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Spanning trails with variations of Chvátal–Erdős conditions

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ABSTRACT

Let $\alpha(G)$, $\alpha'(G)$, $\kappa(G)$ and $\kappa'(G)$ denote the independence number, the matching number, connectivity and edge connectivity of a graph G, respectively. We determine the finite graph families $\mathcal{F}_1$ and $\mathcal{F}_2$ such that each of the following holds.

(i) If a connected graph G satisfies $\kappa'(G) \geq \alpha(G) - 1$, then G has a spanning closed trail if and only if G is not contractible to a member of $\mathcal{F}_1$.

(ii) If $\kappa'(G) \geq \max\{2, \alpha(G) - 3\}$, then G has a spanning trail. This result is best possible.

(iii) If a connected graph G satisfies $\kappa'(G) \geq 3$ and $\alpha'(G) \leq 7$, then G has a spanning closed trail if and only if G is not contractible to a member of $\mathcal{F}_2$.

1. Introduction

In this paper, graphs considered are finite and loopless. We follow [5] for undefined terms and notation. Let $\boldsymbol{N_G(u)}$ be the set of vertices adjacent to u in G, and $\boldsymbol{D_i(G)} = \{\boldsymbol{v} \in \boldsymbol{V(G)} : \boldsymbol{d(v)} = \boldsymbol{i}\}$. As in [5], for a graph G, let $\boldsymbol{\alpha(G)}$, $\boldsymbol{\alpha'(G)}$, $\boldsymbol{\kappa(G)}$, $\boldsymbol{\kappa'(G)}$ denote independence number, matching number, connectivity and edge connectivity of G, respectively. An edge cut E of a graph G is **essential** if $G - E$ contains two nontrivial components. We use $\boldsymbol{O(G)}$ to denote the set of all odd degree vertices of G. A cycle on n vertices is often called an $\boldsymbol{n}$**-cycle**. For $A \subseteq V(G) \cup E(G)$, $\boldsymbol{G[A]}$ is the subgraph of G induced by A, and $\boldsymbol{G - A}$ is the subgraph of G obtained by deleting the elements in A. Let H be a graph. We say G is $\boldsymbol{H}$**-free** if G does not contain H as a subgraph.

As in [5], a graph G is **eulerian** if G is a closed trail. Equivalently, G is eulerian if G is connected with $O(G) = \emptyset$. A graph is **supereulerian** if it has a spanning closed trail. Boesch et al. [3] first posed the problem of characterizing supereulerian graphs. Pulleyblank [19] proved that determining if a 3-edge-connected planar graph is supereulerian is NP-complete. Catlin [8] gave a survey on supereulerian graphs, which was supplemented and updated in [14,18].

Motivated by a well-known result of Chvátal and Erdős [15] that every graph G with $\kappa(G) \geq \alpha(G)$ is Hamiltonian, there have been researches on conditions analogous to this Chvátal–Erdős Theorem to assure the existence of spanning trials in a graph utilizing relationship among independence number, matching number and edge-connectivity. See [1,16,17] and [21], among others. Let $\boldsymbol{P(10)}$ denote the Petersen graph and let $\boldsymbol{K_{2,3}(1,2,2)}$, $\boldsymbol{S_{1,2}}$, $\boldsymbol{K'_{2,3}}$ be the graphs depicted in Fig. 1. Let $\boldsymbol{P^n}$ be a path of order n. Define

$$\mathcal{F}_1 = \{K_2, P^3, P^4, K_{2,3}, K_{2,3}(1,2,2), S_{1,2}, P(10)\} \quad \text{and} \quad \mathcal{F}_2 = \{P(10), P(14)\}.$$

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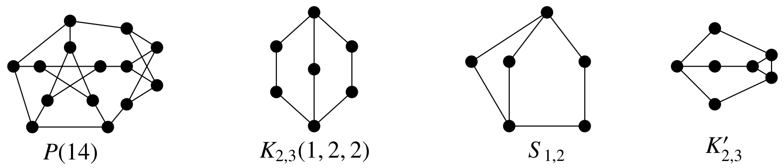

Fig. 1. $P(14)$ and some graphs in $\mathcal{F}_1$.

Theorem 1.1 (*Han et al., Theorem 3 of [16]*). *Let G be a simple graph with $\kappa(G) \geq 2$. If $\kappa(G) \geq \alpha(G) - 1$, then exactly one of the following holds.*

(i) *G is supereulerian.*

(ii) $G \in \{P(10), K_{2,3}, K_{2,3}(1, 2, 2), S_{1,2}, K'_{2,3}\}$.

(iii) *G is a 2-connected graph obtained from $K_{2,3}$ (resp. $S_{1,2}$) by replacing a vertex whose neighbors have degree three in $K_{2,3}$ (resp. $S_{1,2}$) with a complete graph of order at least three.*

Theorem 1.2 (*Tian and Xiong, Theorem 4 of [21]*). *If G is a 2-connected graph with $\alpha(G) \leq \kappa(G) + 3$, then G has a spanning trail.*

The supereulerian property for graphs G with $\alpha'(G) \leq 2$ and $\kappa'(G) \geq 2$ has been completely determined in [1] and [17].

The purpose of this paper is to investigate the existence of spanning trails in graphs with given relationship between independence number and edge-connectivity, or matching number with edge-connectivity. In this paper, we determine the finite graph families $\mathcal{F}_1$ and $\mathcal{F}_2$ such that each of the following holds.

Theorem 1.3. *If a graph G satisfies $\kappa'(G) \geq \alpha(G) - 1$, then G has a spanning closed trail if and only if G is not contractible to a member of $\mathcal{F}_1$.*

Theorem 1.4. *If $\kappa'(G) \geq \max\{2, \alpha(G) - 3\}$, then G has a spanning trail.*

Theorem 1.5. *If a graph G satisfies $\kappa'(G) \geq 3$ and $\alpha'(G) \leq 7$, then G has a spanning closed trail if and only if G is not contractible to a member of $\mathcal{F}_2$.*

In Section 2, we display the mechanism we will use in our arguments. In the subsequent sections, we prove the main results.

2. Preliminaries

For a subset $Y \subseteq E(G)$, the **contraction** G/Y is the graph obtained from G by identifying the two ends of each edge in Y and then by deleting the resulting loops. If H is a subgraph of G, we often use G/H for $G/E(H)$. A graph G is called **collapsible** if for any $R \subseteq V(G)$ with $|R|$ is even, G has a spanning subgraph S_R with $O(S_R) = R$. By definition, collapsible graphs are supereulerian. In [7], Catlin showed that every graph G has a unique collection of maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The **reduction** of G, denoted by G', is the graph $G/(H_1 \cup H_2 \cup \cdots \cup H_c)$. A graph G is reduced if $G' = G$.

Theorem 2.1 (*Catlin, Theorem 2 of [7]*). *Every graph G with $\kappa'(G) \geq 4$ is collapsible.*

Theorem 2.2 (*Catlin, Theorem 3 of [7]*). *Let G be a connected graph, H be a collapsible subgraph of G and let G' be the reduction of G. Then*

(i) *G is collapsible if and only if G/H is collapsible.*

(ii) *G is supereulerian if and only if G/H is supereulerian.*

(iii) *G has a spanning trail if and only if G/H has a spanning trail.*

(iv) *Any subgraph of a reduced graph is reduced.*

Let $\boldsymbol{F(G)}$ be the minimum number of extra edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The following results on the structures of reduced graphs will be needed.

Theorem 2.3. *Let G be a connected reduced graph. Then*

(i) *(Catlin, Theorem 7 of [6]) If $|V(G)| \geq 3$, then $F(G) = 2(|V(G)| - 1) - |E(G)|$.*

(ii) *(Catlin, Theorem 8 of [7]) G is simple and K_3-free.*

(iii) *(Catlin, Theorem 8 of [7]) $\delta(G) \leq 3$.*

(iv) *(Catlin et al., Theorem 1.3 of [9]) Either $G \in \{K_1, K_2\} \cup \{K_{2,t} : t \geq 1\}$ or $F(G) \geq 3$ and $|E(G)| \leq 2|V(G)| - 5$.*

3. Spanning trails with bounded independence numbers

In the first half of this section, we investigate the relationship between minimum degree and independence number that assures supereulerian property.

Lemma 3.1. *Let G be a reduced graph with $\delta(G) \geq 2$ and $\alpha(G) \leq 3$. Then G is supereulerian if and only if $G \notin \{K_{2,3}, K_{2,3}(1, 2, 2), S_{1,2}\}$.*

Proof. Since G is reduced, by Theorem 2.3(ii), G is simple and K_3-free. Thus, $\Delta(G) \leq \alpha(G) \leq 3$. Assume that G has a cut vertex u. Since $\Delta(G) \leq 3$, at least one of the edges incident with u is a cut edge of G. Let uv denote this cut edge. Suppose G_1 and G_2 are two connected components in $G - uv$. Since $\delta(G) \geq 2$, $|D_1(G_i)| \leq 1$ $(i = 1, 2)$. Since G is K_3-free, G_i is K_3-free. Hence we may assume that, for $1 \leq i \leq 2$, $V(G_i) - \{u, v\}$ has two vertices u_i and v_i with $u_iv_i \notin E(G_i)$. It follows that $\{u_1, v_1, u_2, v_2\}$ is an independent set in G, contrary to the assumption $\alpha(G) \leq 3$. Thus we may assume that $\kappa(G) \geq 2$ and so $\kappa(G) \geq \alpha(G) - 1$. Since G is reduced with $\alpha(G) \leq 3$, by Theorem 1.1 and $\alpha(P(10)) = 4$, either G is supereulerian or $G \in \{K_{2,3}, K_{2,3}(1, 2, 2), S_{1,2}\}$. □

Theorem 3.2 (*Corollary 5.2 of [11]*). *Let G be a connected simple graph with $|V(G)| \leq 13$ and $\delta(G) \geq 3$, and G' be the reduction of G. Then $G' \in \{K_1, K_2, P^3, K_{1,2}, K_{1,3}, P(10)\}$.*

Theorem 3.3 (*Theorem 1 of [12]*). *Let G be a connected reduced graph with order n.*
(i) *If $\alpha(G) = 2$, then $n \leq 5$.*
(ii) *If $\alpha(G) = 3$, then $n \leq 8$.*
(iii) *If $\alpha(G) \geq 4$, then $\frac{\delta(G)\alpha(G)+4}{2} \leq n \leq 4\alpha(G) - 5$.*

Theorem 3.4. *Let G be a connected reduced graph with $\delta(G) \geq \alpha(G) - 1$. Then G is supereulerian if and only if $G \notin \mathcal{F}_1$.*

Proof. It is routine to verify that every graph in $\mathcal{F}_1$ is not supereulerian. It suffices to prove that under the assumption of the theorem, if $G \notin \mathcal{F}_1$, then G must be supereulerian. Since K_1 is supereulerian, we assume that $|V(G)| \geq 2$. Since G is reduced, by Theorem 2.3(iii), we have $\delta(G) \leq 3$. Assume first that $\delta(G) = 3$, implying $\alpha(G) \leq \delta(G) + 1 = 4$. By Theorem 3.3(iii), $|V(G)| \leq 11$. Since $\delta(G) = 3$, by Theorem 3.2, we have $G \cong P(10)$, which is in $\mathcal{F}_1$, contrary to the assumption $G \notin \mathcal{F}_1$. Hence we must have $\delta(G) \leq 2$. Assume that $\delta(G) = 2$ and so $\alpha(G) \leq \delta(G) + 1 = 3$. By Lemma 3.1, we have $G \in \{K_{2,3}, K_{2,3}(1, 2, 2), S_{1,2}\} \subseteq \mathcal{F}_1$, contrary to the assumption $G \notin \mathcal{F}_1$. Thus we must have $\delta(G) = 1$, forcing $\alpha(G) \leq 2$. By Theorem 2.3(ii), G is K_3-free. So $\Delta(G) \leq \alpha(G) \leq 2$. Since G is a connected graph with $\delta(G) = 1$, $\Delta(G) \leq 2$ and $\alpha(G) \leq 2$, G must be a path with length at most 4. Thus, $G \in \{K_2, P^3, P^4\} \subseteq \mathcal{F}_1$, again, contrary to the assumption $G \notin \mathcal{F}_1$. These contradictions justify Theorem 3.4. □

Corollary 3.5. *Let G be a connected graph with $\kappa'(G) \geq \alpha(G) - 1$. Let G' be the reduction of G. Then G is supereulerian if and only if $G' \notin \mathcal{F}_1$.*

Proof. By the definition of graph contractions, we have $\kappa'(G') \geq \kappa'(G) \geq \delta(G) \geq \alpha(G) - 1 \geq \alpha(G') - 1$. By Theorem 3.4, G' is supereulerian if and only if $G' \notin \mathcal{F}_1$. □

In the following, we will investigate the relationship between $\alpha(G)$ and $\kappa'(G)$ which may warrant the existence of (possibly open) spanning trails. We need the assistance of some former results.

Lemma 3.6 (*Corollary 2.1 of [11]*). *Let G be a simple 2-edge-connected graph with order $n \leq 9$. If $|D_2(G)| \leq 2$ and G is K_3-free, then G is collapsible.*

Lemma 3.7 (*Corollary 2.3 of [11]*). *Let G be a simple 2-edge-connected graph with order n. If $n \leq 10$ and $|D_2(G)| \leq 1$, then either G is collapsible or $G \cong P(10)$.*

Lemma 3.8. *Let G be a connected reduced graph with order $n \leq 10$ and $\delta(G) \geq 2$. If $|D_2(G)| \leq 2$, then $\kappa'(G) \geq 2$.*

Proof. By contradiction, we assume that G has a cut edge e, and H_1, H_2 are the two connected components in $G - e$ with $|V(H_1)| \leq |V(H_2)|$. Since $n = |V(G)| \leq 10$ and $\delta(G) \geq 2$, $1 < |V(H_1)| \leq 5$. Because $\delta(G) \geq 2$ and $|D_2(G)| \leq 2$, $0 \leq |D_1(H_1)| \leq 1$. And we have either $|D_2(H_1)| \leq 3$ if $|D_1(H_1)| = 0$ or $|D_2(H_1)| \leq 1$ if $|D_1(H_1)| = 1$. In either case, we have $2|D_1(H_1)| + |D_2(H_1)| \leq 3$. Then

$$\begin{aligned} 2|E(H_1)| &= \sum_{i\geq 1} i|D_i(H_1)| = |D_1(H_1)| + 2|D_2(H_1)| + \sum_{j\geq 3} j|D_j(H_1)| \\ &\geq |D_1(H_1)| + 2|D_2(H_1)| + 3[|V(H_1)| - (|D_1(H_1)| + |D_2(H_1)|)] \\ &= 3|V(H_1)| - (2|D_1(H_1)| + |D_2(H_1)|) \\ &\geq 3|V(H_1)| - 3. \end{aligned}$$

Since G is reduced, by Theorem 2.2(iv) and Theorem 2.3(i), we have,

$$F(H_1) = 2|V(H_1)| - |E(H_1)| - 2 \le 2|V(H_1)| - \frac{3|V(H_1)| - 3}{2} - 2 = \frac{|V(H_1)| - 1}{2} \le 2.$$

By Theorem 2.3(iv), and since $|V(H_1)| \le 5$ and $\delta(G) \ge 2$, we must have $H_1 \cong K_{2,3}$. It follows that $|V(H_2)| = |V(H_1)| = 5$, and so by the symmetry between H_1 and H_2, we also have $H_2 \cong K_{2,3}$. Since the cut edge e in G is incident with at most one vertex in $D_2(H_1)$ and at most one vertex in $D_2(H_2)$, it follows that $|D_2(G)| \ge (|D_2(H_1)| - 1) + (|D_2(H_2)| - 1) = 4$, contrary to the assumption that $|D_2(G)| \le 2$. Hence we must have $\kappa'(G) \ge 2$. □

Theorem 3.9. *If G is a graph with $\kappa'(G) \ge \max\{2, \alpha(G) - 3\}$, then G has a spanning trail.*

Proof. We argue by contradiction and assume that

$$G \text{ does not have a spanning trail, and } |V(G)| \text{ is minimized.} \tag{1}$$

Suppose G' is the reduction of G. Since $\kappa'(G') \ge \kappa'(G) \ge \max\{2, \alpha(G)-3\} \ge \max\{2, \alpha(G')-3\}$, by Theorem 2.2(iii), it suffices to prove the case when $G' = G$. By Theorem 2.1, we have $2 \le \kappa'(G) \le 3$. By Theorem 1.2, we may assume that $\kappa'(G) > \kappa(G)$. Let X be a vertex cut of G and let

$$H_1, H_2, \ldots, H_t \text{ be the components of } G - X, \text{ and for } 1 \le i \le t,\ G_i = G[V(H_i) \cup X]. \tag{2}$$

Claim 1. *Suppose that $|X| \le 2$ and $t \ge 2$. Then each of the following holds.*

(i) *If Γ_1 and Γ_2 are two subgraphs of G with $V(\Gamma_1) \cap V(\Gamma_2) \subseteq X$ and $E(\Gamma_1) \cap E(\Gamma_2) \subseteq E(G[X])$, then $\alpha(G[V(\Gamma_1) \cup V(\Gamma_2)]) \ge \alpha(\Gamma_1) + \alpha(\Gamma_2) - |X|$.*

(ii) *$\sum_{i=1}^{t} \alpha(G_i) \le \alpha(G) + |X|(t-1)$.*

(iii) *If $|X| = 1$, then for any $1 \le i \le t$, G_i is not supereulerian.*

(iv) *If $|X| = 1$, then for each $1 \le i \le t$, we have $\kappa'(G_i) \ge \kappa'(G)$ and $\alpha(G_i) \ge \kappa'(G) + 1$. And*

$$t(\kappa'(G) + 1) \le \sum_{i=1}^{t} \alpha(G_i) \le \alpha(G) + (t-1). \tag{3}$$

Proof of Claim 1. We prove (i) first. For $1 \le i \le 2$, suppose S_i is a maximum independent set of Γ_i. Let $X' = \{x \in X : N_G(x) \cap (S_1 \cup S_2) \ne \emptyset\}$ and S be obtained from $S_1 \cup S_2 - X'$. Since $V(\Gamma_1) \cap V(\Gamma_2) \subseteq X$ and $E(\Gamma_1) \cap E(\Gamma_2) \subseteq E(G[X])$, S is an independent set of $G[V(\Gamma_1) \cup V(\Gamma_2)]$. Since $\alpha(G[V(\Gamma_1) \cup V(\Gamma_2)]) \ge |S| \ge |S_1| + |S_2| - |X|$, and so (i) follows. Consequently, (ii) of Claim 1 follows from (i) by induction on t. To show (iii), we assume that for some i_0, G_{i_0} is supereulerian. Let $\Gamma = G/G_{i_0}$. Then we have $\kappa'(G/G_{i_0}) \ge \kappa'(G) \ge \max\{2, \alpha(G) - 3\} \ge \max\{2, \alpha(G/G_{i_0}) - 3\}$. By (1), G_{i_0} has a spanning trail T passing through the only vertex in X, and so T can be extended to a spanning trail of G by including a spanning eulerian subgraph of G_{i_0} to T. This justifies (iii). Finally we note that if $|X| = 1$, then every edge cut of G_i is also an edge cut of G, and so $\kappa'(G_i) \ge \kappa'(G)$. If for some i, $\alpha(G_i) \le \kappa'(G)$, then by Corollary 3.5, G_i is supereulerian, contrary to Claim 1(iii). This proves $\alpha(G_i) \ge \kappa'(G) + 1$. Hence (3) follows from Claim 1(ii) and $\alpha(G_i) \ge \kappa'(G) + 1$, for each $1 \le i \le t$. This proves Claim 1.

Throughout the rest of the proof, when a vertex cut X of G is specified, the notation in (2) will be used in the arguments.

Case 1. $\kappa'(G) = 2$.

Since $\kappa'(G) \ge \max\{2, \alpha(G) - 3\}$, we have $\alpha(G) \le 5$. By Theorem 1.2, we may assume that $\kappa(G) = 1$, and so G has a vertex cut $X = \{x\}$. By (3) with $\kappa'(G) = 2$ and $\alpha(G) \le 5$, we have $3t \le t + 4$, and so $t = 2$. Again by (3), $\alpha(G_1) + \alpha(G_2) = 6$, and so $(\alpha(G_1), \alpha(G_2)) \in \{(2, 4), (3, 3), (4, 2)\}$. Since $|X| = 1$, by Claim 1(iv), $\kappa'(G_i) \ge \kappa'(G) = 2$ $(i = 1, 2)$. If $(\alpha(G_1), \alpha(G_2)) = (2, 4)$ (resp. $(\alpha(G_1), \alpha(G_2)) = (4, 2)$), then by Theorem 3.4, G_1 (resp. G_2) is supereulerian, contrary to Claim 1(iii). Hence we must have $(\alpha(G_1), \alpha(G_2)) = (3, 3)$. By Theorem 3.4, each of G_1 and G_2 is either supereulerian or is in $\{K_{2,3}, K_{2,3}(1, 2, 2), S_{1,2}\}$. Since $|X| = 1$, it follows that G has a spanning trail, contrary to (1). Thus Case 1 will always lead to a contradiction.

Case 2. $\kappa'(G) = 3$.

As $\kappa'(G) \ge \max\{2, \alpha(G) - 3\}$, we have $\alpha(G) \le 6$. By Theorem 1.2, we may assume that $\kappa(G) \in \{1, 2\}$. Let X be a vertex cut of G with $|X| = \kappa(G)$. If $\kappa(G) = 1$, then $\kappa'(G_i) \ge \kappa'(G) = 3$ $(1 \le i \le t)$, and so by (3) and $\alpha(G) \le 6$, we have $4t \le t + 5$, forcing $t = 1$, contrary to the fact that $t \ge 2$. Hence we must have $\kappa(G) = 2$. Denote $X = \{u_1, u_2\}$. Since $\kappa'(G) = 3$ and since G is reduced, for each $i \in \{1, 2\}$, (recall that notation in (2) is used here),

$$|D_1(G_i)| \le 1,\ D_1(G_i) \cup D_2(G_i) \subseteq \{u_1, u_2\} \text{ and } |V(H_i)| \ge 3. \tag{4}$$

Let $G_i^- = G_i - D_1(G_i)$ and $|V(G_i^-)| = n_i^-$ $(1 \le i \le t)$. By $\kappa'(G) = 3$ and (4),

$$\delta(G_i^-) \ge 2 \text{ and } |D_2(G_i^-)| \le 2. \tag{5}$$

By Theorem 2.2(iv), G_i^- is reduced. If $n_i^- \le 9$, by (5) and Lemma 3.8, $\kappa'(G_i) \ge 2$. By Theorem 2.3(ii) and Lemma 3.6, G_i^- is collapsible, contrary to the fact that G_i^- is reduced. Then $n_i^- \ge 10$ $(1 \le i \le t)$. Since $\alpha(G) \le 6$, by Theorem 3.3(iii), $n \le 19$. Since $|X| = 2$, we have $t = 2$.

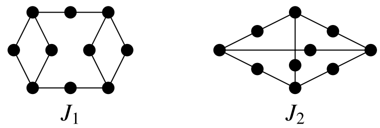

Fig. 2. The graphs J_1 and J_2.

Claim 2. *For each i with $1 \le i \le 2$, both $n_i^- = 10$ and $G_i = G_i^-$.*

Proof of Claim 2. By symmetry, it suffices to prove the case when $i = 1$. By contradiction, assume that $n_1^- > 10$ or $G_1 \ne G_1^-$. Since $G_1^- = G_1 - D_1(G_1)$, each of these assumptions leads to $|V(G_1)| \ge 11$. Hence we have $|V(G_1)| \ge 11$ and $|V(G_2)| \ge 10$. As $G_i = G[V(H_i) \cup X]$ and $|X| = 2$, we have $|V(H_1)| \ge 9$ and $|V(H_2)| \ge 8$. By Theorem 3.3, both $\alpha(H_1) \ge 4$ and $\alpha(H_1) \ge 3$, and so $\alpha(G) \ge \alpha(H_1) + \alpha(H_2) \ge 7$, contrary to $\alpha(G) \le 6$. This proves Claim 2.

Claim 3. *For each i with $1 \le i \le 2$ and for any $u \in \{u_1, u_2\}$, there is a spanning trail starting from u in G_i.*

Proof of Claim 3. Without loss of generality, suppose $i = 1$. By Claim 2, $G_1 = G_1^-$, and so (5) applies to G_1. If $|D_2(G_1)| \le 1$, by Claim 2 and Lemma 3.8, $\kappa'(G_1) \ge 2$. By Lemma 3.7, we have $G_1 \cong P(10)$ which has a spanning trail starting from any vertex. Hence we assume that $|D_2(G_1)| = 2$. By (4), $D_2(G_1) = \{u_1, u_2\}$. By Lemma 3.8, $\kappa'(G_1) \ge 2$. For any $u \in \{u_1, u_2\}$, since $d_{G_1}(u) = 2$, by Claim 2, there is a $v \in V(G_1)$ such that $uv \notin E(G_1)$. Let $G_1^+ = G_1 + uv$. Then $\delta(G_1^+) \ge 2$, $|D_2(G_1^+)| \le 1$ and $\kappa'(G_1^+) \ge \kappa'(G_1) \ge 2$. By Lemma 3.7, either G_1^+ is collapsible or $G_1^+ \cong P(10)$. If G_1^+ is collapsible, then there is a spanning closed trail T in G_1^+, and so $T - uv$ is a spanning trail starting from u in G_1. If $G_1^+ \cong P(10)$, then $G_1^+ - u$ is supereulerian, implying that G_1 has a spanning trail starting from u. This proves Claim 3.

Since $t = 2$, by Claim 3, G has a spanning trail. This completes the proof of Theorem 3.9. □

As shown in [21], the inequality of Theorem 3.9 is sharp. Let J_1 and J_2 be the graphs depicted in Fig. 2. For $i \in \{1, 2\}$, let $t \ge 3$ be an integer and J_i^* be a graph obtained from J_i by replacing at most two 2-degree vertices in J_i by a complete graph K_t. Then $\kappa'(J_i^*) \ge 2$ and $\alpha(J_i^*) = 6$. But J_i^* does not have a spanning trail.

4. Spanning trails with bounded matching numbers

In this section, we will investigate supereulerian graphs with a bounded size of maximum matchings. A component H of G is an **odd component** if $|V(H)| \equiv 1 \pmod 2$. Let $q(G) = |\{Q : Q \text{ is an odd component of } G\}|$. Tutte [22] and Berge [2] proved the following theorem.

Theorem 4.1 (*Tutte, [22]; Berge, [2]*). *Let G be a graph with n vertices. If*

$$t = \max_{S \subset V(G)} \{q(G - S) - |S|\}, \tag{6}$$

then $\alpha'(G) = (n - t)/2$.

In [13], a lower bound of the size of maximum matching has been found for reduced graphs.

Theorem 4.2 (*Theorem 1 of [13]*). *Let G be a reduced graph with n vertices and $\delta(G) \ge 3$. Then $\alpha'(G) \ge \min\{\frac{n-1}{2}, \frac{n+4}{3}\}$.*

Following a similar idea in [13], the lower bond in Theorem 4.2 can be slightly improved as shown in Theorem 4.4. We start with a lemma on reduced graphs.

Lemma 4.3. *Let G be a connected reduced graph with $\delta(G) \ge 3$. Suppose that $S \subseteq V(G)$ is a vertex subset attaining the maximum in (6) with $|S| > 0$, $m = q(G - S)$ and that $G_1, G_2, \ldots, G_m$ are the components in $G - S$ with odd number of vertices such that $|V(G_1)| \le |V(G_2)| \le \cdots \le |V(G_m)|$. Define*

$$X = \{G_i : |V(G_i)| = 1, 1 \le i \le m\}, \qquad Y = \{G_i : |V(G_i)| = 3,\ 1 \le i \le m\}, \qquad x = |X|, y = |Y|.$$

$$V^* = \cup_{k=1}^{x+y} V(G_k),\ G^* = G[V^* \cup S^*] \quad \text{and} \quad s^* = |S^*|, \quad \text{where } S^* = \{s \in S : v^* s \in E(G),\ v^* \in V^*\}. \tag{7}$$

Thus G^* is spanned by a bipartite subgraph with (V^*, S^*) being its vertex bipartition with $|V^*| = x + 3y \ge 1$. By the definition of x, V^* contains x isolated vertices in $G^*[V^*]$. Then each of the following holds.

(i) $n \ge \sum_{i=1}^{m} |V(G_i)| + |S| \ge m|V(G_1)| + |S|$.
(ii) *If $x > 0$, then $s^* \ge 3$.*
(iii) $m \le \frac{n+4x+2y-|S|}{5}$.
(iv) $G^* \notin \{K_1, K_2, K_{1,2}, K_{2,2}\}$.
(v) $|E(G^*)| \ge 3x + 7y$.

Proof. Statement (i) follows from the definition of m and G_i. If $x > 0$, then by $\delta(G) \geq 3$, there must be at least 3 vertices in S^* adjacent to the only vertex in G_1, and so $s^* \geq 3$. This justifies (ii). By (7), we have $n \geq |S| + x + 3y + 5(m - x - y)$, and so (iii) follows. As $\delta(G) \geq 3$, every vertex in G_1 must have degree at least 3 in G^*, and so (iv) must hold. Since $\delta(G) \geq 3$ and G does not contain a 3-cycle, every vertex in $\cup_{G_i \in X} V(G_i)$ is incident with at least 3 edges in G^*; and every component in $G^*[\cup_{G_i \in Y} V(G_i)]$ is a $K_{1,2}$ and is incident with at least 5 edges with one end in S^* plus two edges in $E(G_i)$. Hence $|E(G^*)| \geq 3x + 7y$. This proves (v). □

Theorem 4.4. *Let G be a connected reduced graph with n vertices and $\delta(G) \geq 3$. Then $\alpha'(G) \geq \min\{\frac{n}{2}, \frac{n+5}{3}\}$.*

Proof. Let t be defined as in (6). By Theorem 4.1, if $t = 0$, $\alpha'(G) = \frac{n}{2} \geq \min\{\frac{n}{2}, \frac{n+5}{3}\}$. Hence we assume that $t \geq 1$. If $n \leq 11$, then since $\delta(G) \geq 3$, by Theorem 3.2, $G \cong P(10)$. As $\alpha'(P(10)) = 5 = \frac{10}{2}$, Theorem 4.4 holds when $n \leq 11$.

Hence we assume that $n \geq 12$, and so $\frac{n+5}{3} < \frac{n}{2}$. By Theorem 4.1, to prove Theorem 4.4, it suffices to show that

$$\alpha'(G) \geq \frac{n-t}{2} \geq \frac{n+5}{3}, \text{ or equivalently, } t \leq \frac{n-10}{3}. \tag{8}$$

In the rest of the proof, we shall show that (8) always holds in any case, which implies the validity of Theorem 4.4. Define $S, m, G_1, G_2, \ldots, G_m, V^*, S^*, s^*$ and G^* as in Lemma 4.3. Since G is reduced, by Theorem 2.3(ii), G is simple and K_3-free. If $|S| = 0$, as G is connected and as $n \geq 12$, we have $t = 1$ and so $|V(G)|$ is odd and $n \geq 13$. By Theorem 4.1 and as $n \geq 13$, $\alpha'(G) \geq \frac{n-1}{2} \geq \frac{n+5}{3}$, and so (8) holds. Hence we assume that $|S| \geq 1$.

Case 1. $x = 0$, i.e. $|V(G_1)| \geq 3$.

Subcase 1.1. $|V(G_1)| = 3$.

Since G is K_3-free, $G_1 \cong K_{1,2}$. By $\delta(G) \geq 3$, we have $|S| \geq 3$. It follows by $n \geq 3m + |S|$ that

$$t = m - |S| \leq \frac{n - |S|}{3} - |S| = \frac{n - 4|S|}{3} \leq \frac{n-12}{3},$$

and so (8) must hold.

Subcase 1.2. $|V(G_1)| = 5$.

If $|S| = 1$, then as G is K_3-free and $\delta(G) \geq 3$, we have $|E(G[V(G_1) \cup S])| \geq \frac{15}{2} > 7$. On the other hand, $G[V(G_1) \cup S] \notin \{K_1, K_2\} \cup \{K_{2,\ell} : \ell \geq 1\}$ since $\delta(G) \geq 3$. By Theorem 2.3(iv), we have $|E(G[V(G_1) \cup S])| \leq 2(5+1) - 5 = 7$. A contradiction is obtained. Hence we assume that $|S| \geq 2$. As $n \geq 5m + |S|$, we have $m \leq \frac{n-|S|}{5}$. It follows by $n \geq 12$ and $t = m - |S| \geq 1$ that

$$t = m - |S| \leq \frac{n - |S|}{5} - |S| = \frac{n - 6|S|}{5} \leq \frac{n-12}{5} < \frac{n-10}{3},$$

and so (8) must hold.

Subcase 1.3. $|V(G_1)| \geq 7$.

Since $t = m - |S| \geq 1$, we have $m \geq 2$, and so $n \geq 7m + |S| \geq 15$ and $m \leq \frac{n-|S|}{7}$. It follows that

$$t = m - |S| \leq \frac{n - |S|}{7} - |S| = \frac{n - 8|S|}{7} \leq \frac{n-8}{7} < \frac{n-10}{3},$$

and so (8) must hold.

Case 2. $x \geq 1$.

By Lemma 4.3(iv), G^* is not in $\{K_1, K_2, K_{1,2}, K_{2,2}\}$, and so by Theorem 2.3(iv), either for some integer $\ell \geq 3$, $G^* \cong K_{2,\ell}$ or $F(G^*) \geq 3$.

Subcase 2.1. For some integer $\ell \geq 3$, $G^* \cong K_{2,\ell}$.

Since $\delta(G) \geq 3$, every vertices in V^* must have degree at least 3 in G^*, $|V^*| = x = 2$ and $s^* = \ell \geq 3$. By the definition of y, we must have $y = 0$ and $|S| \geq |S^*|$. It follows by Lemma 4.3(iii) that $1 \leq t = m - |S| \leq \frac{n+8+2y-|S|}{5} - |S| \leq \frac{n+8-6s^*}{5}$, and so $n \geq 6s^* - 3$. As $s^* \geq 3$, we have $n \geq 6s^* - 3 \geq 15 \geq 32 - 9s^*$, or $5(n-10) \geq 3(n+8-6s^*)$. Hence

$$t \leq \frac{n+8-6s^*}{5} < \frac{n-10}{3},$$

and so (8) must hold.

Subcase 2.2. $F(G^*) \geq 3$.

By Theorem 2.3(i) and by Lemma 4.3(v), $3x + 7y \leq |E(G^*)| \leq 2(|V(G^*)| - 1) - 3 = 2(x + 3y + |S|) - 5$. This implies that $|S| \geq \frac{x+y+5}{2}$, and so $n \geq x + 3y + |S| \geq \frac{3x+7y+5}{2}$. It follows that

$$t = m - |S| \leq \frac{n + 4x + 2y - |S|}{5} - |S| \leq \frac{n + x - y - 15}{5} \leq \frac{n-10}{3},$$

and so (8) must hold. This completes the proof of the theorem. □

The following theorem for 3-edge-connected graphs with order at most 15 will be needed.

Theorem 4.5 (*Theorem 1.1 of [11]*). *Let G be a 3-edge-connected graph and G' be the reduction of G.*
(i) *If $|V(G)| \leq 13$, then either G is supereulerian or $G' \cong P(10)$.*
(ii) *If $|V(G)| \leq 14$, then either G is supereulerian or $G' \in \mathcal{F}_2$.*
(iii) *If $|V(G)| = 15$, G is not supereulerian and $G' \notin \mathcal{F}_2$, then G is an essentially 4-edge-connected reduced graph with girth at least 5, $\kappa(G) \geq 2$ with $V(G) = D_3(G) \cup D_4(G)$ where $|D_4(G)| = 3$.*

A few more former results are needed in the proof of the main theorem in this section.

Theorem 4.6 (*Theorem 3.1 of [10]*). *Let G be a 3-edge-connected reduced graph with $F(G) = 3$. Then either G is supereulerian or each of the following holds:*
(i) *G has no edge joining two vertices of even degree;*
(ii) *G has girth at least 5;*
(iii) *G has no 2-edge-connected subgraph G with $F(H) = 2$.*

Theorem 4.7 (*Reiman, [20]; Bollobás, [4]*). *Let G be a connected bipartite C_4-free graph with vertex bipartition $\{X, Y\}$, where $|X| \leq |Y|$. Then*

$$|E(G)| \leq \sqrt{|Y| \cdot |X|(|X| - 1) + \frac{|Y|^2}{4}} + \frac{|Y|}{2}.$$

Lemma 4.8. *Let G be a reduced graph with order $n \geq 15$. If $\kappa'(G) \geq 3$ and $\alpha'(G) \leq 7$, then G is supereulerian.*

Proof. By contradiction, assume that G is not supereulerian. As $\alpha'(G) \leq 7$ and $n \geq 15$, by Theorem 4.4,

$$15 \leq n \leq 3\alpha'(G) - 5 \leq 16. \tag{9}$$

Let t be the integer satisfying (6) in Theorem 4.1. Then $\alpha'(G) = \frac{n-t}{2}$. By (9) and Theorem 4.4, we have $7 \geq \alpha'(G) = \frac{n-t}{2} \geq \frac{n+5}{3}$. Thus $\frac{n-10}{3} \geq t \geq n - 14$. By (9), we have $t \geq 1$ when $n = 15$ and $t \geq 2$ when $n = 16$. We shall show that neither case can occur to reach a contradiction to the assumption that G is not supereulerian, thereby proving the theorem. Define S, m, $G_1, G_2, \ldots, G_m, V^*, S^*, s^*$ and G^* as in Lemma 4.3.

Claim 4. $G^* \notin \{K_1, K_2\} \cup \{K_{2,\ell}, \ell \geq 1\}$.

Proof of Claim 4. By Lemma 4.3(iv), $G^* \notin \{K_1, K_2, K_{1,2}, K_{2,2}\}$. Suppose that $G^* \cong K_{2,\ell}$, for some $\ell \geq 3$. By the definition of G^*, we have $x \geq \min\{2, \ell\} = 2$. This implies that $x = 2$, $y = 0$ and $|S| = s^* = \ell \geq 3$. By Lemma 4.3(i), $n \geq |S| + x + 5(m - x) = |S| + 5m - 4x$. As $|S| \geq 3$, $n \in \{15, 16\}$, $x = 2$ and $m = |S| + t$, we have $16 \geq n \geq 6|S| + 5t - 8 \geq 18 - 5t - 8 = 10 - 5t$, and so $t \leq \frac{6}{5} < 2$. Thus, $t = 1$ and $n = 15$. By Theorem 4.5(iii), G does not have cycles of length at most 4, contrary to the assumption that $G^* \cong K_{2,\ell}$. This justifies Claim 4.

Claim 5. *Each of the following holds.*
(i) $|S| \geq \frac{x+y+5}{2}$.
(ii) $x - y \geq 5t + 15 - n$.

Proof of Claim 5. By Claim 4, $G^* \notin \{K_1, K_2\} \cup \{K_{2,\ell}, \ell \geq 1\}$. By Theorem 2.3(iv), $F(G^*) \geq 3$. As G is reduced, G^* is also reduced, and so by Theorem 2.3(i) and Lemma 4.3(v), $3x + 7y \leq |E(G^*)| \leq 2(x + 3y + |S|) - 5$. Hence (i) must hold.

By Lemma 4.3(iii) and by $m - |S| = t$, we have $\frac{n+4x+2y-|S|}{5} - |S| \geq t$. It follows by Claim 5(i) that

$$\frac{n + 4x + 2y - \frac{x+y+5}{2}}{5} - \frac{x + y + 5}{2} \geq \frac{n + 4x + 2y - |S|}{5} - |S| \geq t$$

which implies $x - y \geq 5t + 15 - n$. Hence (ii) holds as well. This proves Claim 5.

Case 1. $t \geq 1$ when $n = 15$.

By Claim 5(ii) with $n = 15$, $x \geq 5 + y \geq 5$. Assume that $|S| \geq x + 1$. By the choice of S, we have $1 \leq t = m - |S|$, and so $m \geq |S| + 1 \geq x + 2$. By Lemma 4.3(i) and by $|S| \geq x + 1$, $n \geq \sum_{i=1}^{m} |V(G_i)| + |S| \geq x + |V(G_{x+1})| + |V(G_{x+2})| + |S| \geq x + 3 + 3 + (x + 1) \geq 17$, contrary to $n = 15$. Hence $|S| \leq x$. Let $E^+ = \{uv \in E(G) : u \in [\cup_{G_i \in X} V(G_i)], v \in S\}$, and $G^+ = G[E^+]$. By (7) and the definition of G^+, G^+ is a bipartite graph with a vertex bipartition $\{\cup_{G_i \in X} V(G_i), S\}$. Since $\delta(G) \geq 3$, $|E(G^+)| \geq 3x$. Since $n = 15$, by Theorem 4.5, G^+ is C_4-free. Since $|S| \leq x$, by Theorem 4.7,

$$3x \leq |E(G^+)| \leq \sqrt{x \cdot |S|(|S| - 1) + \frac{x^2}{4}} + \frac{x}{2} \leq \sqrt{x^2(x - 1) + \frac{x^2}{4}} + \frac{x}{2}. \tag{10}$$

Solving (10) for x to get $\frac{25x^2}{4} \leq x^3 - x^2 + \frac{x^2}{4}$, and so $x \geq 7$. In particular, when $x = 7$, the equality in (10) holds. Thus, if $x = 7$, $|S| = x = 7$ and so $m = |S| + t = 8$. By Lemma 4.3(i), $|S| \leq 15 - x - \sum_{i=x+1}^{m} |V(G_i)| \leq 15 - 7 - 3 = 5$, contrary to that $|S| = 7$. Hence we must have $x \geq 8$. As $n = 15$ and $x - |S| \leq m - |S| = t = 1$, we have $|S| = 7$ and $x = 8$. By Theorem 4.7, $|E(G^+)| < 23$. As $\delta(G) \geq 3$, $|E(G^+)| \geq 3x = 24$, a contradiction. This proves that Case 1 does not occur.

Case 2. $t \geq 2$ when $n = 16$.

By Claim 5(ii), $x \geq 9 + y$. By Claim 5(i), $|S| \geq 7 + y$. Since $n = 16$, we must have $x = 9 = |V^*|$, $|S| = 7$ and $V(G^*) = V(G)$. As $\delta(G) \geq 3$, we have $|E(G)| \geq |E(G^*)| \geq 3x = 27$. By Theorem 2.3(i) and (iv), $|E(G)| \leq 2|V(G)| - 5 = 27$. Therefore, $|E(G)| = 27$, $F(G) = 3$ and $G^* \cong G$ is a bipartite graph with bipartition V^* and S. By Theorem 4.6, the girth of G is at least 5. By Theorem 4.7, $|E(G)| \leq 24$, contrary to $|E(G)| = 27$. This proves that Case 2 does not occur as well, and completes the proof. □

Theorem 4.9. *Let G be a connected graph with n vertices and $\kappa'(G) \geq 3$, and G' be the reduction of G. If $\alpha'(G) \leq 7$, then G is supereulerian if and only if $G' \notin \mathcal{F}_2 = \{P(10), P(14)\}$.*

Proof. As $P(10)$ and $P(14)$ are not supereulerian, the necessity is clear. By Theorem 2.2, G is supereulerian if and only if G' is supereulerian. By the definition of contractions, we have $\kappa(G') \geq \kappa'(G) \geq 3$ and $\alpha'(G') \leq \alpha'(G) \leq 7$. Hence it suffices to prove that

$$\text{if a reduced graph } G \text{ with } \kappa'(G) \geq 3 \text{ and } \alpha'(G) \leq 7 \text{ is not supereulerian, then } G \in \mathcal{F}_2\,. \tag{11}$$

By Lemma 4.8, (11) holds if $|V(G)| \geq 15$. By Theorem 4.5, (11) holds if $|V(G)| \leq 14$. This completes the proof of the theorem. □

Corollary 4.10. *Let G be a connected graph. If $|V(G)| \leq 15$ and $\kappa'(G) \geq 3$, then G is supereulerian if and only if the reduction of G is not in $\mathcal{F}_2$.*

Proof. If $|V(G)| \leq 15$, then $\alpha'(G) \leq \frac{15}{2}$. So $\alpha'(G) \leq 7$. By Theorem 4.9, this corollary holds. □

Corollary 4.11. *Let G be a connected reduced graph. Each of the following holds.*

(i) *If $|V(G)| \leq 15$ and $\delta(G) \geq 3$, then G is supereulerian if and only if $G \notin \mathcal{F}_2$.*

(ii) *If $\delta(G) \geq 3$ and $\alpha(G) \leq 5$, then G is supereulerian if and only if $G \neq P(10)$.*

Proof. First we prove (i). Suppose that $\kappa'(G) \leq 2$. Let X be a minimal edge cut in G with $|X| \leq 2$. Let G_1 and G_2 be the two components in $G - X$ with $|V(G_1)| \leq |V(G_2)|$. Since $|V(G)| \leq 15$, $|V(G_1)| \leq 7$. Since $|X| \leq 2$ and $\delta(G) \geq 3$, either $|D_1(G_1)| = 0$ and $|D_2(G_1)| \leq 2$ or $|D_1(G_1)| \leq 1$ and $|D_2(G_1)| = 0$. Then $G \notin \{K_1, K_2\} \cup \{K_{2,\ell} : \ell \geq 1\}$ and $|E(G_1)| \geq \frac{4+3(|V(G_1)|-2)}{2}$. By Theorem 2.3, $\frac{4+3(|V(G_1)|-2)}{2} \leq |E(G_1)| \leq 2|V(G_1)| - 5$. Then $|V(G_1)| \geq 8$, contrary to that $|V(G_1)| \leq 7$. Thus, $\kappa'(G) \geq 3$. Statement (i) follows from Corollary 4.10.

Now we prove (ii). If $\alpha(G) \leq 5$, by Theorem 3.3, $|V(G)| \leq 15$. Since $\alpha(P(14)) = 6$, the statement follows from (i) above. □

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