Characterization of Digraphic Sequences with Strongly Connected Realizations

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Abstract: Let $\mathbf{d} = \{(d_1^+, d_1^-), \ldots, (d_n^+, d_n^-)\}$ be a sequence of of nonnegative integers pairs. If a digraph D with $V(D) = \{v_1, v_2, \ldots, v_n\}$ satisfies $d_D^+(v_i) = d_i^+$ and $d_D^-(v_i) = d_i^-$ for each i with $1 \le i \le n$, then $\mathbf{d}$ is called a degree sequence of D. If D is a strict digraph, then $\mathbf{d}$ is called a strict digraphic sequence. Let $\langle \mathbf{d} \rangle$ be the collection of digraphs with degree sequence $\mathbf{d}$. We characterize strict digraphic sequences $\mathbf{d}$ for which there

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exists a strict strong digraph $D \in \langle \mathbf{d} \rangle$. J. Graph Theory 84: 191–201, 2017

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1. INTRODUCTION

Digraphs in this article are finite and loopless. We follow [1] for undefined terminologies and notations. As in [1], $V(D)$ and $A(D)$ denote the vertex set and the arc set of a digraph D; and (u, v) represents an arc oriented from a vertex u to a vertex v. A digraph D is strict if D has neither loops nor parallel arcs; and D is nontrivial if $A(D) \neq \emptyset$. If X and Y are vertex subsets (not necessarily disjoint) of a digraph D, then let $A(X, Y) = \{(u, v) \in A(D) | x \in X$ and $y \in Y\}$. For a subset $X \subseteq V(D)$, define

$$\partial_D^+(X) = A(F, V(D) - X) \text{ and } \partial_D^-(X) = \partial_D^+(V(D) - X).$$

We use $D[X]$ to denote the subdigraph of D induced by X. If F is a subdigraph of D, then for notational convenience, we often use $\partial_D^+(F)$, $\partial_D^-(F)$ for $\partial_D^+(V(F))$, $\partial_D^-(V(F))$, respectively.

For a vertex u of D, define the *out-degree* $d_D^+(u)$ (*in-degree* $d_D^-(u)$, respectively) of u to be $|\partial_D^+(\{u\})|$ ($|\partial_D^-(\{u\})|$, respectively). Let $V(D) = \{v_1, \ldots, v_n\}$. The sequence of integer pairs $\{(d_D^+(v_1), d_D^-(v_1)), (d_D^+(v_2), d_D^-(v_2)), \ldots, (d_D^+(v_n), d_D^-(v_n))\}$ is called a *degree sequence* of D. Throughout this article, we always assume in the sequence $\mathbf{d} = \{(d_1^+, d_1^-), \ldots, (d_n^+, d_n^-)\}$, the first components are so ordered that $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$.

A sequence of integer pairs $\mathbf{d} = \{(d_1^+, d_1^-), \ldots, (d_n^+, d_n^-)\}$ is *digraphic* (*multidigraphic* or *strict digraphic*, respectively) if there exists a digraph (a multidigraph or a strict digraph, respectively) D with degree sequence $\mathbf{d}$, where D is called a $\mathbf{d}$-realization. Let $\langle \mathbf{d} \rangle$ be the set of all $\mathbf{d}$-realizations. The following theorem is well known, which can be found in [2, 8, 13], among others.

Theorem 1.1 (Fulkerson-Ryser). *Let $\mathbf{d} = \{(d_1^+, d_1^-), \ldots, (d_n^+, d_n^-)\}$ be a sequence of nonnegative integer pairs with $d_1^+ \geq \cdots \geq d_n^+$. Then $\mathbf{d}$ is strict digraphic if and only if each of the following holds:*

(i) $d_i^+ \leq n - 1, d_i^- \leq n - 1$ *for all* $1 \leq i \leq n$;
(ii) $\sum_{i=1}^n d_i^+ = \sum_{i=1}^n d_i^-$;
(iii) $\sum_{i=1}^k d_i^+ \leq \sum_{i=1}^k \min\{k-1, d_i^-\} + \sum_{i=k+1}^n \min\{k, d_i^-\}$ *for all* $1 \leq k \leq n$.

For any sequence $\mathbf{d} = \{(d_1^+, d_1^-), \ldots, (d_n^+, d_n^-)\}$ satisfying $\sum_{i=1}^n d_i^+ = \sum_{i=1}^n d_i^-$, we associate with a bipartite graph G with vertex bipartition (X, Y) such that $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ and such that for each i with $1 \leq i \leq n$, $d_G(x_i) = d_i^+$ and $d_G(y_i) = d_i^-$. Obtain a digraph D'' from G by orienting each edge $x_i y_j \in E(G)$ to an arc (x_i, y_j). Then obtain a digraph D' on n vertices from D'' by identifying each x_i with y_i, for every i with $1 \leq i \leq n$. By the construction, D' is a digraph with degree sequence $\mathbf{d}$. Note that it is possible that this D may have parallel arcs and loops. We shall call this digraph D' a *pseudo* $\mathbf{d}$*-realization*. This construction of D' will be utilized in the proof of the following multidigraphic version of Theorem 1.1.

Proposition 1.2. *Let $\mathbf{d} = \{(d_1^+, d_1^-), \ldots, (d_n^+, d_n^-)\}$ be a sequence of nonnegative integer pairs. Then $\mathbf{d}$ is multidigraphic if and only if each of the following holds:*

(i) $\sum_{i=1}^{n} d_i^+ = \sum_{i=1}^{n} d_i^-$;
(ii) *for* $k = 1, \ldots, n,\ d_k^+ \leq \sum_{i \neq k} d_i^-$.

Proof. We assume first that that a multidigraph D is a **d**-realization. Then $\sum_{i=1}^{n} d_i^+ = |A(D)| = \sum_{i=1}^{n} d_i^-$ and so (i) follows. For each $u \in V(D)$, we have $\partial_D^+(\{u\}) = \partial_D^-(V(D) - \{u\}) \subseteq \bigcup_{v \in V(D)-\{u\}} \partial_D^-(\{v\})$, implying (ii).

Conversely, suppose that $\mathbf{d} = \{(d_1^+, d_1^-), \ldots, (d_n^+, d_n^-)\}$ satisfies (i) and (ii). We will construct a multidigraph **d**-realization D with $V(D) = \{v_1, v_2, \ldots, v_n\}$. Since **d** satisfies (i), as commented right before Proposition 1.2, there exists a pseudo **d**-realization D', possibly with parallel arcs and loops. Let D be a pseudo **d**-realization whose number of loops is minimized.

If D is loopless, then D is a **d**-realization, and so **d** is multidigraphic. Hence we assume that D has at least one loop. If D has two distinct vertices v_i and v_j (say), both of which are incident with loops, then obtain a new digraph D_1 from D by replacing two loops $\ell_i = (v_i, v_i)$ and $\ell_j = (v_j, v_j)$ by two arcs $a_i = (v_i, v_j)$ and $a_j = (v_j, v_i)$. It follows that D_1 is also a pseudo **d**-realization with fewer number of loops than D, contradicts the choice of D. Hence we may assume D has exactly one vertex, say v_i, incident with a loop $\ell = (v_i, v_i)$. If for some $j, k \neq i$, $a = (v_j, v_k) \in A(D)$, then obtain a new digraph D_2 from $D - \{a, \ell\}$ by adding two new arcs $a_1 = (v_j, v_i)$ and $a_2 = (v_i, v_k)$. Thus D_2 is also a pseudo **d**-realization with fewer number of loops than D, contradicts the choice of D. This leads to the assumption that every arc in D must be incident with v_i. Since $\ell = (v_i, v_i) \in A(D)$, we conclude that $d_i^+ > \sum_{j \neq i} d_j^-$, contradicts (ii). This contradiction indicates that D must be loopless, and so **d** is multidigraphic. □

Graphic degree sequences for undirected graphs have been characterized by Havel [11], Erdös and Gallai [5], and Hakimi [9], among others. Characterizations of multigraphic degree have been given by Senior [15] and Hakimi [9]. Characterizations for graphic sequences and multigraphic sequences with realizations having prescribed edge connectivity have been studied by many, as seen in Edmonds [4], Wang [16], Wang and Kleitman [17], and Chou and Frank [3], among other. For more in the literature on degree sequences, see surveys [10] and [12].

The purpose of this study is to seek analogous characterizations in digraphs. A digraph D is *strongly connected* (or just strong) if for any $u, v \in V(D)$, D has a (u, v)-dipath. For an integer $k > 0$, D is *k-arc-connected* if for any arc set S with $|S| < k$, the subdigraph $D - S$ is strongly connected. Thus a digraph is 1-arc-connected if and only if it is strongly connected. The k-arc-connector characterization of Frank in [6, 7] (see also Theorem 63.3 in [14]) leads to a characterization for multidigraphic sequences with k-arc-connected realizations. In Section 2 of this article, we shall present a characterization for strict digraphic sequences to have a strongly connected realization. For $k \geq 2$, attempts to obtain similar characterizations for strict digraphic sequences with k-arc-connected realizations are discussed in the last section.

We conclude this section with a special notation used in this article. A *2-switching* of a digraph D is an operation on two arcs $(u_1, v_1), (u_2, v_2) \in A(D)$ to obtain a new digraph D' from $D - \{(u_1, v_1), (u_2, v_2)\}$ by adding new arcs $\{(u_1, v_2), (u_2, v_1)\}$. The resulted D' is usually denoted by $D \otimes \{(u_1, v_1), (u_2, v_2)\}$. By definition,

$$D \otimes \{(u_1, v_1), (u_2, v_2)\} \text{ and } D \text{ have the same degree sequence .} \tag{1.1}$$

Thus digraphic degree sequences will remains unchanged under 2-switchings. This operation will be a main tool in the arguments of this article.

2. STRICT DIGRAPH

In this section, we will present a characterization of strict digraphic degree sequences that have strongly connected strict digraph realizations. By the definition of strong digraphs, we observe that for a digraph D,

$$D \text{ is strongly connected if and only if for any } \emptyset \neq X \subset V(D), |\partial_D^+(X)| \geq 1. \quad (2.1)$$

Throughout this section, $\mathbf{d} = \{(d_1^+, d_1^-), \ldots, (d_n^+, d_n^-)\}$ denotes a strict digraphic sequence with $d_1^+ \geq \cdots \geq d_n^+$. For any $k \in \{1, \ldots, n\}$, define

$$f(k) = \sum_{i=1}^{k}(d_i^- - d_i^+) + \sum_{i=k+1}^{n} \min\{k, d_i^-\}. \quad (2.2)$$

Theorem 2.1. *Let $\mathbf{d} = \{(d_1^+, d_1^-), \ldots, (d_n^+, d_n^-)\}$ be a strict digraphic sequence with $d_1^+ \geq \ldots \geq d_n^+$. Then $\mathbf{d}$ has a strong strict $\mathbf{d}$-realization if and only if both of the following hold.*

(i) $d_i^+ \geq 1, d_i^- \geq 1$ *for all* $1 \leq i \leq n$;
(ii) $f(k) \geq 1$ *for all* $1 \leq k \leq n-1$.

Proof. Assume that $D \in \langle \mathbf{d} \rangle$ is a strong strict digraph. Then (i) follows from (2.1). Let $F \subset V(D)$ be a nonempty proper subset of $V(D)$. Then by (2.1),

$$\sum_{v_i \in F} d_i^- = |A(D[F])| + |\partial_D^-(F)| \geq |A(D[F])| + 1 \text{ and}$$
$$\sum_{v_i \in F} d_i^+ = |A(D[F])| + |\partial_D^+(F)| = |A(D[F])| + \sum_{v_i \notin F} |N_D^-(v_i) \cap F|.$$

Thus $\sum_{v_i \in F}(d_i^- - d_i^+) \geq 1 - \sum_{v_i \notin F} |N_D^-(v_i) \cap F| \geq 1 - \sum_{v_i \notin F} \min\{|F|, d_i^-\}$, and so (ii) follows by letting $F = \{v_1, \ldots, v_k\}$ for $k = 1, \ldots, n-1$. This justifies the necessity.

Now we prove the sufficiency. For any digraph H, let $c(H)$ be the number of strong components of H. Since $\mathbf{d}$ is a strict digraphic sequence, we assume that $D \in \langle \mathbf{d} \rangle$ is so chosen that D is a strict digraph and that

$$c(D) \text{ is minimized.} \quad (2.3)$$

If $c(D) = 1$, then done, and so we may assume $c(D) \geq 2$. We shall show that D must have certain structure that leads to a contradiction to (ii). Since $c(D) \geq 2$, D has a strong component L_1 such that

$$N_D^-(L_1) = \emptyset. \quad (2.4)$$

By Theorems 2.1(i) and (2.4), $|V(L_1)| \geq 2$ and so,

$$L_1 \text{ is a nontrivial strong component of } D. \quad (2.5)$$

Claim 1. *For any $u \in V(L_1)$, and for any subset $X \subseteq V(D) - V(L_1)$ with $|X| \geq 2$, if $D[X]$ is strong, then $X \subseteq N_D^+(u)$.*

Let $X \subseteq V(D) - V(L_1)$ with $D[X]$ being strong, and let L_2 be the strong component of D such that $X \subseteq V(L_2)$. Suppose, to the contrary, that there exist a vertex $u \in V(L_1)$ and a vertex $v \in V(L_2) \subseteq V(D) - V(L_1)$ such that $(u, v) \notin A(D)$. By (2.5) there exists a vertex $u' \in N^+_{L_1}(u)$, and by the assumption of Claim 1, there exists a vertex $v' \in N^-_{L_2}(v)$. Let $D' = D \otimes \{(u, u'), (v', v)\}$. Since $\partial^-_D(L_1) = \emptyset$ and since $(u, v) \notin A(D)$, D' is strict. By (1.1), $D' \in \langle \mathbf{d} \rangle$. As $D'[V(L_1) \cup V(L_2)]$ is strongly connected, we have $c(D') = c(D) - 1$, contrary to (2.3). This proves Claim 1.

Claim 2. *For any $u \in V(L_1)$ and $(v_1, v_2) \in A(D - V(L_1))$, if $(u, v_1) \in A(D)$, then $(u, v_2) \in A(D)$.*

Suppose, to the contrary, that there exist $u \in V(L_1)$ and $(v_1, v_2) \in A(D - V(L_1))$ such that $(u, v_1) \in A(D)$ but $(u, v_2) \notin A(D)$. If v_1, v_2 lie in the same strong component of D, then by Claim 1, we would have $(u, v_1), (u, v_2) \in A(D)$, contrary to the assumption that $(u, v_2) \notin A(D)$. Thus v_1, v_2 must be in different strong components of D, and so $c(D - \{v_1, v_2\}) = c(D)$. Let L' be the strong component of D containing v_1.

Since L_1 is strong and by (2.5), there exists a vertex $u' \in N^+_{L_1}(u)$. Let $D' = D \otimes \{(u, u'), (v_1, v_2)\}$. Since $\partial^-_D(L_1) = \emptyset$ and $(u, v_2) \notin A(D)$, D' is also strict. By (1.1), $D' \in \langle \mathbf{d} \rangle$. Furthermore, as both L_1 and L' are strong, and as $(u, v_1), (v_1, u') \in A(D')$, it follows by definition that $D'[V(L_1) \cup V(L')]$ is strong. This leads to $c(D') < c(D)$, a contradiction to (2.3). This completes the proof of Claim 2.

Let $F_1 = V(L_1)$, $F_2 = \{v \notin F_1 |$ there exists a nontrivial strong component N of $D - F_1$ and a vertex $u \in V(N)$ such that $D - F_1$ has a (u, v)-dipath$\}$, and let

$$\begin{aligned}
F_3 &:= \{v \in V(D) - (F_1 \cup F_2) \mid F_1 \subseteq N^-_D(v)\};\\
F_{31} &:= \{v \in F_3 \mid N^-_D(v) \cap F_3 = \emptyset\};\\
F_{32} &:= F_3 - F_{31};\\
F_4 &:= V(D) - (F_1 \cup F_2 \cup F_3);\\
F_{41} &:= \{v \in F_4 \mid N^+_D(v) \cap F_4 \neq \emptyset\};\\
F_{42} &= F_4 - F_{41}.
\end{aligned} \tag{2.6}$$

It is possible that some of these subset defined above might be empty. Claim 3 follows from Claims 1 and 2.

Claim 3. *For any $v \in F_2 \cup F_3$, $F_1 \subseteq N^-_D(v)$.*

Claim 4. *For any $v \in F_{41}$, $F_2 \cup F_{32} \subseteq N^+_D(v)$.*

Suppose that there exist a vertex $v \in F_{41}$ and a vertex $u' \in F_2 \cup F_{32}$ such that $(v, u') \notin A(D)$. Let $v' \in N^+_D(v) \cap F_4$. By (2.6), F_1 contains a vertex w such that $(w, v') \notin A(D)$. By (2.5), there exists a vertex $w' \in N^+_D(w) \cap F_1$. If $u' \in F_2$, then by the definition of F_2, there exists a vertex $u \in N^-_D(u') \cap F_2$; if $u' \in F_{32}$, then by (2.6), there exists a vertex $u \in N^-_D(u') \cap F_3$. In either case, a vertex $u \in N^-_D(u') \cap (F_2 \cup F_3)$ exists. Define $D' = D - \{(u, u'), (v, v'), (w, w')\} + \{(u, w'), (v, u'), (w, v')\}$. As $w' \in F_1$ and $\partial^-_D(F_1) = \emptyset$, D' is also a strict digraph in $\langle \mathbf{d} \rangle$.

If both u and u' are in the same strong component C' of D, then in $D - \{(u, u'), (v, v'), (w, w')\}$, every vertex in $V(C') - \{u\}$ has a dipath to u. By Claim 1, $F_1 \cup V(C')$ induces a strongly connected subdigraph in D', whence $c(D') < c(D)$, contrary to (2.3).

If u and u' are in different strong components of D, then the strong components of $D - \{(u, u'), (v, v')\}$ are also the strong components of D. Furthermore, by Claim 3, $(w, u) \in A(D)$. Thus F_1 and the component containing u in D are contained in one strong component of D', implying $c(D') < c(D)$, again contrary to (2.3). This verifies Claim 4.

Claim 5. *There exists a vertex subset $Z \subseteq V(D)$ with $F_{41} \subseteq Z \subseteq F_4$ such that*

(i) *for any $v \in Z$, $F_2 \cup F_{32} \subseteq N_D^+(v)$ and $N_D^+(v) \cap (F_4 \cup F_{31}) \neq \emptyset$, and*
(ii) *for any $v \in F_{31}$, either $Z \subseteq N_D^-(v) \cap F_4$ or $N_D^-(v) \cap F_4 \subseteq Z$.*

To prove this claim, we start with some notation. Let $X_0 := F_{41}$, $Y_0 = \emptyset$ and for $i = 1, 2, \ldots$, define

$$X_i := X_{i-1} \cup (N_D^-(Y_{i-1}) \cap F_4); \tag{2.7}$$

$$Y_i := \{v \in F_{31} \mid N_D^-(v) \cap F_4 \not\subseteq X_i \text{ and } X_i \not\subseteq N_D^-(v) \cap F_4\}. \tag{2.8}$$

By definition, $X_1 = X_0 = F_{41}$. We first justify the following subclaim (5A).

(5A). For any strict **d**-realization D satisfying (2.3), we define the sets $F_1, F_2, F_3, F_{31}, F_{32}, F_4, F_{41}, F_{42}$ as in (2.6), and X_i, Y_i with $i \geq 0$ as in (2.7) and (2.8). Thus for any $i \geq 1$ and for any $v \in X_i$, $F_2 \cup F_{32} \subseteq N_D^+(v)$.

We argue by induction on i to prove (5A). When $i = 1$ the result follows from Claim 4. Assume that for some $k > 1$, (5A) holds for any $i < k$. We want to prove (5A) holds for $i = k$ as well. Suppose, to the contrary, that X_k has a vertex v such that $F_2 \cup F_{32} - N_D^+(v) \neq \emptyset$. By induction, for any $z \in X_{k-1}$, $F_2 \cup F_{32} \subseteq N_D^+(z)$. Hence $v \in X_k - X_{k-1}$. By (2.7), $v \in N_D^-(Y_{k-1}) \cap F_4 - X_{k-1}$. It follows that there exists a vertex $v' \in Y_{k-1}$ such that $(v, v') \in A(D)$. Moreover, by (2.8), X_{k-1} contains a vertex u such that $(u, v') \notin A(D)$. For these vertices u and v, we shall show that

$$\text{There exists a } u' \in N_D^+(u) \text{ such that } (v, u') \notin A(D). \tag{2.9}$$

In fact, if $k = 2$, then $u \in X_{k-1} = F_{41}$. By the definition of F_{41}, there is a vertex $u_1' \in F_4$ such that $(u, u_1') \in A(D)$. As $v \notin X_{k-1} = F_{41}$, $v \in F_{42}$. By the definition of F_{42}, $(v, u_1') \notin A(D)$, and so (2.9) holds. Now we assume that $k \geq 3$. We first show that $u \in X_{k-1} - X_{k-2}$. If $u \in X_{k-2}$, then as $(u, v') \notin A(D)$ and $(v, v') \in A(D)$, we conclude that $v' \in Y_{k-2}$ and so $v \in X_{k-1}$, contrary to the assumption that $v \in X_k - X_{k-1}$. Hence we must have $u \in X_{k-1} - X_{k-2}$. By the definition of X_{k-1}, there exists a vertex $u_2' \in Y_{k-2}$ such that $(u, u_2') \in A(D)$. If $(v, u_2') \in A(D)$, then as $u_2' \in Y_{k-2}$ and by (2.7), we must have $v \in X_{k-1}$, contrary to the assumption that $v \in X_k - X_{k-1}$. Hence $(v, u_2') \notin A(D)$, and so (2.9) must hold.

By (2.9), there always exists a vertex $u' \in N_D^+(u)$ such that $(v, u') \notin A(D)$. Let $D' = D \otimes \{(u, u'), (v, v')\}$. Then since $(u, v'), (v, u') \notin A(D)$, D' is also a strict **d**-realization. As the two arcs $(u, u'), (v, v')$ are not in any strong components of D, $c(D') \leq c(D - \{(u, u'), (v, v')\}) = c(D)$. By (2.3), we have $c(D') = c(D)$. Thus D' is also a strict **d**-realization satisfying (2.3).

To complete the proof of Subclaim (5A), we work on the strict **d**-realization D' instead of D. Since $D' - \{(u, v'), (v, u')\} = D - \{(u, u'), (v, v')\}$ and since $\partial_{D'}^-(F_1) = \partial_D^-(F_1) = \emptyset$, we choose $F_1' = F_1$, and define the corresponding sets $F_2', F_3', F_{31}', F_{32}', F_4', F_{41}', F_{42}'$ as in (2.6), and X_j', Y_j' for $j \geq 0$ as in (2.7) and (2.8) for the digraph D'. Then by the definitions of these sets, we observe that $F_2' = F_2, F_3' = F_3, F_{31}' = F_{31}, F_{32}' = F_{32}, F_4' = F_4$.

If $k = 2$, then by the definition of D', we have $N_{D'}^+(v) \cap F_4 \neq \emptyset$, and so $v \in F_{41}'$. Applying Claim 4 to D', we conclude that $F_2' \cup F_{32}' \subseteq N_{D'}^+(v)$, and so $F_2 \cup F_{32} \subseteq N_D^+(v)$. If $k \geq 3$, then by the definitions of X_j' and Y_j', we observe that $X_i' = X_i, Y_i' = Y_i$ for $i =$

$1, \ldots, k-2$. However, as $u' \in Y_{k-2}$, by (2.7), we conclude that $v \in X'_{k-1}$. By induction, $F'_2 \cup F'_{32} \subseteq N^+_{D'}(v)$. It follows that $F_2 \cup F_{32} \subseteq N^+_D(v)$, which completes the proof of (5A).

We are now ready to finish the proof of Claim 5. By (2.7), we have $F_{41} = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_i \subseteq F_4$. The finiteness of the graph warrants that there is a constant integer h such that $X_h = X_j$ for any $j \geq h$. Define $Z = X_h$. Then $F_{41} \subseteq Z \subseteq F_4$. By (5A), for any $v \in Z$, $F_2 \cup F_{32} \subseteq N^+_D(v)$. Also, by the definitions of X_i, $N^+_D(v) \cap (F_4 \cup F_{31}) \neq \emptyset$. This justifies Claim 5(i). By (2.7) and (2.8), we have $N^-_D(Y_h) \cap F_4 \subseteq X_h$ and $Y_{h+1} = Y_h$. It follows that $N^-_D(Y_{h+1}) \cap F_4 \subseteq X_h$. If $Y_h \neq \emptyset$, then for any $v \in Y_h = Y_{h+1}$, we have $N^-_D(v) \cap F_4 \subseteq X_h$. On the other hand, by (2.8), $N^-_D(v) \cap F_4 \nsubseteq X_h$, and so a contradiction is obtained. Hence $Y_{h+1} = Y_h = \emptyset$. By (2.8), we have, for any vertex $v \in F_{31}$, either $N^-_D(v) \cap F_4 \subseteq Z$ or $Z \subseteq N^-_D(v) \cap F_4$. This proves Claim 5.

We continue letting $Z = X_k$. Choose $z_0 \in Z$ such that $d^+_D(z_0) = \min_{u \in Z}\{d^+_D(u)\}$ if $Z \neq \emptyset$. Define

$$Z' = \begin{cases} \{v \in F_4 - Z \mid d^+_D(v) \geq d^+_D(z_0)\} & \text{if } Z \neq \emptyset \\ \emptyset & \text{if } Z = \emptyset. \end{cases}$$

Claim 6. *If $Z' \neq \emptyset$, then for any $v \in Z'$, $N^+_D(v) = N^+_D(z_0)$.*

Let $v \in Z'$ be an arbitrary vertex. First, we show that $N^+_D(v) \cap F_{31} \subseteq N^+_D(z_0) \cap F_{31}$. Take any vertex $u \in N^+_D(v) \cap F_{31}$. By (2.6), we have $v \in N^-_D(u) \cap F_4$ and so $N^-_D(u) \cap F_4 \nsubseteq Z$. By Claim 5, we have $Z \subseteq N^-_D(u) \cap F_4$. It follows that $z_0 \in N^-_D(u)$ and so $u \in N^+_D(z_0)$. Hence $N^+_D(v) \cap F_{31} \subseteq N^+_D(z_0) \cap F_{31}$. Since $v \notin F_{41}$, $N^+_D(v) = \big(N^+_D(v) \cap F_{31}\big) \cup (N^+_D(v) \cap (F_2 \cup F_{32})) \subseteq (N^+_D(z_0) \cap F_{31}) \cup (F_2 \cup F_{32}) \subseteq N^+_D(z_0)$. This, together with $d^+_D(v) \geq d^+_D(z_0)$, implies $N^+_D(v) = N^+_D(z_0)$.

Define $F = F_1 \cup Z \cup Z'$. We make the following two claims.

Claim 7. *For any $v \notin F$, either $F \subseteq N^-_D(v)$ or $N^-_D(v) \subseteq F$.*

Pick an arbitrary vertex $v \notin F$. By (2.6), we have $v \in (F_2 \cup F_{32}) \cup F_{31} \cup (F_4 - (Z \cup Z'))$. We will justify Claim 7 by showing that any subset in the union containing the vertex v will lead to the conclusion of Claim 7. If $v \in F_2 \cup F_{32}$, then by Claims 3 and 5, $F_1 \cup Z \subseteq N^-_D(v)$. It follows that $v \in N^+_D(z_0)$. If $Z' = \emptyset$, then $F = F_1 \cup Z \cup Z' \subseteq N^-_D(v)$, and so Claim 7 holds. Now assume that $Z' \neq \emptyset$. By Claim 6, for any $z' \in Z'$, $v \in N^+_D(z_0) = N^+_D(z')$. Thus $Z' \subseteq N^-_D(v)$. It follows that $F = F_1 \cup Z \cup Z' \subseteq N^-_D(v)$, and so Claim 7 holds. If $v \in F_{31}$, then by Claim 5, either $N^-_D(v) \cap F_4 \subseteq Z$ or $Z \subseteq N^-_D(v) \cap F_4$. In fact, if $Z \subseteq N^-_D(v) \cap F_4$, then $(z_0, v) \in A(D)$. Thus by Claim 6, $(z', v) \in A(D)$ for any $z' \in Z'$, implying $Z \cup Z' \subseteq N^-_D(v)$. Hence we have either $N^-_D(v) \cap F_4 \subseteq Z \cup Z'$ or $Z \cup Z' \subseteq N^-_D(v) \cap F_4$. Furthermore, by the definition of F_{31} in (2.6) and by Claim 3, we must have $N^-_D(v) = F_1 \cup (N^-_D(v) \cap F_4)$. Thus either $N^-_D(v) \subseteq F_1 \cup Z \cup Z' = F$ or $F \subseteq N^-_D(v)$. In either case, Claim 7 holds. Therefore, we may assume that $v \in F_4 - Z - Z'$. By the definition of F_{41} in (2.6), and by the fact $F_{41} \subseteq Z$, it follows that $F_4 - Z - Z'$ is an independent set of D. Furthermore, by the definitions of F_2, F_3 and by Claim 2, we have $N^-_D(v) \cap (F_2 \cup F_3) = \emptyset$. It follows that $N^-_D(v) \subseteq F_1 \cup Z \cup Z' = F$. Hence Claim 7 is justified.

Claim 8. *For any $u \in F$ and $v \notin F$, $d^+_D(u) > d^+_D(v)$.*

Let $u \in F$ and $v \notin F$ be two vertices. Since $F = F_1 \cup Z \cup Z'$, we will justify Claim 8 by examining the cases when the vertex u lies in different subsets of F. If $u \in F_1$, then by Claim 3 and by the fact that $D[F_1]$ is strong, we have $d^+_D(u) \geq |F_2 \cup F_3| + 1$.

By the definition of F_{41} or by Claim 2, $N_D^+(v) \cap F_4 = \emptyset$. Thus $d_D^+(v) \le |F_2 \cup F_3| \le d_D^+(u) - 1$, and so Claim 8 holds. Hence we may assume $u \in Z \cup Z' - F_1$. By Claims 5 and 6, $d_D^+(u) \ge |F_2 \cup F_{32}| + 1$. If $v \in F_2 \cup F_3$, then by the definitions of F_{31} and F_3, we have $N_D^+(v) \cap F_{31} = \emptyset$. By Claim 2, we also have $N_D^+(v) \cap F_4 = \emptyset$. Thus $N_D^+(v) \subseteq F_2 \cup F_{32}$. This leads to $d_D^+(v) \le |F_2 \cup F_{32}| < d_D^+(u)$, and so Claim 8 holds. Hence we may assume that $v \in F_4 - Z - Z'$. By the definition of Z', we observe that $d(v) < d(z_0) = \min_{w \in Z} d_D^+(w) = \min_{w \in Z \cup Z'} d_D^+(w) \le d_D^+(u)$. Claim 8 is proved.

We are now ready to complete the proof of the theorem. We adopt the notation that $V(D) = \{v_1, \ldots, v_n\}$ with $d_D^+(v_i) = d_i^+$ and $d_D^-(v_i) = d_i^-$ for $i = 1, \ldots, n$. Assume that $|F| = t$. By Claim 8, $F = \{v_1, \ldots, v_t\}$. By Theorem 2.1(ii), we have

$$\sum_{u \in F} (d_D^-(u) - d_D^+(u)) + \sum_{u \notin F} \min\{|F|, d_D^-(u)\} = \sum_{i=1}^{t} (d_i^- d_i^+) + \sum_{i=t+1}^{n} \min\{t, d_i^-\} \ge 1. \tag{2.10}$$

This inequality (2.10) and Claim 7 imply that

$$\begin{aligned} |\partial_D^-(F)| &= \sum_{u \in F} d_D^-(u) - |A(D[F])| \\ &= \sum_{u \in F} d_D^-(u) - \sum_{u \in F} d_D^+(u) + |\partial_D^+(F)| \\ &= \sum_{u \in F} \left(d_D^-(u) - d_D^+(u)\right) + \sum_{u \notin F} \min\{|F|, d_D^-(u)\} \ge 1. \end{aligned}$$

As $F_1 = V(L_1)$, by (2.4), we must have $\partial_D^-(F_1) = \emptyset$, and so there is an arc $(x, y) \in A(D)$ such that $x \in \overline{F} = F_2 \cup F_3 \cup (F_4 - Z - Z')$ and $y \in Z \cup Z' \subseteq F_4$. This is a contradiction to Claim 2 or to the definition of F_{41}. This completes the proof of Theorem 2.1. □

3. AN EXAMPLE

As shown in Theorem 63.3 of [14], Frank in [6, 7] has obtained characterizations for multidigraphic degree sequences to have strongly k-arc-connected realizations. It is natural to seek similar characterizations of strict digraphic sequences that have a strongly k-arc-connected strict realization. The purpose of this section is to present an example to show that it might be difficult to find such a characterization.

Define the function f as in (2.2). In Theorem 2.1, it is shown that a necessary condition for a strict digraphic sequence **d** to have a strongly 1-arc-connected strict realization is that $f(i) \ge 1$ for all i with $1 \le i \le n - 1$. In fact, a slightly stronger necessary condition is also presented in the arguments to prove Theorem 2.1. For any subset I with $\emptyset \ne I \subset \{1, \ldots, n\}$, define

$$g(I) = \sum_{i \in I} (d_i^- - d_i^+) + \sum_{i \notin I} \min\{|I|, d_i^-\}. \tag{3.1}$$

By definition, it is routine to verify that the function f defined in (2.2) satisfies $f(i) = g(\{1, 2, \ldots, i\})$. In the justification of (2.10), we have shown that a necessary condition for a strict digraphic sequence **d** to have a strongly 1-arc-connected strict realization is that $g(I) \ge 1$, for any subset I with $\emptyset \ne I \subset \{1, \ldots, n\}$. With a similar argument as in the proof of Theorem 2.1, it is routine to show that both $f(i) \geq k$ and $g(I) \geq k$ are necessary conditions for a strict digraphic sequence **d** to have a strongly k-arc-connected strict realization. In this section, we will give a strict digraphic sequence **d** to show that it is possible that a strict digraphic sequence **d** satisfies the condition $f(i) \to \infty$ (or the condition $g(I) \to \infty$), but **d** to does not have a strongly k-arc-connected strict realization.

Example 3.1. *Let $t > 1$ be an integer and $\mathbf{d} = \{(n-1,t)^t, (3t+1, 2t+1), (2t+1, 3t+1), (t+2,t)^t, (t,t+2)^t, (t, n-1)^t\}$. Then each of the following holds.*

(i) *There is only one strict digraph D with degree sequence $\mathbf{d}$.*
(ii) *The sequence $\mathbf{d}$ satisfies the condition that $g(I) \geq t$ for any $\emptyset \subset I \subset \{1, \ldots, n\}$.*
(iii) *The only strict digraph D with degree sequence* **d** *is not strongly 2-arc-connected.*

Proof. By Theorem 1.1, **d** is strict digraphic. Let D be a strict digraph with degree sequence **d**, X_1 be the set of the t vertices with out-degree $n-1$ and in-degree t, X_2 be the set of the t vertices with out-degree $t+2$ and in-degree t, X_3 be the set of t vertices with out-degree t and in-degree $t+2$, X_4 be the set of t vertices with out-degree t and in-degree $n-1$. Then $|X_1| = |X_2| = |X_3| = |X_4| = t$ and $n = 4t+2$. Let $u, v \notin \bigcup_{i=1}^4 X_i$ be the two additional vertices satisfying $d_D^+(u) = 3t+1$, $d_D^-(u) = 2t+1$, $d_D^+(v) = 2t+1$, $d_D^-(v) = 3t+1$. Thus $V(D) = \{u, v\} \cup (\cup_{i=1}^4 X_i)$.

In order to determine the structure of D, we only need to find out $N_D^+(x)$ for every vertex $x \in V(D)$. We make the Observations (A)–(F) as follows.

(A) For each vertex $x \in V(D)$, as vertices in X_1 have out-degree $n-1$ and vertices in X_4 have in-degree $n-1$, both $X_4 - \{x\} \subseteq N_D^+(x)$ and $X_1 - \{x\} \subseteq N_D^-(x)$.
(B) For each $x_2 \in X_2$ and $x_3 \in X_3$, it follows by (A) and by the fact that vertices in X_2 have in-degree t and vertices in X_3 have out-degree t, that $N_D^-(x_2) = X_1$ and $N_D^+(x_3) = X_4$.
(C) By Observations (A) and (B), we have both $N_D^+(u) \subseteq X_1 \cup X_3 \cup X_4 \cup \{v\}$ and $N_D^-(v) \subseteq X_1 \cup X_2 \cup X_4 \cup \{u\}$. As $d_D^+(u) = d_D^-(v) = 3t+1$, we must also have $N_D^+(u) = X_1 \cup X_3 \cup X_4 \cup \{v\}$ and $N_D^-(v) = X_1 \cup X_2 \cup X_4 \cup \{u\}$.
(D) For each $x_4 \in X_4$, since X_4 is the set of t vertices with out-degree t and in-degree $n-1$, and by (A)–(C), we conclude that $N_D^+(x_4) = (X_4 - \{x_4\}) \cup \{v\}$.
(E) It follows from (A)–(D) that for each $x_1 \in X_1$, we have $N_D^-(x_1) = X_1 \cup \{v\} - \{x_1\}$, and so $N_D^+(v) \subseteq X_3 \cup X_4 \cup \{u\}$. These, together with the fact $d_D^+(v) = 2t+1$, implies $N_D^+(v) = X_3 \cup X_4 \cup \{u\}$.
(F) For every $x_2 \in X_2$, we have $X_4 \cup \{u, v\} \subseteq N_D^+(x_2)$. As $d_D^+(x_2) = t+2$, we must have $N_D^+(x_2) = X_4 \cup \{u, v\}$.

From Observations (A)–(F), we conclude that for each $x \in V(D)$, the set $N_D^+(x)$ is uniquely determined. This implies (i).

Next we justify (ii). For each nonempty $I \subset \{1, \ldots, n\}$, we will show that $g(I) \geq t$. Let $X = \{v_i \mid i \in I\}$, and let $\alpha_i = |X \cap X_i|$, for $i = 1, 2, 3, 4$, $\alpha_u = |X \cap \{u\}|$ and $\alpha_v = |X \cap \{v\}|$. Then $0 \leq \alpha_i \leq t$ for $i = 1, 2, 3, 4$ and $0 \leq \alpha_u, \alpha_v \leq 1$. Let $\alpha = |X|$. Then $\alpha = \sum_{i=1}^4 \alpha_i + \alpha_u + \alpha_v$. By (3.1),

$$\begin{aligned} g(I) &= \sum_{i \in I} (d_i^- - d_i^+) + \sum_{i \notin I} \min\{|I|, d_i^-\} \\ &= (t-(n-1))\alpha_1 + (t-(t+2))\alpha_2 + ((t+2)-t)\alpha_3 + (n-1-t)\alpha_4 \\ &\quad + (2t+1-(3t+1))\alpha_u + (3t+1-(2t+1))\alpha_v + (t-\alpha_1)\min\{\alpha, t\} \end{aligned}$$

$$
\begin{aligned}
&+ (t-\alpha_2)\min\{\alpha, t\} + (t-\alpha_3)\min\{\alpha, t+2\} \\
&+ (t-\alpha_4)\min\{\alpha, n-1\} + (1-\alpha_u)\min\{\alpha, 2t+1\} + (1-\alpha_v)\min\{\alpha, 3t+1\} \\
=\ & (3t+1)(\alpha_4-\alpha_1) + 2(\alpha_3-\alpha_2) + t(\alpha_v-\alpha_u) + (2t-\alpha_1-\alpha_2)\min\{\alpha, t\} \\
&+ (t-\alpha_3)\min\{\alpha, t+2\} + \alpha(t-\alpha_4) + (1-\alpha_u)\min\{\alpha, 2t+1\} \\
&+ (1-\alpha_v)\min\{\alpha, 3t+1\}.
\end{aligned}
$$

Case 1. $1 \le \alpha \le t$. In this case,

$$
\begin{aligned}
g(I) =\ & (3t+1)(\alpha_4-\alpha_1) + 2(\alpha_3-\alpha_2) + t(\alpha_v-\alpha_u) + (4t+2 \\
& -\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_u-\alpha_v)\alpha \\
=\ & (4t+2-\alpha)\alpha + (3t+1)(\alpha_4-\alpha_1) + 2(\alpha_3-\alpha_2) + t(\alpha_v-\alpha_u) \\
=\ & (t-\alpha)\alpha + \alpha_1 + 3t\alpha_2 + (3t+4)\alpha_3 + (6t+3)\alpha_4 + (2t+2)\alpha_u \\
& + (4t+2)\alpha_v.
\end{aligned}
$$

If at least one of $\alpha_2, \alpha_3, \alpha_4, \alpha_u, \alpha_v$ is at least 1, then $g(I) > t$. Hence we may assume that $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_u = \alpha_v = 0$. Thus $\alpha = \alpha_1$, and so $g(I) = (t-\alpha)\alpha + \alpha = \alpha(t+1-\alpha) \ge t$ as $1 \le \alpha \le t$.

Case 2. $t+1 \le \alpha \le t+2$. In this case,

$$
\begin{aligned}
g(I) =\ & (3t+1)(\alpha_4-\alpha_1) + 2(\alpha_3-\alpha_2) + t(\alpha_v-\alpha_u) + (2t-\alpha_1-\alpha_2)t \\
& +(2t+2-\alpha_3-\alpha_4-\alpha_u-\alpha_v)\alpha \\
=\ & -(4t+1)\alpha_1 - (t+2)\alpha_2 - (\alpha-2)\alpha_3 + (3t+1-\alpha)\alpha_4 - (t+\alpha)\alpha_u \\
& - (\alpha-t)\alpha_v + 2t^2 + (2t+2)\alpha \\
=\ & -(4t+1)\alpha_1 - (t+2)\alpha_2 - (\alpha-2)\alpha_3 + (3t+1-\alpha)\alpha_4 - (t+\alpha)\alpha_u \\
& - (\alpha-t)\alpha_v + 2t^2 - (2t-1)\alpha + (4t+1)(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_u+\alpha_v) \\
=\ & -(2t-1)\alpha + (3t-1)\alpha_2 + (4t+3-\alpha)\alpha_3 + (7t+2-\alpha)\alpha_4 \\
& + (3t+1-\alpha)\alpha_u + (5t+1-\alpha)\alpha_v + 2t^2 \\
\ge\ & 2t^2 - (2t-1)\alpha + (2t-1)(\alpha_2+\alpha_3+\alpha_4+\alpha_u+\alpha_v) \\
\ge\ & 2t^2 - (2t-1)\alpha + (2t-1)(\alpha-t) = t.
\end{aligned}
$$

Case 3. $t+3 \le \alpha \le n-1 = 4t+1$. In this case,

$$
\begin{aligned}
g(I) =\ & (3t+1)(\alpha_4-\alpha_1) + 2(\alpha_3-\alpha_2) + t(\alpha_v-\alpha_u) + (2t-\alpha_1-\alpha_2)t \\
& + (t-\alpha_3)(t+2) + \alpha t - \alpha\alpha_4 + (1-\alpha_u)\min\{\alpha, 2t+1\} \\
& + (1-\alpha_v)\min\{\alpha, 3t+1\}.
\end{aligned}
$$

Since $\min\{\alpha, 2t+1\}\cdot(1-\alpha_u) \ge 0$ and $\min\{\alpha, 3t+1\}\cdot(1-\alpha_v) \ge (1-\alpha_v)(\alpha_1+\alpha_2)$, it follows that

$$
\begin{aligned}
g(I) \ge\ & (3t+1)(\alpha_4-\alpha_1) + 2(\alpha_3-\alpha_2) + t(\alpha_v-\alpha_u) + (2t-\alpha_1-\alpha_2)t \\
& +(t-\alpha_3)(t+2) + (\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_u+\alpha_v)t - \alpha\alpha_4 \\
& + (1-\alpha_v)(\alpha_1+\alpha_2) \\
=\ & -(3t+\alpha_v)\alpha_1 - (1+\alpha_v)\alpha_2 + (4t+1-\alpha)\alpha_4 + 2t\alpha_v + 3t^2 + 2t \\
\ge\ & -(3t+\alpha_v)t - (1+\alpha_v)t + 2t\alpha_v + 3t^2 + 2t = t.
\end{aligned}
$$

As in any case, we always have $g(I) \geq t$ for all $\emptyset \subset I \subset \{1, \ldots, n\}$. This proves (ii).

To see that this strict digraph D is not 2-arc-connected, it suffices to observe that direct computation yields $|\partial_D^-(X_1 \cup X_2 \cup \{u\})| = 1$. Thus (iii) must hold. □

Example 3.1 shows that the necessary condition $g(I) \geq k$ fails to be a sufficient condition for **d** to have a strict k-arc-connected realization. It remains open to find an necessary and sufficient condition for such degree sequences.

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