}$, then $T - \{e, e', v_1v_m\} + \{v_0v_m\}$ is a (v_0, v_m)-trail, and so $T_7 = T - \{v_1v_m\} + \{v_0v_m\}$ is an (e, e')-trail with $E(T_7) \subseteq \partial(T) \cup \{f\} \subseteq \partial(T_7)$. By Lemma 2.2, this is a contradiction to (5). Hence $T - \{e, e', v_1v_m\}$ has two components L_1' and L_2' with $v_1 \in V(L_1')$ and $v_m \in V(L_2')$. By Claim 2(iii) and (iv), $N_H(v_0) - V(C_e) = \emptyset$ and $A_G(H) = \{v_0, v_m\}$. Thus either $\{e, v_1v_m\}$ is an edge cut of G, or there is an edge $z_1z_2 \in \partial(T) - E(T)$ with $z_1 \in V(L_1')$ and $z_2 \in V(L_2')$. Since G is essentially 3-edge-connected, if $\{e, v_1v_m\}$ is an edge cut of G, then one side of $G - \{e, v_1v_m\}$ is a singleton v_1. In this case, as $T = v_0ev_1e_2v_2e_3v_3 \cdots v_{m-1}e'v_m$ with $v_2 = v_m$, we obtain an (e, e')-trail $T_8 = v_1ev_0e_2'v_2e_3v_3 \cdots v_{m-1}e'v_m$ where $e_2' = v_0v_m \in E(C_r)$. Since $E(T_8) \subseteq \partial(T) \cup \{f\} \subseteq \partial(T_8)$, this leads to a contradiction to (5). Assume then that there is an edge $z_1z_2 \in \partial(T) - E(T)$ with $z_1 \in V(L_1')$ and $z_2 \in V(L_2')$. Since G is triangular, G has a cycle $C_{z_1z_2}$ of length 2 or 3 containing z_1z_2, and so $T_9 = G[E(T)\Delta E(C_e)\Delta(C_{z_1z_2})]$ is an (e, e')-trail violating (7). As in either case, a contradiction is always obtained, we conclude that both $v_1 = v_{m-1}$ and $v_m \notin T^o$. Since v_1 is not a cut vertex of G, we must have $N_G(v_1) - V(C_e) = \emptyset$. It follows that we must have $s = 2$, and $T_{10} = v_1ev_0e''v_me'v_{m-1}$, having $f \in \partial(T_{10})$, which leads to a contradiction to (5). This proves (vi).

We continue our proof of Theorem 3.1. If $v_0 \neq v_m$, then by Claim 3(v), v_1 is a cut vertex of G. By Claim 2(iv), $\{v_0v_1, v_1v_m\}$ is an essential edge cut of G, contradicting the assumption that G is essentially 3-edge-connected. Therefore, we must have $v_0 = v_m$. By Claim 2(iii), v_0 is a cut vertex of G and $V(C_e) = \{v_0, v_1, v_{m-1}\}$. By Claim 2(iv) and by the existence of f and v_1v_{m-1}, $\{v_0v_1, v_0v_{m-1}\}$ is an essential edge-cut of G, which is a contradiction to the assumption that G is essentially 3-edge-connected. This final contradiction indicates that (5) does not hold, which proves Theorem 3.1. $\square$

4. Main results

We start with some former results and lemmas. Recall that if $G \in \mathcal{G}$, then $\ell(G)$ is defined in (1).

Lemma 4.1 (*Zhang et al., Lemma 3.2 [20]*). *If $G \in \mathcal{G}$, then $L^{\ell(G)}(G)$ is triangular.*

Lemma 4.2 (*Zhang et al., Proposition 2.3 [21]*). *Let G be a simple connected triangular graph. Each of the following holds.*

(i) *The line graph $L(G)$ is triangular.*
(ii) *If G is k-connected, then $L(G)$ is $(k+1)$-connected.*
(iii) *If G is essentially k-edge-connected, then $L(G)$ is essentially $(k+1)$-edge-connected.*

From the definition of line graphs, we make the following observations.

Observation 4.3. *Let $G \in \mathcal{G}$ be a graph, let $\mathcal{H}(G)$ denote the collection of all edge-induced subgraphs of G and let $\mathcal{L}(G)$ denote the collection of all induced subgraphs of $L(G)$.*

(i) *For any $H \in \mathcal{H}(G)$, by the definition of line graphs, $L(H) = L(G[E(H)])$ is an induced subgraph of $L(G)$, and so $L(H) \in \mathcal{L}(G)$. Conversely, if $\Gamma \in \mathcal{L}(G)$, then $H = G[V(\Gamma)] \in \mathcal{H}(G)$. Hence there exists a bijection between $\mathcal{H}(G)$ and $\mathcal{L}(G)$. We also use $L : \mathcal{H}(G) \mapsto \mathcal{L}(G)$ to denote this bijection, and L^{-1} denotes the inverse mapping of L. By the definition of iterated line graphs, for any integer $s > 1$, L^s is an operator mapping subgraphs in $\mathcal{H}(G)$ into subgraphs in $L^s(G)$; and L^{-s} pulls back induced subgraphs in $L^s(G)$ to subgraphs in $\mathcal{H}(G)$.*
(ii) *In particular, if $e \in E(G)$, we define $v_e = L(e)$. Thus $v_e \in V(L(G))$ is a cut vertex of $L(G)$ if and only if $\{e\}$ is an essential edge-cut of G; if $v_{e_1}v_{e_2} \in E(L(G))$ is an edge which is not lying in a K_3 of $L(G)$, then $L^{-1}(v_{e_1}v_{e_2}) = G[\{e_1, e_2\}]$ is a divalent path of G.*
(iii) *By (i), we conclude that if P is a divalent path of length $h > 0$, the for any integer k with $0 \le k < h$, $L^k(P)$ is a divalent path of length $h - k$ in $L^k(G)$; and $L^h(P)$ is a vertex of $L^h(G)$.*
(iv) *By (ii), we observe that for integers $s \ge t \ge 2$, if v is a cut vertex of $L^s(G)$, then $\{L^{-1}(v)\}$ is an essential edge cut of $L^{s-1}(G)$; and $L^{-2}(v)$ is a bridge divalent path of length 2 in $L^{s-2}(G)$. Inductively, if $s - t \ge 0$, then $L^{-t}(v)$ is a bridge divalent path of length t in $L^{s-t}(G)$.*
(v) *Similarly, if e is an edge which is not in a complete subgraph of order at least 3 in $L(G)$, $L^{-1}(e)$ is a divalent path of length 2 in G. For integers $s \ge t \ge 2$, if e is an edge which is not in a complete subgraph of order at least 3 in $L^s(G)$, then $\{L^{-1}(e)\}$ is a divalent path of length 2 in $L^{s-1}(G)$. Inductively, if $s - t \ge 0$, then $L^{-t}(e)$ is a divalent path of length $t + 1$ in $L^{s-t}(G)$.*

We are now ready to prove the main results, restated below as Theorem 4.5. We observe that if G is a triangular graph, then G is connected and every edge of G lies in a cycle. Hence

every triangular graph is 2-edge-connected. (9)

Lemma 4.4. *Let $G \in \mathcal{G}$ be a graph with $\ell = \ell(G) \ge 2$. Each of the following holds.*

(i) *If G has a bridge divalent $(3, t)$-path of length ℓ for some integer $t \ge 3$, then $pc(G) = \ell(G) + 2$.*
(ii) *If G does not have any bridge divalent $(3, t)$-path of length ℓ, then $pc(G) \le \ell(G) + 1$.*

Proof. (i) Suppose first that $P = v_0e_1v_1 \cdots v_{\ell-1}e_\ell v_\ell$ is a bridge divalent $(3, t)$-path for some integer $t \geq 3$ with $d_G(v_0) = 3$. Let e'_1, e'_2, e_1 be the three edges of G incident with v_0. By the definition of line graphs, the neighbors of vertex e_1 in $L(G)$ are the vertices $\{e'_1, e'_2, e_2\}$, and so $L(P)$ is bridge divalent $(3, t')$-path in $L(G)$ for some integer $t' \geq 3$. Inductively, we conclude that $L^{\ell-1}(P)$ is a cut edge z_1z_2 of $L^{\ell-1}(G)$ such that $d_{L^{\ell-1}(G)}(z_1) = 3$ (say) and $d_{L^{\ell-1}(G)}(z_2) \geq 3$. Thus $\{z_1z_2\}$ is an essential edge cut of $L^{\ell-1}(G)$. By Observation 4.3(i), the cut edge z_1z_2 in $L^{\ell-1}(G)$ is a cut vertex v of $L^\ell(G)$. Since $d_{L^{\ell-1}(G)}(z_1) = 3$, the three edges in $N_{L^{\ell-1}(G)}(z_1)$ form a 3-cycle C of $L^\ell(G)$ containing the cut vertex v. Since v is a cut vertex of $L^\ell(G)$, the two edges in C incident with v form an essential edge cut of $L^\ell(G)$. By Observation 4.3, $L^{s+1}(G)$ is not 3-connected. By (4), $L^{\ell+1}(G)$ is not panconnected. Hence by (4), $\mathrm{pc}(G) \geq \ell(G) + 2$. On the other hand, by (9) and by Lemmas 4.1 and 4.2, $L^{\ell+1}$ is triangular and essentially 3-edge-connected, and so by Theorem 3.1, $\mathrm{pc}(G) = \ell(G) + 2$. This proves (i).

(ii) Assume that G does not have any bridge divalent $(3, t)$-path of length ℓ. Let $P = v_0e_1v_1 \cdots v_{\ell-1}e_\ell v_\ell$ be a divalent (s, t)-path of length $\ell(G)$. Since P is a maximal divalent path of G, $s \neq 2$ and $t \neq 2$. Let $Q(G)$ be the collection of all divalent paths of G of length ℓ. We have the following cases.

Case 1. Every bridge divalent path of G of length ℓ is either an (s, t)-path with $s \geq t \geq 4$, or with $s \geq 3$ and $t = 1$.

By Observation 4.3(iii), for every $Q \in Q(G)$, $L^\ell(Q)$ is a vertex of $L^\ell(G)$. Moreover, if Q is a bridge divalent path, then $L^\ell(Q)$ is a cut vertex of $L^\ell(G)$. Since every bridge divalent path of G is either an (s, t)-path with $s \geq t \geq 4$, or with $s \geq 3$ and $t = 1$, $L^\ell(G)$ does not have any essential edge cut of size 2. By (9), $L^\ell(G)$ is essentially 3-edge-connected. By Lemma 4.1, $L^\ell(G)$ is triangular. By Theorem 3.1, $L(L^\ell(G))$ is panconnected, and so $\mathrm{pc}(G) \leq \ell + 1$. Hence (ii) holds for Case 1.

Case 2. G does not have a bridge divalent path of length ℓ.

By Lemma 4.1, $L^\ell(G)$ is triangular. By (9), $L^\ell(G)$ is 2-edge-connected. If $L^\ell(G)$ has an essential edge cut X of size 2, then since $L^\ell(G)$ is triangular, X must be in a cycle of size 3, and so the vertex incident with both edges in X must be a cut vertex v of $L^\ell(G)$. By Observation 4.3(vi), $L^{-\ell}(v)$ is a bridge divalent path of length ℓ of G, contradicting the assumption that G does not have a bridge divalent path of length ℓ. This contradiction implies that $L^\ell(G)$ is essentially 3-edge-connected. By Theorem 3.1, $L^\ell(G)$ is panconnected, and so $\mathrm{pc}(G) \leq \ell + 1$. Hence (ii) holds for Case 2 as well. $\square$

Theorem 4.5. *For a graph $G \in \mathcal{G}$, $pc(G) \leq \ell(G) + 2$. Furthermore, if $\ell(G) \geq 2$, then $pc(G) = \ell(G) + 2$ if and only if G has a bridge divalent $(3, t)$-path of length $\ell(G)$, for some integer $t \geq 3$.*

Proof. Let $G \in \mathcal{G}$ and $\ell = \ell(G)$. By Lemma 4.1, $L^\ell(G)$ is triangular. By (9), $L^\ell(G)$ is 2-edge-connected, so $L^\ell(G)$ is essentially 2-edge-connected. By Lemma 4.2, $L^{\ell+1}(G)$ is both triangular and essentially 3-edge-connected. It follows from Theorem 3.1 that $L^{\ell+2}(G)$ is panconnected.

Now assume that $\ell \geq 2$. If for some integer $t \geq 3$, G has a bridge divalent $(3, t)$-path of length $\ell(G)$, then by Lemma 4.4(i), $\mathrm{pc}(G) = \ell + 2$. Therefore we will assume that $\mathrm{pc}(G) = \ell + 2$. Let $Q(G)$ be the collection of all divalent path of G of length ℓ. If every path in $Q(G)$ is not a bridge divalent path, or if every bridge divalent path $Q \in Q(G)$ is an (s, t) path such that either $\min\{s, t\} \geq 4$, or both $\max\{s, t\} \geq 3$ and $\min\{s, t\} = 1$, then by Lemma 4.4(ii), $\mathrm{pc}(G) = \ell + 1$, contradicting the assumption that $\mathrm{pc}(G) = \ell + 2$. Hence we must have a bridge divalent $(3, t)$-path of length $\ell(G)$, for some integer $t \geq 3$. This completes the proof of the theorem. $\square$

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