# On *k*-Maximal Strength Digraphs

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**Abstract:** Let $k > 0$ be an integer and let $D$ be a simple digraph on $n > k$ vertices. We prove that If

$$|A(D)| > k(2n - k - 1) + \binom{n-k}{2},$$

then $D$ must have a nontrivial subdigraph $H$ such that the strong arc connectivity of $H$ is at least $k + 1$. We also show that this bound is best possible and present a constructive characterization for the extremal graphs.  J. Graph Theory 84: 17–25, 2017

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## 1. INTRODUCTION

We consider finite simple graphs and simple digraphs. Usually, we use $G$ to denote a graph and $D$ a digraph. Undefined terms and notation will follow [3] for graphs and [2] for digraphs. In particular, $\kappa'(G)$ denotes the edge connectivity of a graph $G$ and $\lambda(D)$ denotes the arc-strong connectivity of a digraph $D$. If $G$ is a simple graph, then $G^c$ denotes the complement of $G$. If $X \subseteq E(G^c)$, then $G + X$ is the simple graph with vertex set $V(G)$ and edge set $E(G) \cup X$. We will use $G + e$ for $G + \{e\}$. Likewise, if $D$ is a simple digraph, let $D^c$ denotes the complement of $D$. For $X \subseteq A(D^c)$ and $e \in A(D^c)$, we similarly define the simple digraphs $D + X$ and $D + e$, respectively. Throughout this article, we use the notation $(u, v)$ to denote an arc oriented from $u$ to $v$ in a digraph. If $W \subseteq V(D)$ or if $W \subseteq A(D)$, then $D[W]$ denotes the subdigraph of $D$ induced by $W$. For $v \in V(D)$, we use $D - v$ for $D[V(D) - \{v\}]$. For graphs $H$ and $G$, we denote $H \subseteq G$ when $H$ is a subgraph of $G$. Similarly, for digraphs $H$ and $D$, $H \subseteq D$ when $H$ is a subdigraph of $D$. We use $D \cong H$ when the digraphs $D$ and $H$ are isomorphic.

Given a graph $G$, Matula [5] first studied the quantity

$$\overline{\kappa}'(G) = \max\{\kappa'(H) \ : \ H \subseteq G\}.$$

He called $\overline{\kappa}'(G)$ the **strength** of $G$. Mader [4] considered an extremal problem related to $\overline{\kappa}'(G)$. For an integer $k > 0$, a simple graph $G$ with $|V(G)| \geq k + 1$ is $k$-**maximal** if $\overline{\kappa}'(G) \leq k$ but for any edge $e \in E(G^c)$, $\overline{\kappa}'(G + e) > k$. In [4], Mader proved the following.

**Theorem 1.1.** *(Mader [4]) If G is a k-maximal graph on $n > k \geq 1$ vertices, then*

$$|E(G)| \leq (n - k)k + \binom{k}{2}.$$

*Furthermore, this bound is best possible.*

The purpose of this article is to investigate the same for simple digraphs. Naturally, for a digraph $D$, we define

$$\overline{\lambda}(D) = \max\{\lambda(H) \ : \ H \subseteq D\}.$$

Let $k \geq 0$ be an integer. A simple digraph $D$ with $|V(D)| \geq k+1$ is $k$-**maximal** if $\overline{\lambda}(D) \leq k$ but for any arc $e \in A(D^c)$, $\overline{\lambda}(D+e) \geq k+1$. For positive integer $n$ and $k$ with $n \geq k+1$, define

$$\mathcal{D}(n,k) = \{D \ : \ D \text{ is a simple digraph with } |V(D)| = n \text{ and } D \text{ is } k\text{-maximal}\}.$$

Our goal is to determine $\max\{|A(D)| : D \in \mathcal{D}(n,k)\}$. If $h < k$, we define $\binom{h}{k} = 0$. Our main result is the following.

**Theorem 1.2.** *Let $k \geq 0$ and $n \geq k+1$ be nonnegative integers. If $D \in \mathcal{D}(n,k)$, then*

$$|A(D)| \leq k(2n-k-1) + \binom{n-k}{2}.$$

*Furthermore, the bound is best possible.*

**Corollary 1.3.** *Let $k > 0$ be an integer and let $D$ be a simple digraph on $n > k$ vertices. If*

$$|A(D)| > k(2n-k-1) + \binom{n-k}{2},$$

*then $D$ must have a subdigraph $H$ such that $\lambda(H) \geq k+1$.*

The corollary follows immediately from Theorem 1.2. In the next section, we investigate properties of $k$-maximal digraphs. The main result will be proved in the last section.

## 2. PROPERTIES OF $k$-MAXIMAL DIGRAPHS

Throughout this section, let $k \geq 0$ be an integer. Define $\mathcal{D}(k) = \cup_{n \geq k+1}\mathcal{D}(n,k)$. Thus $\mathcal{D}(k)$ is the family of all $k$-maximal digraphs. Recall that a tournament on $n$ vertices is an orientation of the complete graph $K_n$ on $n$ vertices. The following lemma indicates that the $k=0$ case has a clear structure.

**Lemma 2.1.** *([1]) A digraph $D \in \mathcal{D}(0)$ if and only if $D$ is an acyclic tournament.*

The easy proof is left to the reader (and found in [1]).

For any integer $n \geq 0$, let $K_n^*$ denote the complete digraph on $n$ vertices. By definition, we have

$$K_{k+1}^* \in \mathcal{D}(k) \text{ and if } H \in \mathcal{D}(k) \text{ and } |V(H)| = k+1, \text{ then } H \cong K_{k+1}^*. \tag{1}$$

Following [2], if $D$ is a digraph and if $X, Y \subseteq V(D)$, then define

$$(X,Y)_D = \{(x,y) \in A(D) \ : \ x \in X, y \in Y\}.$$

We further define that, for $X \subseteq V(D)$,

$$\partial_D^+(X) = (X, V(D)-X)_D \text{ and } \partial_D^-(X) = (V(D)-X, X)_D.$$

For each $v \in V(D)$, we define

$$N_D^+(v) = \{u \in V(D) \ : \ (v,u) \in A(D)\} \text{ and } N_D^-(v) = \{u \in V(D) \ : \ (u,v) \in A(D)\}.$$

When the digraph $D$ is understood from the context, we sometimes omit the subscript $D$ in the notation above. By the definition of arc-strong connectivity in [2], a digraph $D$ satisfies $\lambda(D) \geq k$ if and only if for any nonempty proper subset $X \subset V(D)$, $|\partial_D^+(X)| \geq k$.

**Lemma 2.2.** *If $D \in \mathcal{D}(k)$ and if $D$ is not a complete digraph, then for any proper nonempty subset $X \subset V(D)$ such that $|\partial_D^+(X)| \leq k$, each of the following holds:*

(i) $(X, V(D) - X)_{D^c} \neq \emptyset$.
(ii) $|\partial_D^+(X)| = k$.
(iii) $(V(D) - X, X)_D = \{(y, x) \ : \ \textit{for any } y \in V(D) - X \textit{ and for any } x \in X\}$.

***Proof.*** Let $Y = V(D) - X$. Suppose that $(X, Y)_{D^c} = \emptyset$. Then the arcs in $\partial^+(X)$ induce an underlying complete bipartite graph with a vertex bipartition $\{X, Y\}$. It follows from $|\partial^+(X)| \leq k$ that we must have

$$|X|(n - |X|) = |X| \cdot |Y| \leq k, \text{ and } |X| + |Y| = n \geq k + 1.$$

As the minimum of $|X|(n - |X|)$ must be attained at the boundary point of the domain $1 \leq |X| \leq n - 1$, we observe that $k \leq n - 1 \leq |X|(n - |X|) \leq k$. it follows that we must have $n = k + 1$, and so by (1) that $D$ must be a $K_{k+1}^*$, contrary to the assumption that $D$ is not a complete digraph. This proves (i).

Therefore there must be an arc $(x, y) \in (X, Y)_{D^c}$. The existence of this arc and the fact that $D \in \mathcal{D}(k)$ imply that $|\partial_D^+(X)| = k$. This proves (ii).

To prove (iii), we again argue by contradiction and assume that for some $x \in X$ and $y \in Y$, $(y, x) \notin A(D)$. Then as $D \in \mathcal{D}(k)$, $\overline{\lambda}(D + (y, x)) \geq k + 1$. It follows that $D + (y, x)$ has a subdigraph $H'$ with $\lambda(H') \geq k + 1$ and with $(y, x) \in A(H')$. Hence both $X \cap V(H') \neq \emptyset$ and $Y \cap V(H') \neq \emptyset$. As $\partial_{H'}^+(X \cap V(H')) \subseteq \partial_D^+(X)$, we have $k + 1 \leq |\partial_{H'}^+(X \cap V(H))| \leq |\partial_D^+(X)| \leq k$, a contradiction. This proves (iii). ■

**Lemma 2.3.** *Suppose that $D \in \mathcal{D}(k) - \{K_{k+1}^*\}$, for any proper nonempty subset $X \subset V(D)$ such that $|\partial_D^+(X)| = k$, define $Y = V(D) - X$. One of the following must hold:*

(i) $|X| = 1$ *and* $D[Y] \in \mathcal{D}(k)$, *and*, $|V(D) - X| \geq k + 1$.
(ii) $|X| \geq k + 1$ *and* $D[X] \in \mathcal{D}(k)$, *and*, $|V(D) - X| = 1$.
(iii) *Both* $D[X] \in \mathcal{D}(k)$ *and* $D[Y] \in \mathcal{D}(k)$, *and*, *both* $|X| \geq k + 1$ *and* $|V(D) - X| \geq k + 1$.

***Proof.*** Let $Y = V(D) - X$. We make the following claims.
**Claim 1.** If $D[X]$ (or $D[Y]$, respectively) is a complete digraph, then $|X| \in \{1, k + 1\}$, (or $|Y| \in \{1, k + 1\}$, respectively).

By symmetry, we prove the case when $D[X]$ is complete. Let $m = |X|$. Then $D[X] = K_m^*$. Since $m - 1 = \lambda(K_m^*) \leq \overline{\lambda}(D) \leq k$, we have $m \leq k + 1$. Assume that $1 < m \leq k$. By Lemma 2.2(i), there exists an arc $(x, y) \in (X, Y)_{D^c}$. As $D \in \mathcal{D}(k)$, $D + (x, y)$ has a subdigraph $H$ with $\lambda(H) \geq k + 1$ and $(x, y) \in A(H)$. Hence $X \cap V(H) \neq \emptyset$. Let $X' = X \cap V(H)$ and $m' = |X'|$, and denote $X' = \{x_1, x_2, \cdots, x_{m'}\}$. For each $x_i$, let $k_i$ denote the number of arcs in $(X', V(H) - X')_H$ incident with $x_i$. Then $\sum_{i=1}^{m'} k_i = |\partial_H^+(X')| \geq \lambda(H) \geq k + 1$, and so $m'(k + 1) \leq \sum_{i=1}^{m'} d_H^+(x_i) \leq \sum_{i=1}^{m'} (k_i + m' - 1) = (k + 1) + m'(m' - 1)$, which leads to $(m' - 1)(k + 1) \leq m'(m' - 1)$. If $m' > 1$, then $k + 1 \leq m' \leq k$, a contradiction. Therefore, we must have $m' = 1$. Assume that $X' = \{x_1\}$. Then all arcs in $(X, Y)_D$ must be incident with $x_1$ in $X$. Since $m \geq 2$. $x_2$ is not incident with any arcs in $(X, Y)_D$, and so for any $y \in Y$, $(x_2, y) \in A(D^c)$. As $D \in \mathcal{D}(k)$, $D + (x_2, y)$ must have a subdigraph $H''$ such that $\lambda(H'') \geq k+1$ and $(x_2, y) \in A(H'')$. Since $m \leq k$, $d^+_{H''}(x_2) \leq |\partial^+_{D[X]}(x_2) \cup \{x_2, y)\}| = (m-1)+1 \leq k$, contrary to the assumption that $\lambda(H'') \geq k+1$. Therefore if $D[X]$ is complete, then $|X| \in \{1, k+1\}$. This proves Claim 1.

**Claim 2.** If $D[X]$ (or $D[Y]$, respectively) is not complete digraph, then $D[X] \in \mathcal{D}(k)$, (or $D[Y] \in \mathcal{D}(k)$, respectively).

Again by symmetry, it suffices to show that $D[X] \in \mathcal{D}(k)$. The case for showing $D[Y] \in \mathcal{D}(k)$ is similar and will be omitted. Since $D[X]$ is not complete, $A(D[X]^c) \neq \emptyset$. For any $e \in A(D[X]^c)$, since $A(D[X]^c) \subseteq A(D^c)$ and since $D \in \mathcal{D}(k)$, it follows by definition of $\mathcal{D}(k)$ that $D+e$ has a subdigraph $H_e$ with $e \in A(H_e)$ and with $\lambda(H_e) \geq k+1$. If for any $e \in A(D[X]^c)$, we always have $V(H_e) \subseteq X$, then by definition of $\mathcal{D}(k)$, we have $D[X] \in \mathcal{D}(k)$ and so $|X| \geq k+1$. Hence by contradiction, we assume that there exists an arc $(u, v) \in A(D[X]^c)$ such that $D+(u, v)$ has a subdigraph $H$ with $\lambda(H) \geq k+1$ and with $(u, v) \in A(H)$, and such that $V(H) \cap Y \neq \emptyset$. Then as $u, v \in X$ and as $V(H) \cap Y \neq \emptyset$, we have $\partial^+_H(X \cap V(H)) = (X \cap V(H), Y \cap V(H))_H \subseteq (X, Y)_D$. It follows that $k+1 \leq \lambda(H) \leq |\partial^+_H(X \cap V(H))| \leq |(X, Y)_D| = |\partial^+_D(X)| = k$, a contradiction. This proves Claim 2.

**Claim 3.** If $|X| = 1$, then $|Y| \geq k+1$; and If $|Y| = 1$, then $|X| \geq k+1$.

By symmetry, we shall assume that $|X| = 1$ to prove $|Y| \geq k+1$. The other case can be done with symmetric arguments. By contradiction, we assume that $X = \{x\}$ and $Y = \{y_1, y_2, \cdots, y_h\}$ with $h \leq k$. By Lemma 2.2 (iii), for each $i$ with $1 \leq i \leq h$, we have $(y_i, x) \in A(D)$. Since $D$ is simple, and since $|(X, Y)_D| = |\partial^+_D(X)| = k$, we must have $h = k$ and $(x, y_i) \in A(D)$ for every $i$ with $1 \leq i \leq k$. It follows that $D = K^*_{k+1}$, contrary to the assumption that $D \neq K^*_{k+1}$. This proves Claim 3.

With these claims, we now to prove the lemma. If $|X| = 1$ or $|Y| = 1$, then by Claim 3, Lemma 2.3 (i) or (ii) must hold. Assume that both $|X| > 1$ and $|Y| > 1$. If $D[X]$ is a complete digraph, then by Claim 1, $D[X] \cong K^*_{k+1} \in \mathcal{D}(k)$. If $D[X]$ is not a complete graph, then by Claim 2, $D[X] \in D(k)$ as well. Hence if $|X| = 1$, then $D[Y] \in \mathcal{D}(k)$, which implies $|Y| \geq k+1$. Similarly, in any case, $D[X] \in \mathcal{D}(k)$ and $|X| \geq k+1$. This proves the lemma. ∎

**Definition 2.4.** *Let $H \in \mathcal{D}(k)$ and let $\{v_1, v_2, \cdots, v_k\} \subset V(H)$ be a subset of $k$ distinct vertices. Let $u$ be a vertex not in $V(H)$. Define a digraph $[H, K_1]_k$ ($[K_1, H]_k$, respectively) as follows:*

(i) $V([H, K_1]_k) = V([K_1, H]_k) = V(H) \cup \{u\}$.
(ii) $A([H, K_1]_k) = A(H) \cup \{(v_1, u), (v_2, u), \cdots, (v_k, u)\} \cup (\bigcup_{v \in V(H)} \{(u, v)\})$.
*($A([K_1, H]_k) = A(H) \cup \{(u, v_1), (u, v_2), \cdots, (u, v_k)\} \cup (\bigcup_{v \in V(H)} \{(v, u)\})$, respectively).*

*Note that each of $[H, K_1]_k$ and $[K_1, H]_k$ represents a family of graphs as the set $\{v_1, v_2, \cdots, v_k\} \subset V(H)$ may vary. For notational convenience, we often use $[H, K_1]_k$ ($[K_1, H]_k$, respectively) to denote any member in the family $[H, K_1]_k$ ($[K_1, H]_k$, respectively).*

**Definition 2.5.** *Let $H_1, H_2 \in \mathcal{D}(k)$ with $H_1$ and $H_2$ being vertex disjoint, and let $\{u_1, u_2, \cdots, u_k\} \subset V(H_1)$ be a multiset of $V(H_1)$ and $\{v_1, v_2, \cdots, v_k\} \subset V(H_2)$ be a multiset of $V(H_2)$ such that all the arcs $(u_1, v_1), (u_2, v_2), \cdots, (u_k, v_k)$ are distinct. Define a digraph $D = D(H_1, H_2; k)$ as follows.*

(i) $V(D) = V(H_1) \cup V(H_2)$.
(ii) $A(D) = A(H_1) \cup A(H_2) \cup \{(u_1, v_1), (u_2, v_2), \cdots, (u_k, v_k)\}$
$\cup \left(\bigcup_{u \in V(H_1), v \in V(H_2)} \{(v, u)\}\right)$.

*Let $[H_1, H_2]_k$ be the family of all such digraphs $D = D(H_1, H_2; k)$ defined above. For notational convenience, we often use $[H_1, H_2]_k$ to denote any member in the family $[H_1, H_2]_k$.*

**Corollary 2.6.** *Let $D \in \mathcal{D}(k) - \{K^*_{k+1}\}$ be a digraph. Then there exists a nonempty proper subset $X \subseteq V(D)$ such that one of the following holds.*

(i) $|X| = 1$, *and for some $H \in \mathcal{D}(k)$, $D = [H, K_1]_k$.*
(ii) $|V(D) - X| = 1$ *and and for some $H \in \mathcal{D}(k)$, $D = [K_1, H]_k$.*
(iii) *For some $H_1, H_2 \in \mathcal{D}(k)$, $D = [H_1, H_2]_k$.*

***Proof.*** Since $D \in \mathcal{D}(k) - \{K^*_{k+1}\}$, by Lemma 2.2, there exists a proper nonempty subset $X \subset V(D)$ such that $|\partial^+_D(X)| = k$. Let $Y = V(D) - X$. By Lemma 2.3, one of the conclusions of Lemma 2.3 must hold.

If Lemma 2.3(i) holds, then $|X| = 1$ and $D[Y] \in \mathcal{D}(k)$. Let $X = \{x\}$ and $H = D[Y]$. As $|(\{x\}, Y)_D| = k$ and as, by Lemma 2.2(iii), for each $y \in Y$, $(y, x) \in A(D)$, it follows by Definition 2.4 that $D = [K_1, H]_k$, implying Corollary 2.6 (ii). If Lemma 2.3(ii) holds, then $|Y| = 1$ and $D[X] \in \mathcal{D}(k)$. Let $H = D[X]$. With a symmetric argument, Corollary 2.6 (i) holds. If Lemma 2.3(iii) holds, then let $H_1 = D[X]$ and $H_2 = D[Y]$. As $|(X, Y)_D| = k$, and $H_1, H_2 \in \mathcal{D}(k)$, and by Lemma 2.2(iii), $D = [H_1, H_2]_k$. ■

## 3. THE EXTREMAL FUNCTION

In this section, we shall determine the extremal function as shown in Theorem 3.2 below. This clearly implies Theorem 1.2. We start with a definition.

**Definition 3.1.** *Let $\mathcal{M}(k)$ be the family of digraphs such that*

(i) $K^*_{k+1} \in \mathcal{M}(k)$ *and such that*
(ii) *a digraph $D \neq K^*_{k+1}$ is in $\mathcal{M}(k)$ if and only if for some $H \in \mathcal{M}(k)$, $D = [H, K_1]_k$ or $D = [K_1, H]_k$.*

**Theorem 3.2.** *Let $D \in \mathcal{D}(n, k)$. Then*

$$|A(D)| \le k(2n - k - 1) + \binom{n-k}{2}. \tag{2}$$

*Furthermore, each of the following holds.*

(i) *If $k = 1$, then every digraph $D \in \mathcal{D}(1)$ satisfies equality in* (2).
(ii) *If $k \ge 2$, then a digraph $D \in \mathcal{D}(k)$ satisfies equality in* (2) *if and only if $D \in \mathcal{M}(k)$.*

***Proof.*** By Definition 3.1, every digraph $D$ in $\mathcal{M}(k)$ satisfies $\overline{\lambda}(D) \le k$. It is routine to verify that every digraphs $D \in \mathcal{M}(k)$ is $k$-maximal. A straightforward inductive computation shows that if $D \in \mathcal{M}(k)$ with $n = |V(D)| \ge k + 1$, then

$$|A(D)| = k(2n - k - 1) + \binom{n-k}{2}. \tag{3}$$

To proceed the proof of the theorem, we first prove (2) by induction on $n = |V(D)|$. If $n = k+1$, then $D = K^*_{k+1}$. As $|A(K^*_{k+1})| = k(k+1)$, we observe that (2) holds when $n = k+1$. Assume that $n > k+1$ and (2) holds for smaller values of $n$. Since $n > k+1$, $D$ cannot be a complete digraph. By Corollary 2.6, we make the following claims.

**Claim 1.** If Corollary 2.6 (i) or (ii) holds, then (2) holds as well.

Without loss of generality, we assume that $D = [H, K_1]_k$ for some $H \in \mathcal{D}(k)$ with $V(K_1) = \{v\}$. As $|V(H)| = n-1$, by Definition 2.4, $|\partial^+_D(v)| = k$ and $|\partial^-_D(v)| = n-1$. It follows by induction on $n$,

$$\begin{aligned}|A(D)| &= |A(H)| + k + (n-1) \le k(2(n-1)-k-1) + \binom{(n-1)-k}{2} \\ &\quad + k + (n-1) \\ &= k(2n-k-1) - 2k + \left(\sum_{i=1}^{n-k-2} i\right) + 2k + (n-k-1) \\ &= k(2n-k-1) + \binom{n-k}{2}.\end{aligned}$$

**Claim 2.** If Corollary 2.6 (iii) holds, then (2) holds. Furthermore, if $k \ge 2$, then (2) holds with strict inequality.

Without loss of generality, we may assume that $D = [H_1, H_2]_k$ for some $H_1, H_2 \in \mathcal{D}(k)$. Let $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. It follows by induction on $n$,

$$\begin{aligned}|A(D)| &= |A(H_1)| + k + n_1n_2 + |A(H_2)| \\ &\le k(2n_1-k-1) + \binom{n_1-k}{2} + k + n_1n_2 + k(2n_2-k-1) + \binom{n_2-k}{2} \\ &= k(2n-k-1) + \binom{n-k}{2} - k^2 + n_1n_2 + \binom{n_1-k}{2} + \binom{n_2-k}{2} \\ &\quad - \binom{n-k}{2}.\end{aligned}$$

We further observe the following.

$$\begin{aligned}&2\left[n_1n_2 + \binom{n_1-k}{2} + \binom{n_2-k}{2} - \binom{n-k}{2}\right] \\ &= 2n_1n_2 + (n_1-k)(n_1-k-1) + (n_2-k)(n_2-k-1) - (n-k)(n-k-1) \\ &= 2n_1n_2 + \left[\sum_{i=1}^{2}(n_i^2 - 2n_ik + k^2 - n_i + k)\right] - (n^2 - 2nk + k^2 - n + k) \\ &= (n_1+n_2)^2 - 2nk + 2k^2 - (n_1+n_2) + 2k - n^2 + 2nk - k^2 + n - k = k(k+1).\end{aligned}$$

As $k \ge 2$, we have $-k^2 + \frac{k(k+1)}{2} = -\frac{k^2-k}{2} < 0$, and so $|A(D)| \le k(2n-k-1) + \binom{n-k}{2} - \frac{k^2-k}{2}$. This implies Claim 2.

**Claim 3.** If $k \ge 2$, and if $D \in \mathcal{D}(k)$ satisfies equality in (2), then $D \in \mathcal{M}(k)$.

Let $D \in \mathcal{D}(k)$ be a digraph satisfying equality in (2). We argue by induction on $n = |V(D)| \ge k+1$ to prove Claim 3. If $n = k+1$, then $D = K^*_{k+1} \in \mathcal{M}(k)$. Hence we assume that $n > k+1$ and that Claim 3 holds for smaller values of $n$.

If $D$ satisfies Corollary 2.6(iii), then by Claim 2, $D$ cannot satisfy equality in (2), contrary to the assumption of Claim 3. Hence $D$ must satisfy Corollary 2.6(i) or (ii). By symmetry, we assume that for some digraph $H \in \mathcal{D}(k)$ and vertex $v \in V(D)$, we have $D = [\{v\}, H]_k$. By Definition 2.4, we have $|A(D)| = |A(H)| + k + n - 1$. As $D$ satisfies equality in (2), this implies that

$$|A(H)| = k(2n-k-1) + \binom{n-k}{2} - (n+k-1) = k(2(n-1)-k-1) + \binom{(n-1)-k}{2}.$$

It follows by induction that $H \in \mathcal{M}(k)$. By Definition 3.1, $D \in \mathcal{M}(k)$, and so Claim 3 is proved by induction.

**Claim 4.** If $k = 1$, then for any $D \in \mathcal{D}(k)$, we always have

$$|A(D)| = k(2n-k-1) + \binom{n-k}{2} = 2(n-1) + \frac{(n-1)(n-2)}{2}. \tag{4}$$

We again argue by induction. Since $|A(K_2^*)| = 2$, (4) holds when $n = k+1 = 2$. Assume that $n > 2$ and (4) holds for smaller values of $n$. Hence $D$ is not a complete digraph. By Corollary 2.6, we have the following observations.

Suppose that Corollary 2.6(i) or (ii) holds. Then $D \in \{[K_1, H]_k, [H, K_1]_k\}$ for some $H \in \mathcal{D}(1)$. By induction,

$$|A(D)| = 2[(n-1)-1] + 1 + (n-1) + \frac{(n-2)(n-3)}{2} = 2(n-1) + \frac{(n-1)(n-2)}{2}.$$

Thus (4) is justified in this case.

Now suppose that for some $H_1, H_2 \in \mathcal{D}(1)$, we have $D = [H_1, H_2]_k$. Let $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. By induction, and with the same computation in the proof of Claim 2, we have

$$\begin{aligned}|A(D)| &= |A(H_1)| + 1 + n_1n_2 + |A(H_2)| = \sum_{i=1}^{2}\left[2(n_i-1) + \frac{(n_i-1)(n_i-1)}{2}\right] + 1 + n_1n_2 \\ &= 2(n_1+n_2-1) - 1 + \frac{(n-1)(n-2)}{2} + \frac{2n_1n_2 + (n_1-1)(n_1-2) + (n_2-1)(n_2-2) - (n-1)(n-2)}{2} \\ &= 2(n_1+n_2-1) - 1 + \frac{(n-1)(n-2)}{2} + \frac{1+1}{2} = 2(n_1+n_2-1) + \frac{(n-1)(n-2)}{2}.\end{aligned}$$

Therefore, (4) is also justified in this case, and so Claim 4 is proved by induction.

Now Theorem 3.2 follows from (3), and Claims 1, 2, 3 and 4. ∎

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