Linear and Multilinear Algebra

ISSN: 0308-1087 (Print) 1563-5139 (Online) Journal homepage: http://www.tandfonline.com/loi/glma20

The (signless) Laplacian spectral radii of c-cyclic graphs with n vertices, girth g and k pendant vertices

Muhuo Liu, Kinkar Ch. Das & Hong-Jian Lai

To cite this article: Muhuo Liu, Kinkar Ch. Das & Hong-Jian Lai (2017) The (signless) Laplacian spectral radii of c-cyclic graphs with n vertices, girth g and k pendant vertices, Linear and Multilinear Algebra, 65:5, 869-881, DOI: 10.1080/03081087.2016.1211082

To link to this article: http://dx.doi.org/10.1080/03081087.2016.1211082

Published online: 01 Aug 2016.

Submit your article to this journal

Article views: 68

View related articles

View Crossmark data

Full Terms & Conditions of access and use can be found at
http://www.tandfonline.com/action/journalInformation?journalCode=glma20

Download by: [West Virginia University Libraries] **Date:** 27 March 2017, At: 07:20

LINEAR AND MULTILINEAR ALGEBRA, 2017
VOL. 65, NO. 5, 869–881
http://dx.doi.org/10.1080/03081087.2016.1211082

Taylor & Francis
Taylor & Francis Group

The (signless) Laplacian spectral radii of *c*-cyclic graphs with *n* vertices, girth *g* and *k* pendant vertices

Muhuo Liu[a], Kinkar Ch. Das[b], Hong-Jian Lai[c]

[a]Department of Mathematics, South China Agricultural University, Guangzhou, China; [b]Department of Mathematics, Sungkyunkwan University, Suwon, Republic of Korea; [c]Department of Mathematics, West Virginia University, Morgantown, WV, USA

ABSTRACT

Let $\Gamma_g(n,k;c)$ denote the class of c-cyclic graphs with n vertices, girth $g \geq 3$ and $k \geq 1$ pendant vertices. In this paper, we determine the unique extremal graph with largest signless Laplacian spectral radius and Laplacian spectral radius in the class of connected c-cyclic graphs with $n \geq c(g-1)+1$ vertices, girth g and at most $n-c(g-1)-1$ pendant vertices, respectively, and the unique extremal graph with largest signless Laplacian spectral radius of $\Gamma_g(n,k;c)$ when $n \geq c(g-1)+k+1$ and $c \geq 1$, and we also identify the unique extremal graph with largest Laplacian spectral radius in $\Gamma_g(n,k;c)$ in the case $c \geq 1$ and either $n \geq c(g-1)+k+1$ and g is even or $n \geq \frac{1}{2}(g-1)k+cg$ and g is odd. Our results extends the corresponding results of [Sci. Sin. Math. 2010;40:1017–1024, Electron. J. Combin. 2011; 18:p.183, Comput. Math. Appl. 2010;59:376–381, Electron. J. Linear Algebra. 2011;22:378–388 and J. Math. Res. Appl. 2014;34:379–391].

ARTICLE HISTORY
Received 3 March 2016
Accepted 5 July 2016

COMMUNICATED BY
J.Y. Shao

KEYWORDS
c-cyclic graph; girth; pendant vertices; adjacency spectral radius; (signless) Laplacian spectral radius

AMS SUBJECT CLASSIFICATIONS
05C50; 05C35; 05C75

1. Introduction

Throughout this paper, unless specially indicated, we are concerned with connected undirected simple graph only. Suppose G is a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For a vertex v of G, we use $N_G(v)$ and $d_G(v)$ to denote the neighbour set and degree of v in G, respectively. If there is no confusion, we always simply write $d(u)$ and $N(u)$ instead of $d_G(u)$ and $N_G(u)$, respectively. The sequence $\pi = (d_1, d_2, \ldots, d_n)$ is called the *degree sequence* of G if $d_i = d(v_i)$ holds for $1 \leq i \leq n$. Throughout this paper, we enumerate the degrees in non-increasing order, i.e. $d_1 \geq d_2 \geq \cdots \geq d_n$, and we suppose that $d(v_i) = d_i$, where $1 \leq i \leq n$. Especially, we use $\Delta(G)$ to denote the maximum degree of G. From the definition, it follows that $\Delta(G) = d_1(G)$. We call u a *pendant vertex* if $d(u) = 1$, and call u a *maximum degree vertex* if $d(u) = \Delta(G)$. Suppose that P is a path. If one end vertex of P is a pendant vertex while all the internal vertices of P are vertices with degrees two, then P is called a *pendant path*.

Throughout this paper, k and c are two nonnegative integers, and n is a positive integer. If G is connected with n vertices and $n+c-1$ edges, then G is called a *c-cyclic graph*. In particular, G is called a tree, unicyclic graph, bicyclic graph or a tricyclic graph if $c = 0$,

CONTACT Hong-Jian Lai hjlai@math.wvu.edu
© 2016 Informa UK Limited, trading as Taylor & Francis Group

1, 2 or 3, respectively. The length of a shortest cycle of G is called the *girth* of G and denoted by $g(G)$. Let $\Gamma(n,k;c)$ denote the class of c-cyclic graphs with n vertices and k pendant vertices, let $\Gamma_g(n;c)$ denote the class of c-cyclic graphs with n vertices and girth g, and let $\Gamma_g(n,k;c)$ denote the class of c-cyclic graphs with n vertices, k pendant vertices and girth g, where g is an integer being at least three hereafter. It is easy to see that $\Gamma(n,k;c)=\bigcup_{g=3}^{n}\Gamma_g(n,k;c)$ and $\Gamma_g(n;c)=\bigcup_{k=0}^{n-1}\Gamma_g(n,k;c)$. For simplification, let $\mathbb{T}(n,k)$, $\mathbb{U}(n,k)$, $\mathbb{B}(n,k)$ and $\mathbb{S}(n,k)$ be the class of trees, unicyclic graphs, bicyclic graphs and tricyclic graphs with n vertices and k pendant vertices, respectively.

As usual, K_n, C_n, P_n and $K_{s,n-s}$ define, respectively, the complete graph, cycle, path and complete bipartite graph on n vertices. Suppose v is a vertex of G, and $P_s = w_1w_2\cdots w_s$, where $V(P_s)\cap V(G)=\emptyset$. If we obtain a new graph G^* from G and P_s by adding two edges vw_1 and vw_s, then we say that G^* is obtained from G by *sewing* the path P_s to v of G. If we obtain a new graph G' from G and P_s by adding one edge vw_1, then we say that G' is obtained from G by *attaching* the path P_s to v of G. In the sequel, if we say that we attach or sew k paths to one vertex of G, then we agree that these k paths are vertex disjoint each other, and they are also vertex disjoint with G.

If q is a positive integer and G is a connected graph, qG denote the graph consisting of q copies of the graph G, and $q^{(p)}$ means p copies of the integer q, where p is a nonnegative integer. Paths $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ are said to *have almost equal lengths* if $l_1, l_2, \ldots, l_k$ satisfy $|l_i - l_j| \le 1$ for $1 \le i \le j \le k$. Denoted by $C(q_1,q_2,\ldots,q_c)$, the graph on $\sum_{i=1}^{c}(q_i - 1)+1$ vertices obtained by sewing the paths $P_{q_1-1}, P_{q_2-1}, \ldots, P_{q_c-1}$, to a common vertex, where $q_i \ge 2$ and $1 \le i \le c$. Let $F_n(k, C_{q_1}^{(s)}, C_{q_2}^{(c-s)})$ be the c-cyclic graph on n vertices obtained from $C(q_1^{(s)}, q_2^{(c-s)})$ by attaching k paths of almost equal lengths to the maximum degree vertex of $C(q_1^{(s)}, q_2^{(c-s)})$. In particular, we always simply write $F_n(k, C_{q_1}^{(c)}, C_{q_2}^{(0)})$ and $F_n(k, C_{q_1}^{(0)}, C_{q_2}^{(c)})$ as $F_n(k, C_{q_1}^{(c)})$ and $F_n(k, C_{q_2}^{(c)})$, respectively. Furthermore, we use the symbol $F_n(k)$ ($\cong F_n(k, C_g^{(0)})$) to denote the unique tree on n vertices obtained by attaching k paths of almost equal lengths to a common vertex. If all cycles of G have exactly one common vertex, then G is called a *bundle graph* (see, e.g. [1]). From the definitions, both $C(q_1,q_2,\ldots,q_c)$ and $F_n(k, C_{q_1}^{(s)}, C_{q_2}^{(c-s)})$ are bundle graphs.

Let $D(G)$ be the diagonal matrix of vertex degrees, and $A(G)$ be the adjacency matrix of G. The *Laplacian matrix* and *signless Laplacian matrix* of G are, respectively, defined as $L(G)=D(G)-A(G)$ and $Q(G)=D(G)+A(G)$. The maximum eigenvalues of $L(G)$ and $Q(G)$ are denoted by $\lambda(G)$ and $\mu(G)$, respectively. Furthermore, $\mu(G)$ and $\lambda(G)$ are called, respectively, the *signless Laplacian spectral radius* and *Laplacian spectral radius* of G. For the relation between $\mu(G)$ and $\lambda(G)$, it is well known that

Theorem 1.1 ([2]): *If G is a connected graph on $n \ge 2$ vertices, then*

$$\Delta(G)+1 \le \lambda(G) \le \mu(G),$$

where the first equality holds if and only if $\Delta(G)=n-1$, and the second equality holds if and only if G is bipartite.

If G has the largest (signless) Laplacian spectral radius in some given category of graphs, then we call G a (*signless*) *Laplacian largest extremal graph*.

Recently, the work on determining the signless Laplacian largest extremal graph, and/or Laplacian largest extremal graph in $\Gamma(n,k;c)$, has attained much attention. For any fixed positive number n and k, it is proved that: $F_n(k)$ is the unique Laplacian largest extremal tree (also the signless Laplacian largest extremal tree by Theorem 1.1) of $\mathbb{T}(n,k)$,[3,4] $F_n(k, C_3^{(1)})$ is the unique signless Laplacian largest extremal unicyclic graph of $\mathbb{U}(n,k)$ [5,6] when $n \geq k+3$ and $F_n(k, C_4^{(1)})$ is the unique Laplacian largest extremal unicyclic graph of $\mathbb{U}(n,k)$ [5,7] when $n \geq k+4$; $F_n(k, C_3^{(2)})$ is the unique signless Laplacian largest extremal bicyclic graph of $\mathbb{B}(n,k)$ [8–10] when $n \geq k+5$ and $F_n(k, C_4^{(2)})$ is the unique Laplacian largest extremal bicyclic graph of $\mathbb{B}(n,k)$ [5,7] when $n \geq k+7$; $F_n(k, C_3^{(3)})$ is the unique signless Laplacian largest extremal tricyclic graph of $\mathbb{S}(n,k)$ [3,9,11] when $n \geq k+7$ and $F_n(k, C_4^{(3)})$ is the unique Laplacian largest extremal tricyclic graph of $\mathbb{S}(n,k)$ [12] when $n \geq k+10$.

In the sequel, one of the present authors extended the above referred results of [3–12] by determining the unique signless Laplacian and Laplacian largest extremal graphs of $\Gamma(n,k;c)$ for $c \geq 0$, $k \geq 1$ and $n \geq 2c+k+1$, namely, he proved that

Theorem 1.2 ([13]): *If $k \geq 1$, $c \geq 0$ and $n \geq 2c+k+1$, then $F_n(k, C_3^{(c)})$ is the unique signless Laplacian largest extremal graph of $\Gamma(n,k;c)$.*

Theorem 1.3: ([13]) *Suppose that $k \geq 1$, $c \geq 0$ and G is a Laplacian largest extremal graph of $\Gamma(n,k;c)$. (i) If $n \geq 3c+k+1$, then $G \cong F_n(k, C_4^{(c)})$. (ii) If $n = 2c+k+1+t$ and $0 \leq t \leq c-1$, then $G \cong F_n(k, C_4^{(t)}, C_3^{(c-t)})$. (iii) If $k+1 \leq n \leq 2c+k$, then G is any graph with $\Delta(G) = n-1$.*

At the same time, the extremal graphs with largest (signless) Laplacian spectral radii in the class of $\Gamma_g(n,k;c)$ and/or $\Gamma_g(n;c)$ were also studied by some scholars. Up to now, for any fixed positive number n, k and $g \geq 3$, the following results are identified: $F_n(k, C_g^{(1)})$ is the unique Laplacian largest extremal unicyclic graph of $\Gamma_g(n,k;1)$ for any $k \geq 1$ and $n \geq k+g$ [14]; $F_n(n-g, C_g^{(1)})$ is the unique signless Laplacian largest extremal unicyclic graph of $\Gamma_g(n;1)$ for any $n \geq g$ [15]; $F_n(n-2g+1, C_g^{(2)})$ is the unique signless Laplacian and Laplacian largest extremal bicyclic graph in the class of bicyclic bundle graphs with n vertices and girth g for any $n \geq 2g-1$, respectively, [15–17]; $F_n(n-3g+2, C_g^{(3)})$ is the unique signless Laplacian largest extremal tricyclic graph in the class of tricyclic bundle graphs with n vertices and girth g for any $n \geq 3g-2$.[18] In this paper, we will extend the corresponding results of [15–18] by showing the following theorem:

Theorem 1.4: *Let $\mathcal{G}$ be the class of graphs pertaining to $\Gamma_g(n;c)$, which contain at most $n-c(g-1)-1$ pendant vertices. If $g \geq 3$, $c \geq 1$ and $n \geq \max\{c(g-1)+1, 6\}$, then $F_n(n-c(g-1)-1, C_g^{(c)})$ is the unique signless Laplacian and Laplacian largest extremal graph of $\mathcal{G}$, respectively.*

Remark 1.1: Since $\lambda(K_{2,3}) = \lambda(F_5(0, C_3^{(2)})) = 5$, the condition '$n \geq 6$' in Theorem 1.4 is necessary. Let H_1 be the bicyclic graph with six vertices obtained from $K_4 - e$ by attaching two isolated vertices to one vertex of degree three of $K_4 - e$. Since $\lambda(F_6(1, C_3^{(2)})) = \lambda(H_1) = 6$, the condition 'contain at most $n-c(g-1)-1$ pendant vertices' in Theorem 1.4 is also necessary.

We will also extend the corresponding result of [14] by proving the following theorem:

Theorem 1.5: *Let $c \geq 1$, $g \geq 3$ and $k \geq 1$, and let G be the Laplacian largest extremal graph of $\Gamma_g(n,k;c)$. (i) If g is even and $n \geq c(g-1)+k+1$, then $G \cong F_n(k, C_g^{(c)})$. (ii) If g is odd and $n \geq \frac{1}{2}(g-1)k+cg$, then $G \cong F_n(k, C_g^{(1)}, C_{g+1}^{(c-1)})$.*

Remark 1.2: Since $\lambda(F_{13}(2, C_5^{(1)}, C_6^{(1)})) < 7.166 < 7.192 < \lambda(F_{13}(2, C_5^{(2)}))$, the condition '$n \geq \frac{1}{2}(g-1)k+cg$' in Theorem 1.5 (*ii*) is necessary.

One can also easily see that Theorem 1.5 extends partially result of Theorem 1.3. Furthermore, the following theorem extends partially result of Theorem 1.2.

Theorem 1.6: *If $k \geq 1$, $g \geq 3$, $c \geq 1$ and $n \geq c(g-1)+k+1$, then $F_n(k, C_g^{(c)})$ is the unique signless Laplacian largest extremal graph of $\Gamma_g(n,k;c)$.*

2. The proof of Theorem 1.6

Let uv be an edge of G and v be a vertex of G. Let $m(v)$ denote the average of the degrees of the vertices being adjacent to v, i.e. $m(v) = \sum_{u \in N(v)} d(u)/d(v)$. Denote by

$$\Psi(uv) = \frac{d(u)(d(u)+m(u)) + d(v)(d(v)+m(v))}{d(u)+d(v)}.$$

Theorem 1.1 presents a well-known lower bound to $\lambda(G)$, while the following result gives a famous upper bound for $\mu(G)$.

Lemma 2.1 ([19]): *Let G be a connected graph with at least three vertices, and let $s = \Psi(u_0v_0) = \max\{\Psi(uv) : uv \in E(G)\}$ and $t = \max\{\Psi(uv) : uv \in E(G) \setminus \{u_0v_0\}\}$. Then*

$$\mu(G) \leq 2 + \sqrt{(s-2)(t-2)}$$

with equality holding if and only if G is a regular graph or a bipartite semiregular graph or a path with four vertices.

When G is connected, by the Perron–Frobenius Theorem of non-negative matrices (see e.g. [20]), it follows that

Lemma 2.2: *If G is connected and $G' \subset G$, then $\mu(G') < \mu(G)$ and $\lambda(G') \leq \lambda(G)$.*

Furthermore, by the Perron–Frobenius Theorem of non-negative matrices, there also exists a unique positive unit eigenvector corresponding to $\mu(G)$. In the sequel, we use $f = (f(v_1), f(v_2), \ldots, f(v_n))^T$ to indicate the unique positive unit eigenvector corresponding to $\mu(G)$, and call f the *Perron vector* of G. As we will see later, the following three operations will play an important role in our proofs.

Let $G-uv$ be the graph obtained from G by deleting the edge $uv \in E(G)$, and let $G+uv$ be the graph obtained from G by adding an edge $uv \notin E(G)$. Similarly, $G-v$ denoted the graph obtained from G by deleting the vertex $v \in V(G)$.

Lemma 2.3 ([4]): *Suppose that u, v are two vertices of a connected graph G, and $w_1, w_2, \ldots, w_k$ $(1 \leq k \leq d(v))$ are some vertices of $N(v) \setminus (N(u) \cup \{u\})$. Let $G' = G + w_1u + w_2u + \cdots + w_ku - w_1v - w_2v - \cdots - w_kv$. If f is the Perron vector of G with $f(u) \geq f(v)$, then $\mu(G') > \mu(G)$.*

Let $G_{u,v}$ define a new graph obtained from G by subdividing the edge uv, i.e. adding a new vertex w and two edges wu, wv in $G - uv$, where $uv \in E(G)$. An *internal path*, say $P = w_1w_2 \cdots w_s$ ($s \geq 2$), is a path joining w_1 and w_s (which need not be distinct) such that the degrees of w_1 and w_s are greater than 2, while all other vertices (if exist) $w_2, w_3, \ldots, w_{s-1}$ are of degree 2.

Lemma 2.4 ([5]): *If G is a connected graph and uv is an edge in an internal path of G, then $\mu(G) > \mu(G_{u,v})$.*

Suppose v is a vertex of a connected graph G with at least two vertices. Let $G_{s,t}$ ($t \geq s \geq 1$) be the graph obtained from G by attaching two new paths $P_s = w_1w_2 \cdots w_s$ and $P_t = u_1u_2 \cdots u_t$, respectively, to v of G. Let $G_{s-1,t+1} = G_{s,t} - w_{s-1}w_s + u_tw_s$.

Lemma 2.5 ([5]): *Let G be a connected graph with at least two vertices. If $t \geq s \geq 1$, then $\mu(G_{s,t}) > \mu(G_{s-1,t+1})$.*

To prove our results, we need to extend Lemma 3.1 of [13] as follows.

Lemma 2.6: *Let G be a graph of $\Gamma(n,k;c)$ and $G \notin \{K_{2,3}, C_n\}$, where $c \geq 1$ and $k \geq 0$. If $\lambda(G) \geq k + 2c + 1$ or $\mu(G) \geq k + 2c + 1$, then G is one of the following candidates: (i) G is obtained by attaching k paths and then sewing another c paths, respectively, to a common vertex; (ii) G is obtained by adding an edge joining one vertex of a cycle to one vertex of degree two of a path; (iii) G is obtained by attaching one path to each of two adjacent vertices of a cycle, respectively; (iv) G is obtained by adding one edge to two nonadjacent vertices of C_n; (v) G is obtained by adding one edge joining one vertex of C_s with one vertex of C_{n-s}, where $n - s \geq s \geq 3$.*

Proof: Suppose the degree sequence of G is $(d_1, d_2, \ldots, d_n)$ and $G \notin \{K_{2,3}, C_n\}$. Since $G \in \Gamma(n,k;c)$, we have

$$2(n + c - 1) = \sum_{i=1}^{n} d_i. \tag{2.1}$$

If $d_1 + d_2 \geq k + 2c + 3$, then

$$2(n + c - 1) = \sum_{i=1}^{n} d_i \geq k + 2c + 3 + 2(n - 2 - k) + k = 2n + 2c - 1, \text{ a contradiction.}$$

If $k \geq 1$ and $d_1 + d_2 \leq k + 2c + 1$, then G is neither regular nor bipartite semiregular. By Theorem 1.1 and Lemma 2.1,

$$\lambda(G) \leq \mu(G) < 2 + \sqrt{(d_1 + d_2 - 2)^2} = d_1 + d_2 \leq k + 2c + 1, \text{ a contradiction.}$$

If $k = 0$ and $d_1 + d_2 \leq 2c$, then by Theorem 1.1 and Lemma 2.1, we have

$$\lambda(G) \leq \mu(G) \leq d_1 + d_2 \leq 2c, \text{ a contradiction.}$$

If $k = 0$ and $d_1 + d_2 = 2c + 1$, then by (2.1), we have $d_1 > d_2 \geq d_3 = 3$. By Theorem 1.1 and Lemma 2.1,

$$\lambda(G) \leq \mu(G) \leq d_1 + d_2 \leq 2c + 1.$$

Furthermore, by Lemma 2.1, $\mu(G) = 2c+1$ implies that $d_1 = 2c-2$ and $d_2 = d_3 = \cdots = d_n = 3$, which contradicts (2.1).
Therefore, if $\lambda(G) \geq k+2c+1$ or $\mu(G) \geq k+2c+1$, then $d_1+d_2 = k+2c+2$. Since G contains exactly k pendant vertices with (2.1), we have

$$d_1+d_2 = k+2c+2,\ \ d_3 = d_4 = \cdots = d_{n-k} = 2 \ \text{ and } \ d_{n-k+1} = \cdots = d_n = 1. \quad (2.2)$$

If $d_1 \leq k+2c-2$, then by Theorem 1.1 and Lemma 2.1 with (2.2),

$$\lambda(G) \leq \mu(G) \leq 2+\sqrt{(d_1+d_2-2)(d_1+d_3-2)} = 2+\sqrt{(k+2c)d_1} < k+2c+1,$$

a contradiction. If $d_1 = k+2c$, by (2.2) we have $d_2 = d_3 = \cdots = d_{n-k} = 2$ and $d_{n-k+1} = d_{n-k+2} = \cdots = d_n = 1$, then G is a graph of ($i$). Notice that $d_1 = k+2c+2-d_2 \leq k+2c$ by (2.2). Thus, the final possibility is $d_1 = k+2c-1$. By (2.2) it follows that $d_2 = 3$. Next, we shall prove that

$$\mu(G) < \max\{\Psi(uv) : uv \in E(G)\}. \quad (2.3)$$

If $c \geq 2$, since $d_1 = k+2c-1 \geq 3 = d_2 > d_3 = 2$ and $G \ncong K_{2,3}$, then G is neither regular nor bipartite semiregular. In this case, (2.3) follows from Lemma 2.1. Now, we consider the case $c = 1$. Since $G \ncong C_n$, $d_n = 1$. Thus, G is neither regular nor bipartite semiregular, and hence (2.3) follows from Lemma 2.1.

Suppose $\Psi(u_0v_0) = \max\{\Psi(uv) : uv \in E(G)\}$, where $d(u_0) \geq d(v_0)$. Since G is connected and $c \geq 1$, $d(u_0) \in \{k+2c-1, 3, 2\}$. If $d_1 = k+2c-1 = 3 = d_2$, then either $c = 2$ and $k = 0$ or $c = 1$ and $k = 2$. If $v_1v_2 \notin E(G)$, then by Theorem 1.1 and (2.3) we have $\lambda(G) \leq \mu(G) < \Psi(u_0v_0) \leq 5$, a contradiction. Thus, $v_1v_2 \in E(G)$. If $c = 2$ and $k = 0$, then G is one graph of (iv) or (v). If $c = 1$ and $k = 2$, then G is one graph of (ii) or (iii). In the following, we only need to consider the case $d_1 = k+2c-1 \geq 4$:

Case 1 $d(u_0) = d_1 = k+2c-1 \geq 4$. If $d(v_0) = 3$, then $d(u_0) = d_1$ and $d(v_0) = d_2$ by (2.2). Therefore,

$$\begin{aligned}\Psi(u_0v_0) &\leq \frac{d_1^2+d_2+2(d_1-1)+d_2^2+d_1+2(d_2-1)}{d_1+d_2}\\ &= k+2c-1+\frac{14}{k+2+2c}\\ &\leq k+2c+1.\end{aligned}$$

If $d(v_0) = 2$, then

$$\Psi(u_0v_0) \leq \frac{d_1^2+3+2(d_1-1)+4+d_1+3}{d_1+2} = k+2c+\frac{6}{k+2c+1} \leq k+2c+1.$$

If $d(v_0) = 1$, then

$$\Psi(u_0v_0) \leq \frac{d_1^2+1+3+2(d_1-2)+1+d_1}{d_1+1} = k+2c+1-\frac{1}{k+2c} < k+2c+1.$$

Case 2 $d(u_0) = 3$. By (2.2), we have $d_1 \geq 4$ and $d(u_0) = 3 = d_2 > d_3$, which implies that $1 \leq d(v_0) \leq 2$. If $d(v_0) = 2$, then

$$\Psi(u_0v_0) \leq \frac{d_2^2 + d_1 + 2(d_2 - 1) + 4 + d_1 + d_2}{d_2 + 2} = \frac{2(k + 2c) + 18}{5} < k + 2c + 1.$$

If $d(v_0) = 1$, then

$$\Psi(u_0v_0) \leq \frac{d_2^2 + d_1 + 2 + 1 + 1 + d_2}{d_2 + 1} = \frac{k + 2c + 15}{4} < k + 2c + 1.$$

Case 3 $d(u_0) = 2$. Then, $1 \leq d(v_0) \leq 2$. If $d(v_0) = 2$, then

$$\Psi(u_0v_0) \leq \frac{2(4 + d_1 + 2)}{2 + 2} = \frac{k + 2c + 5}{2} < k + 2c + 1.$$

If $d(v_0) = 1$, then

$$\Psi(u_0v_0) \leq \frac{4 + d_1 + 1 + 1 + 2}{2 + 1} = \frac{k + 2c + 7}{3} < k + 2c + 1.$$

Now, from Theorem 1.1 with (2.3) and the above discussion, we can conclude that $\lambda(G) \leq \mu(G) < \Psi(u_0v_0) \leq k + 2c + 1$, a contradiction. □

Lemma 2.7: *If G is one graph of (ii) or (iii) as defined in Lemma 2.6 and $g(G) = g$, then $\mu(G) < \mu(F_n(2, C_g^{(1)}))$.*

Proof: Let f be the Perron vector of G. We first suppose that G is a graph of (*iii*). Without loss of generality, we may suppose that $f(v_1) \geq f(v_2)$. Let u be a vertex of $N_G(v_2) \setminus V(C_g)$. Let $G' = G + v_1u - v_2u$. By Lemma 2.3, we have $\mu(G) < \mu(G')$. Since G' is obtained from a cycle C_g by attaching two paths to exactly one vertex of C_g, by Lemma 2.5 we have $\mu(G') \leq \mu(F_n(2, C_g^{(1)}))$. Therefore, $\mu(G) < \mu(F_n(2, C_g^{(1)}))$ holds.

We secondly suppose that G is a graph of (*ii*). Suppose that $v_1 \in V(C_g)$. Then, $v_2 \notin V(C_g)$. If $f(v_1) \geq f(v_2)$, since $d_G(v_2) = 3$, it can be proved similarly with the former case. If $f(v_1) < f(v_2)$, choose u as a vertex of $N_G(v_1) \cap V(C_g)$. Let $G' = G + v_2u - v_1u$. By Lemma 2.3, we have $\mu(G) < \mu(G')$. Suppose that $N_{G'}(u) = \{v_2, v\}$. Let $G_1 = G' + vv_2 - uv - uv_2$, $G_2 = G_1 - u$ and $G_3 = G_1 + xu$, where x is a pendant vertex of G_1. Since $d_{G_2}(v_2) = 4$, vv_2 lies in an internal path of G_2. By Lemma 2.4, $\mu(G') < \mu(G_2)$. Furthermore, since $G_1 \subset G_3$, by Lemma 2.2 we have $\mu(G_2) = \mu(G_1) < \mu(G_3)$, and hence $\mu(G') < \mu(G_3)$. Note that G_3 is obtained by attaching two paths to exactly one vertex of C_g. By Lemma 2.5, we have $\mu(G_3) \leq \mu(F_n(2, C_g^{(1)}))$.

Now, we can conclude that $\mu(G) < \mu(F_n(2, C_g^{(1)}))$ holds. □

Proof of Theorem 1.6: Suppose that G is a signless Laplacian largest extremal graph of $\Gamma_g(n, k; c)$. Since $F_n(k, C_g^{(c)}) \in \Gamma_g(n, k; c)$, by the choice of G and Theorem 1.1 it follows that $\mu(G) \geq \mu(F_n(k, C_g^{(c)})) > k + 2c + 1$. By Lemmas 2.5–2.7, we can conclude that G is obtained by attaching k paths of almost equal lengths to the maximum degree vertex of $C(q_1, q_2, \ldots, q_c)$, where $q_1 \geq q_2 \geq \cdots \geq q_{c-1} \geq q_c = g$.

If $q_1 = g$, then $q_1 = q_2 = \cdots = q_c = g$ and hence $G \cong F_n(k, C_g^{(c)})$, the result already holds. Thus, we only need to consider the case of $q_1 \geq g + 1$.

Suppose that $\{u, v, w\} \in V(C_{q_1})\backslash\{v_1\}$ such that $uv \in E(C_{q_1})$ and $vw \in E(C_{q_1})$. Let x be a pendant vertex of G. Let $G_1 = G + uw - uv - vw$, $G_2 = G_1 - v$ and $G_3 = G_1 + xv$. Then, $G_3 \in \Gamma_g(n, k; c)$. Since $d_G(v_1) = k + 2c \geq 3$, uw lies in an internal path of G_2. By Lemma 2.4, we have $\mu(G) < \mu(G_2)$. Note that $G_1 \subset G_3$. By Lemma 2.2, $\mu(G) < \mu(G_2) = \mu(G_1) < \mu(G_3)$, contradicting the choice of G. Thus, $G \cong F_n(k, C_g^{(c)})$. □

3. The proofs of Theorems 1.4–1.5

By Lemma 2.2, if we add some edges to a connected graph, the signless Laplacian spectral radius will increase strictly. However, the following result shows that additional edges to a connected graph can result for unchanged Laplacian spectral radius:

Lemma 3.1 ([21,22]): *Let v be a vertex of a connected graph G with at least two vertices and let G' be obtained from G by attaching k paths of equal lengths to v. If G'' is obtained from G by adding any s $(1 \leq s \leq \frac{k(k-1)}{2})$ edges among these pendant vertices of G'', which belong to the referred k paths, then $\lambda(G') = \lambda(G'')$.*

Lemma 3.2: *If $k \geq 1$, $c \geq s \geq 1$ and $n \geq c(g-1) + k + s + 1$, then*

$$\mu(F_n(k, C_{g+1}^{(s)}, C_g^{(c-s)})) < \mu(F_n(k, C_{g+1}^{(s-1)}, C_g^{(c-s+1)})).$$

Proof: Suppose v_1 is the maximum degree vertex of $F_n(k, C_{g+1}^{(s)}, C_g^{(c-s)})$, and suppose that $\{u, v, w\} \in V(C_{g+1})\backslash\{v_1\}$ such that $uv \in E(C_{g+1})$ and $vw \in E(C_{g+1})$ in $F_n(k, C_{g+1}^{(s)}, C_g^{(c-s)})$. Let x be a pendant vertex of $F_n(k, C_{g+1}^{(s)}, C_g^{(c-s)})$. Let $G_1 = F_n(k, C_{g+1}^{(s)}, C_g^{(c-s)}) + uw - uv - vw$, $G_2 = G_1 - v$ and $G_3 = G_1 + xv$. Since $d_{G_2}(v_1) = k + 2c \geq 3$, uw is contained in an internal path of G_2. By Lemma 2.4, we have $\mu(F_n(k, C_{g+1}^{(s)}, C_g^{(c-s)})) < \mu(G_2)$. Note that $G_1 \subset G_3$. By Lemma 2.2, $\mu(G_2) = \mu(G_1) < \mu(G_3)$. Furthermore, since G_3 is obtained by attaching k paths to the maximum degree vertex of $C((g+1)^{(s-1)}, g^{(c-s+1)})$, by Lemma 2.5 we have $\mu(G_3) \leq \mu(F_n(k, C_{g+1}^{(s-1)}, C_g^{(c-s+1)}))$. Now, the result follows by combining the above discussion. □

Suppose that $g \geq 3$ is an odd number. Let $F_n^*(k, C_{g+1}^{(c-s)}, C_g^{(s)})$ be a graph obtained from $F_n(k, C_{g+1}^{(c-s)}, C_g^{(s)})$ by deleting every edge, which has the largest distance from the maximum degree vertex of $F_n(k, C_{g+1}^{(c-s)}, C_g^{(s)})$ in each cycle C_g. From the definition, $F_n^*(k, C_{g+1}^{(c-s)}, C_g^{(s)})$ can be also obtained from $F_{n-s(g-1)}(k, C_{g+1}^{(c-s)})$ by attaching $2s$ paths of length $\frac{1}{2}(g-1) - 1$ to the vertex of degree $k + 2(c-s)$ of $F_{n-s(g-1)}(k, C_{g+1}^{(c-s)})$. By Lemma 3.1, the following equation holds for any $c \geq s \geq 1$, $k \geq 1$ and $g \geq 3$:

$$\lambda(F_n^*(k, C_{g+1}^{(c-s)}, C_g^{(s)})) = \lambda(F_n(k, C_{g+1}^{(c-s)}, C_g^{(s)})). \tag{3.1}$$

Lemma 3.3: *Suppose that g is odd, and G is a c-cyclic graph on n vertices obtained by attaching k paths to the vertex with degree $2c$ of $C(g^{(s)}, q_1, q_2, \ldots, q_{c-s})$. If $c \geq s \geq 1$, $k \geq 1$ and $q_1 \geq q_2 \geq \cdots \geq q_{c-s} \geq g+1$, then*

$$\lambda(G) \leq \lambda(F_n^*(k, C_{g+1}^{(c-s)}, C_g^{(s)})) = \lambda(F_n(k, C_{g+1}^{(c-s)}, C_g^{(s)})),$$

where the first equality holds if and only if $G \cong F_n(k, C_{g+1}^{(c-s)}, C_g^{(s)})$.

Proof: Since v_1 is the maximum degree vertex of G, we have $d_G(v_1) = k + 2c$. Let G_1 be a $(c-s)$-cyclic graph on n vertices obtained by deleting every edge, which has the largest distance from v_1 in each C_g. By Lemma 3.1, $\lambda(G) = \lambda(G_1)$.

First we assume that $c = s$. In this case, G_1 is a tree obtained by attaching $k + 2c$ paths (among which at least $2c$ paths are $P_{0.5(g-1)}$) to a common vertex. By (3.1) with Theorem 1.1 and Lemma 2.5, we have

$$\lambda(G) = \lambda(G_1) = \mu(G_1) \le \mu(F_n^*(k, C_g^{(c)})) = \lambda(F_n^*(k, C_g^{(c)})).$$

If $\lambda(G) = \lambda(F_n^*(k, C_g^{(c)}))$, then $\mu(G_1) = \mu(F_n^*(k, C_g^{(c)}))$, and hence $G_1 \cong F_n^*(k, C_g^{(c)})$ by Lemma 2.5. By the definition of G_1, $G \cong F_n(k, C_g^{(c)})$. Conversely, if $G \cong F_n(k, C_g^{(c)})$, then by (3.1) we have $\lambda(G) = \lambda(F_n^*(k, C_g^{(c)}))$. So, the result holds for $c = s$.

Next, we assume that $c - s \ge 1$. In this case, $g(G_1) = q_{c-s} \ge g + 1$.

Case 1 $q_1 = g + 1$. Now, $q_1 = q_2 = \cdots = q_{c-s} = g + 1$ and G_1 is a $(c-s)$-cyclic graph obtained by attaching $k + 2s$ paths (among which at least $2s$ paths are $P_{0.5(g-1)}$) to the vertex of degree $2(c-s)$ of $C((g+1)^{(c-s)})$. By Lemma 2.5,

$$\mu(G_1) \le \mu\left(F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)\right)$$

with equality holding if and only if $G_1 \cong F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)$. By Theorem 1.1 and Lemma 3.1, we have

$$\lambda(G) = \lambda(G_1) = \mu(G_1) \le \mu\left(F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)\right) = \lambda(F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)).$$

If $\lambda(G) = \lambda(F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right))$, then $\mu(G_1) = \mu(F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right))$, and hence $G \cong F_n\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)$. Conversely, if $G \cong F_n\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)$, then by Lemma 3.1 implies that $\lambda(G) = \lambda(F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right))$.

Case 2 $q_1 \ge g + 2$. Suppose that $\{u, v, w\} \in V(C_{q_1})\backslash\{v_1\}$ such that $uv \in E(C_{q_1})$ and $vw \in E(C_{q_1})$. Let x be a pendent vertex pertaining to a longest pendant path of G. Let $G_2 = G_1 + uw - uv - vw$, $G_3 = G_2 - v$ and $G_4 = G_2 + xv$. Since $d_{G_3}(v_1) = k + 2c \ge 3$, uw is contained in an internal path of G_3. By Lemma 2.4, we have $\mu(G_1) < \mu(G_3)$. Note that $G_2 \subset G_4$. By Lemma 2.2, $\mu(G_1) < \mu(G_3) = \mu(G_2) < \mu(G_4)$.

Note that G_4 contains exactly $k + 2s$ pendant vertices, and at least $2s$ pendant vertices are contained in $2s$ pendant paths of lengths $\frac{1}{2}(g-1)$ initial from the maximum degree vertex of G_4. By repeating the above operation, we can obtain a $(c-s)$-cyclic graph G_5 such that $\mu(G_4) \le \mu(G_5)$, where G_5 is obtained by attaching $k + 2s$ paths (among which at least $2s$ paths are $P_{0.5(g-1)}$) to the vertex of degree $2(c-s)$ of $C((g+1)^{(c-s)})$. By Lemma 2.5, $\mu(G_5) \le \mu(F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right))$. Now, from Theorem 1.1, we have $\lambda(G) = \lambda(G_1) \le \mu(G_1) < \mu(G_5) \le \mu(F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)) = \lambda(F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right))$. □

Lemma 3.4: *If g is odd, $c \geq s \geq 1$, $k \geq 1$ and $n \geq \frac{1}{2}(g-1)k + cg + 2 - s$, then*

$$\lambda\left(F_n\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)\right) < \lambda\left(F_n\left(k, C_{g+1}^{(c-s+1)}, C_g^{(s-1)}\right)\right).$$

Proof: Let P be a longest pendant path among these k pendant paths, which are initial from the maximum degree vertex (i.e. v_1) of $F_n\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)$, and let y be the pendant vertex of P. If $|V(P)| \leq \frac{1}{2}(g+1)$, then

$$n \leq s(g-1) + g(c-s) + 1 + \frac{1}{2}(g-1)k = \frac{1}{2}(g-1)k + gc + 1 - s, \text{ a contradiction.}$$

Thus, $|V(P)| \geq \frac{1}{2}(g+1)+1$. Let x be a pendant vertex of a pendant path with length $\frac{1}{2}(g-1)$, which is initial from v_1 in $F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)$. By the definition of $F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)$, such vertex x must exist.

Let $G_1 = F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right) + xy$. Then $g(G_1) \geq g+1$ and G_1 is a $(c-s+1)$-cyclic graph obtained by attaching $k + 2(s-1)$ paths (among which at least $2s-1$ paths are $P_{0.5(g-1)}$) to the maximum degree vertex of $C((g+1)^{(c-s)}, q)$, where $q \geq g+1$. By Lemmas 2.4 and 2.5, $\mu(G_1) \leq \mu(F_n^*(k, C_{g+1}^{(c-s+1)}, C_g^{(s-1)}))$. Since $g+1$ is even, by (3.1), we have

$$\mu(F_n^*(k, C_{g+1}^{(c-s+1)}, C_g^{(s-1)})) = \lambda(F_n^*(k, C_{g+1}^{(c-s+1)}, C_g^{(s-1)})).$$

Furthermore, since $F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right) \subset G_1$, by (3.1) with Theorem 1.1 and Lemma 2.2, we have

$$\begin{aligned}\lambda\left(F_n\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)\right) &= \lambda(F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)) = \mu(F_n^*\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)) \\ &< \mu(G_1) \leq \lambda(F_n^*(k, C_{g+1}^{(c-s+1)}, C_g^{(s-1)})) = \lambda(F_n(k, C_{g+1}^{(c-s+1)}, C_g^{(s-1)})).\end{aligned}$$

Thus, the required inequality holds. □

Lemma 3.5: *If G is one graph of (ii) or (iii) as defined in Lemma 2.6 and $g(G) = g$, then $\lambda(G) < \lambda(F_n(2, C_g^{(1)}))$.*

Proof: When g is even, by Lemma 2.7 and Theorem 1.1, the result already holds. Thus, we may suppose that g is odd and $n \geq 6$ in the following, as $n = 5$ can be checked easily. If $g = 3$ and G is a graph of (iii), it is well-known that [23]

$$\lambda(G) \leq \max\{|N(u) \cup N(v)| : uv \in E(G)\},$$

and hence $\lambda(G) \leq 5 < \lambda(F_n(2, C_g^{(1)}))$ by Theorem 1.1. If $g = 3$ and G is a graph of (ii), let G' be the graph obtained from G by deleting one edge with both end vertices are of degrees two in C_3. By Lemmas 2.1 and 3.1 with Theorem 1.1, we have

$$\lambda(G) = \lambda(G') \leq \max\{\Psi(uv) : uv \in E(G')\} = 5 < \lambda(F_n^*(2, C_g^{(1)})).$$

If $g \geq 5$, then by Lemmas 2.1–2.2 it follows that

$$\lambda(G) \leq \mu(G) \leq 2 + \sqrt{\left(\frac{16}{3} - 2\right)(5-2)} < 5.163 < \lambda(F_7^*(2, C_5^{(1)})) \leq \lambda(F_n^*(2, C_g^{(1)})). \tag{3.2}$$

Now, the result follows from (3.1). □

Proof of Theorem 1.5: When g is even, by Theorems 1.1 and 1.6, we have $\lambda(G) \leq \mu(G) \leq \mu(F_n(k, C_g^{(c)})) = \lambda(F_n(k, C_g^{(c)}))$, where $\lambda(G) = \lambda(F_n(k, C_g^{(c)}))$ holds if and only if $G \cong F_n(k, C_g^{(c)})$. Thus, (i) follows. Now, we turn to prove (ii).

Since $n \geq \frac{1}{2}(g-1)k + cg + 1$, $F_n(k, C_g, C_{g+1}^{(c-1)}) \in \Gamma_g(n, k; c)$. By Theorem 1.1 and the choice of G, $\lambda(G) \geq \lambda(F_n(k, C_g, C_{g+1}^{(c-1)})) \geq k + 2c + 1$. From Lemmas 2.6 and 3.5, it follows that G is obtained from $C(q_1, q_2, \ldots, q_c)$ by attaching k paths to the maximum degree vertex of $C(q_1, q_2, \ldots, q_c)$, where $q_1 \geq q_2 \geq \cdots \geq q_{c-1} \geq q_c = g$.

We suppose that G contains exactly s cycles C_g. By Lemma 3.3, we have

$$\lambda(G) \leq \lambda\left(F_n\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)\right)$$

with equality holding if and only if $G \cong F_n\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)$. If $s = 1$, then the result already holds. Otherwise, $s \geq 2$. By Lemma 3.4, it follows that

$$\lambda\left(F_n\left(k, C_{g+1}^{(c-s)}, C_g^{(s)}\right)\right) < \lambda\left(F_n\left(k, C_{g+1}^{(c-s+1)}, C_g^{(s-1)}\right)\right) \leq \cdots \leq \lambda(F_n(k, C_{g+1}^{(c-1)}, C_g^{(1)})).$$

Thus, $\lambda(G) < \lambda(F_n(k, C_{g+1}^{(c-1)}, C_g^{(1)}))$. This completes the proof of (ii). □

Lemma 3.6: *Let G be one graph of (iv) or (v) as defined in Lemma 2.6 and $g(G) = g$. If $n \geq 2g - 1$, then $\lambda(G) < \lambda(F_n(n - 2g + 1, C_g^{(2)}))$ and $\mu(G) < \mu(F_n(n - 2g + 1, C_g^{(2)}))$.*

Proof: In this case, G is neither regular nor bipartite semiregular. If $n - 2g + 1 \geq 1$, by Lemma 2.1 and Theorem 1.1, we have

$$\lambda(G) \leq \mu(G) < 6 \leq \lambda(F_n(n - 2g + 1, C_g^{(2)})) \leq \mu(F_n(n - 2g + 1, C_g^{(2)})),$$

and the result already holds.

If $n = 2g - 1$, it is easy to check the result holds for $g = 3$. Now, we suppose that $g \geq 4$. Since $F_7^*(2, C_5^{(1)}) \subset F_n(0, C_g^{(2)})$, by Theorem 1.1, Lemmas 2.1–2.2 and (3.2), we have

$$\lambda(G) \leq \mu(G) < 5.163 < \lambda(F_7^*(2, C_5^{(1)})) \leq \lambda(F_n(0, C_g^{(2)})) \leq \mu(F_n(0, C_g^{(2)})).$$

This completes the proof of this result. □

Proof of Theorem 1.4: Let G be a Laplacian or signless Laplacian largest extremal graph of $\mathcal{G}$. Since $F_n(k, C_g^{(c)}) \in \mathcal{G}$, by Lemmas 2.6–2.7 and Lemmas 3.5–3.6, G is obtained from $C(q_1, q_2, \ldots, q_c)$ by attaching k paths to the maximum degree vertex of $C(q_1, q_2, \ldots, q_c)$, where $q_1 \geq q_2 \geq \cdots \geq q_{c-1} \geq q_c = g$ and $0 \leq k \leq n - c(g-1) - 1$. Furthermore, when $n = c(g-1) + 1$, then $G \cong F_n(0, C_g^{(c)})$ and the result already holds. We may suppose that $n \geq c(g-1) + 2$ in the following.

If $k \le n - c(g-1) - 2$, by Lemma 2.1 and Theorem 1.1, we have

$$\begin{aligned}\lambda(G) \le \mu(G) \le d_1 + d_2 = 2c + k + 2 \le 2c + n - c(g-1) \\ \le \lambda(F_n(n - c(g-1) - 1, C_g^{(c)})) \le \mu(F_n(n - c(g-1) - 1, C_g^{(c)})).\end{aligned}$$

By the structure of G and Lemma 2.1, G is regular with $d_1 = 2c + k = 2 = d_2$. Thus, $c = 1$ and $k = 0$, which implies that $G \cong C_n$. In this case, $0 \le k \le n - c(g-1) - 2 = -1$, a contradiction. Therefore, $k = n - c(g-1) - 1$ and hence $G \cong F_n(n - c(g-1) - 1, C_g^{(c)})$. □

Acknowledgements

The authors would like to thank the referees for their valuable comments, corrections and suggestions, which lead to an improvement of the original manuscript.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The first author is partially supported by NSFC project [grant number 11571123], the Training Program for Outstanding Young Teachers in University of Guangdong Province [grant number YQ2015027]; China Scholarship Council, and the second author is partially supported by the National Research Foundation funded by the Korean government with the [grant number 2013R1A1A2009341].

References

[1] Borovićanin B, Petrović M. On the index of cactuses with n vertices. Publ. Inst. Math. Beograd. 2006;79:13–18.
[2] Merris R. Laplacian matrices of graphs: a survey. Linear Algebra Appl. 1994;197–198:143–176.
[3] Li K, Wang L, Zhao G. The signless Laplacian spectral radius of tricyclic graphs and trees with k pendant vertices. Linear Algebra Appl. 2011;435:811–822.
[4] Zhang X-D. The Laplacian spectral radii of trees with degree sequences. Discrete Math. 2008;308:3143–3150.
[5] Liu M, Tan X, Liu B. The (signless) Laplacian spectral radius of unicyclic and bicyclic graphs with n vertices and k pendant vertices. Czech. Math. J. 2010;60:849–867.
[6] Zhang X-D. The signless Laplacian spectral radius of graphs with given degree sequences. Discrete Appl. Math. 2009;157:2928–2937.
[7] Guo J-M. The Laplacian spectral radii of unicyclic and bicyclic graphs with n vertices and k pendant vertices. Sci. China Math. 2010;53:2135–2142.
[8] Feng L. The signless Laplacian spectral radius for bicyclic graphs with k pendant vertices. Kyungpook Math. J. 2010;50:109–116.
[9] Liu M, Tan X, Liu B. The largest signless Laplacian spectral radius of connected bicyclic and tricyclic graphs with n vertices and k pendant vertices. Appl. Math. J. Chinese Univ. Ser. A. 2011;26:215–222. Chinese.
[10] Huang Y, Liu B, Liu Y. The signless Laplacian spectral radius of bicyclic graphs with prescribed degree sequences. Discrete Math. 2011;311:504–511.
[11] Zhang J, Guo J-M. The signless Laplacian spectral radius of tricyclic graphs with k pendant vertices. J. Math. Res. Appl. 2012;32:281–287.

[12] Guo S-G, Wang Y-F. The Laplacian spectral radius of tricyclic graphs with n vertices and k pendant vertices. Linear Algebra Appl. 2009;431:139–147.
[13] Liu M. The (signless) Laplacian spectral radii of c-cyclic graphs with n vertices and k pendant vertices. Electron. J. Linear Algebra. 2012;23:942–952.
[14] Liu H, Lu M, Zhang S. On the (Laplacian) spectral radius of unicyclic graphs with girth g and k pendant vertices. Sci. Sin. Math. 2010;40:1017–1024. Chinese.
[15] Li K, Wang L, Zhao G. The signless Laplacian spectral radius of unicyclic and bicyclic graphs with a given girth. Electron. J. Combin. 2011;18:p.183.
[16] Zhai M, Yu G, Shu J. The Laplacian spectral radius of bicyclic graphs with a given girth. Comput. Math. Appl. 2010;59:376–381.
[17] Zhu Z. The signless Laplacian spectral radius of bicyclic graphs with a given girth. Electron. J. Linear Algebra. 2011;22:378–388.
[18] Lu Q, Wang L. The signless Laplacian spectral radius of tricyclic graphs with a given girth. J. Math. Res. Appl. 2014;34:379–391.
[19] Pan Y-L. Sharp upper bounds for the Laplacian graph eigenvalues. Linear Algebra Appl. 2002;355:287–295.
[20] Horn RA, Johnson CR. Matrix analysis. New York (NY): Cambridge University Press; 1990.
[21] Das KC. The Laplacian spectrum of a graph. Comput. Math. Appl. 2004;48:715–724.
[22] Guo J-M. The Laplacian spectral radius of a graph under perturbation. Comput. Appl. Math. 2007;54:709–720.
[23] Das KC. An improved upper bound for Laplacian graph eigenvalues. Linear Algebra Appl. 2003;368:269–278.