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Minimum degree conditions for the Hamiltonicity of 3-connected claw-free graphs

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ABSTRACT

Settling a conjecture of Kuipers and Veldman posted in Favaron and Fraisse (2001) [9], Lai et al. (2006) [15] proved that if H is a 3-connected claw-free simple graph of order $n \geq 196$, and if $\delta(H) \geq \frac{n+5}{10}$, then either H is Hamiltonian, or the Ryjáček's closure $cl(H) = L(G)$ where G is the graph obtained from the Petersen graph P by adding $\frac{n-15}{10}$ pendant edges at each vertex of P. Recently, Li (2013) [17] improved this result for 3-connected claw-free graphs H with $\delta(H) \geq \frac{n+34}{12}$ and conjectured that similar result would also hold even if $\delta(H) \geq \frac{n+12}{13}$. In this paper, we show that for any given integer $p > 0$ and real number ϵ, there exist an integer $N = N(p, \epsilon) > 0$ and a family $\mathcal{Q}(p)$, which can be generated by a finite number of graphs with order at most $\max\{12, 3p-5\}$ such that for any 3-connected claw-free graph H of order $n > N$ and with $\delta(H) \geq \frac{n+\epsilon}{p}$, H is Hamiltonian if and only if $H \notin \mathcal{Q}(p)$.

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As applications, we improve both results in Lai et al. (2006) [15] and in Li (2013) [17], and give a counterexample to the conjecture in Li (2013) [17].
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1. Introduction

We shall use the notation of Bondy and Murty [1], except when otherwise stated. Graphs considered in this paper are finite and loopless. A graph is called a *multigraph* if it contains multiple edges. A graph without multiple edges is called a *simple graph* or simply a graph. As in [1], $\kappa'(G)$ and $d_G(v)$ (or $d(v)$) denote the edge-connectivity of G and the degree of a vertex v in G, respectively. An edge cut X of a graph G is *essential* if each of the components of $G - X$ contains an edge. A graph G is *essentially k-edge-connected* if G is connected and does not have an essential edge cut of size less than k. An edge $e = uv$ is called a *pendant edge* if either $d(u) = 1$ or $d(v) = 1$. The size of a maximum matching in G is denoted by $\alpha'(G)$. The length of a shortest cycle in G is the *girth* of G. A connected graph G is *Eulerian* if the degree of each vertex in G is even. An Eulerian subgraph Γ in a graph G is called a *spanning Eulerian subgraph* of G if $V(G) = V(\Gamma)$ and is called a *dominating Eulerian subgraph* if $E(G - V(\Gamma)) = \emptyset$. A graph is *supereulerian* if it contains a spanning Eulerian subgraph. The family of supereulerian graphs is denoted by $\mathcal{SL}$. Let $O(G)$ be the set of vertices of odd degree in G. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph Γ_R of G with $O(\Gamma_R) = R$. When $R = \emptyset$, such Γ_R is a spanning Eulerian subgraph. Examples of collapsible graphs include C_2 (a cycle of length 2) and $K_3 = C_3$. But cycles with length at least 4 (C_i with $i \geq 4$) are not collapsible. We use $\mathcal{CL}$ to denote the family of collapsible graphs. Thus, $\mathcal{CL} \subset \mathcal{SL}$. For a graph G, define $D_i(G) = \{v \in V(G) \mid d(v) = i\}$ and define

$$\bar{\sigma}_2(G) = \min\{d(u) + d(v) \mid \text{for every edge } uv \in E(G)\}. \tag{1}$$

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. The following theorem shows a relationship between a graph and its line graph.

Theorem A *(Harary and Nash-Williams [11]). The line graph $H = L(G)$ of a simple graph G with at least three edges is Hamiltonian if and only if G has a dominating Eulerian subgraph.*

A graph H is *claw-free* if H does not contain an induced subgraph isomorphic to $K_{1,3}$. Ryjáček [20] defined the closure $cl(H)$ of a claw-free graph H to be one obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of H as long as this is possible. A graph H is said to be *closed* if $H = cl(H)$.

Theorem B *(Ryjáček [20]). Let H be a claw-free simple graph and $cl(H)$ its closure. Then*

(a) $cl(H)$ is well defined, and $\kappa(cl(H)) \geq \kappa(H)$;
(b) there is a K_3-free simple graph G such that $cl(H) = L(G)$;
(c) both graphs H and $cl(H)$ have the same circumference.

It follows from Theorems A and B that a claw-free simple graph H with its closure $cl(H) = L(G)$ is Hamiltonian if and only if G has a dominating Eulerian subgraph.

Many researches have been done on the minimum degree conditions for claw-free simple graphs to be Hamiltonian (see the surveys [7] and [10]). Matthews and Sumner [19] proved that if H is a 2-connected claw-free simple graph of order n with $\delta(H) \geq \frac{n-2}{3}$, then H is Hamiltonian. Kuipers and Veldman [13] and Favaron et al. [8] proved that if H is a 2-connected claw-free simple graph with sufficiently large order n and $\delta(H) \geq \frac{n+c}{6}$ (where c is a constant), then H is Hamiltonian or is a member of one of the ten well-defined families of graphs. Kovářík et al. [12] proved that if H is a 2-connected claw-free simple graph of order $n \geq 153$ with $\delta(H) \geq \frac{n+39}{8}$, then either H is Hamiltonian or the closure of H is in the five classes of graphs. For 3-connected claw-free simple graphs, the following were proved.

Theorem C *(Li [16]). If H is a 3-connected claw-free simple graph of order n and if $\delta(H) \geq \frac{n+5}{5}$, then H is Hamiltonian.*

Theorem D *(Li, Lu and Liu [18]). If H is a 3-connected claw-free simple graph of order n and if $\delta(H) \geq \frac{n+7}{6}$, then H is Hamiltonian.*

Theorem E *(Kuipers and Veldman [13]). If H is a 3-connected claw-free simple graph with sufficiently large order n, and if $\delta(H) \geq \frac{n+29}{8}$, then H is Hamiltonian.*

Theorem F *(Favaron and Fraisse [9]). If H is a 3-connected claw-free simple graph of order n and if $\delta(H) \geq \frac{n+37}{10}$, then H is Hamiltonian.*

Settling a conjecture of Kuipers and Veldman posted in [9,13], Lai et al. proved:

Theorem G *(Lai et al. [15]). If H is a 3-connected claw-free simple graph of order $n \geq 196$ and if $\delta(H) \geq \frac{n+5}{10}$, then either H is Hamiltonian, or $cl(H) = L(G)$ where G is obtained from the Petersen graph P by adding $\frac{n-15}{10}$ pendant edges at each vertex of P.*

Recently, Li [17] further improved Theorem G and proved the following:

Theorem H *(Li [17]). If H is a 3-connected claw-free simple graph of order $n \geq 363$ and if $\delta(H) \geq (n+34)/12$, then either H is Hamiltonian or $H \in \mathcal{F}_1 \cup \mathcal{F}_2$ (which are defined below).*

Li [17] defined the following families of non-Hamiltonian graphs in which $\overline{K}_{2,t}$ is a connected spanning subgraph of $K_{2,t}$. For each $i \in \{1, 2, 3\}$, let $\mathcal{F}_i = \{H \mid H$ is a 3-connected claw-free graph such that $cl(H) = L(G)$ where G is a graph obtained from the Petersen graph P by replacing exactly $i - 1$ vertices v_j ($j = 1$ to $i - 1$ if $i > 1$) of P with $i - 1$ $W_j = \overline{K}_{2,t}$ $(t \geq 2)$, and each W_j is connected to $P - \cup_{j=1}^{i-1}\{v_j\}$ by the three edges incident with v_j in such a way that G is essentially 3-edge-connected and by adding at least one pendant edge at all other $10 - (i - 1)$ vertices of P and by subdividing m edges of P for $m = 0, 1, 2, \cdots, 15\}$.

In [17], Li claimed that the bound $\delta(H) \geq (n + 34)/12$ in Theorem H can be relaxed to $\delta(H) \geq (n + 6)/12$ and posed the following conjecture:

Conjecture A. *Let H be a 3-connected claw-free simple graph of order $n \geq 483$. If $\delta(H) \geq (n+12)/13$, then either H is Hamiltonian or $H \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.*

The purpose of the current research is to investigate the validity of Conjecture A, to find an approach to unify all the results above, and to seek the best result along this direction.

Let $\mathcal{Q}_0(r)$ be the family of 3-edge-connected nonsupereulerian K_3-free simple graphs of order at most r. For given integer $r \geq p > 0$, let $\mathcal{Q}_p(r)$ denote the graph family of essentially 3-edge-connected graphs such that any graph $G \in \mathcal{Q}_p(r)$ if and only if G is obtained from a member $G_p \in \mathcal{Q}_0(r)$ by replacing t $(1 \leq t \leq p)$ vertices of G_p with nontrivial connected graphs or adding some pendant edges at these t vertices such that G does not have a dominating Eulerian subgraph. Note that the Petersen graph P is the smallest graph in $\mathcal{Q}_0(r)$. In fact, $\mathcal{Q}_0(r) = \{P\}$ if $r \leq 13$ (see Theorem K(a)).

In this paper, using Ryjáček's closure concept [20] and Catlin's reduction method [2], we prove the following:

Theorem 1.1. *Let $p > 0$ be a given integer and let ϵ be a given number. Let $N(p, \epsilon) = 6p^2 + (p+1)|\epsilon|$. Let H be a 3-connected claw-free simple graph of order n. If $n > N(p, \epsilon)$ and $\delta(H) \geq \frac{n+\epsilon}{p}$ then H is Hamiltonian if and only if the closure of H satisfies that $cl(H) = L(G)$, where $G \notin \mathcal{Q}_p(r)$ with $r \leq \max\{12, 3p - 5\}$.*

Using structural properties of the closure concept, Favaron et al. in [8] show a method for characterizing non-Hamiltonian graphs H with $\delta(H) > \frac{n+k^2-4k+7}{k}$ for any integer $k \geq 4$.

Theorem I *(Favaron et al. [8]). Let $k \geq 4$ be an integer. Let H be a 2-connected claw-free simple graph of order n such that $n \geq 3k^2 - 4k - 7$ and $\delta(H) > \frac{n+k^2-4k+7}{k}$. Then either H is Hamiltonian or the closure $cl(H)$ can be covered by at most $k - 1$ cliques.*

Theorem 1.1 and Theorem I share some similarity. However, the structural information for the non-Hamiltonian exceptional graphs is different. Thus, Theorem 1.1 and Theorem I are independent.

Theorems C, D, E, F, G and H are the special cases of Theorem 1.1 with $(p, \epsilon) \in \{(5, 5), (6, 7), (8, 29), (10, 37), (10, 5), (12, 34)\}$. Using Theorem 1.1, we will show an improvement of Theorems G and H as a special case of Theorem 1.1 with $p = 13$ and $\epsilon = 6$ and give a counterexample to Conjecture A. The same example also shows that the bound $\delta(H) \geq (n + 34)/12$ in Theorem H can not be relaxed to $\delta(H) \geq (n + 6)/12$.

As Kuratowski [14] and Wagner [22] characterized planar graphs in terms of graphs without a subgraph contractible to K_5 or $K_{3,3}$, the graphs K_5 or $K_{3,3}$ have been considered as the only obstacles for a graph to be planar. Theorem 1.1 shows that in some sense, with given p and ϵ, for 3-connected claw-free simple graphs H of order n with $\delta(H) \geq (n + \epsilon)/p$, there are only a finite number of obstacles for such graphs to be Hamiltonian.

2. Catlin's reduction method

Let G be a connected multigraph. For $X \subseteq E(G)$, the *contraction* G/X is the multigraph obtained from G by identifying the two ends of each edge $e \in X$ and deleting the resulting loops. Note that even if G is a simple graph, multiple edges may arise by the identification. But we do not replace multiple edges into a single edge. If Γ is a connected subgraph of G, then we write G/Γ for $G/E(\Gamma)$ and use v_Γ for the vertex in G/Γ to which Γ is contracted, and v_Γ is called a *contracted vertex* if $\Gamma \neq K_1$.

Let G and G_T be two connected graphs. We say that G is contractible to G_T if G_T is a graph obtained from G by successively contracting a collection of pairwise vertex disjoint connected subgraphs, and call G_T the *contraction graph* of G. For a vertex $v \in V(G_T)$, there is a connected subgraph $G(v)$ in G such that v is obtained by contracting $G(v)$. We call $G(v)$ the preimage of v in G and call v the contraction image of $G(v)$ in G_T.

Catlin [2] showed that every multigraph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_c$ such that $V(G) = \cup_{i=1}^{c} V(\Gamma_i)$. The *reduction* of G is the graph obtained from G by contracting each Γ_i into a single vertex v_i $(1 \leq i \leq c)$ and is denoted by G'. Thus, the reduction of G is a special type of contraction graph of G. For a vertex $v \in V(G')$, there is a unique maximal collapsible subgraph $\Gamma(v)$ in G such that v is the contraction image of $\Gamma(v)$ and $\Gamma(v)$ is the *preimage* of v. A vertex $v \in V(G')$ is a *contracted* vertex if $\Gamma(v) \neq K_1$. A graph is *reduced* if $G = G'$. By definition, K_1 is a collapsible and supereulerian graph. As K_1 does not have any edge cuts, we define $\kappa'(K_1) = \infty$. By the definition of contraction, we have $\kappa'(G') \geq \kappa'(G)$.

Note that although multiple edges may arise by contracting an edge, contracting a maximal collapsible subgraph will not generate multiple edges. In particular, the girth of the reduction of a multigraph is always larger than 3 (see Theorem J(c) below).

Throughout this paper, we use P to denote the Petersen graph and use P_{14}, $K_{1,3}(1, 1, 1)$ and $J'(1, 1)$ to denote the graphs depicted in Fig. 2.1, respectively.

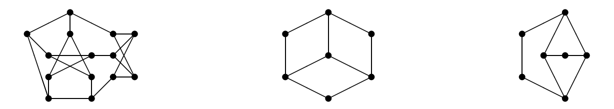

Fig. 2.1. P_{14} and the two reduced graphs G of order 7 with $|D_2(G)| = 3$.

Theorem J *(Catlin [2] and Catlin et al. [3]). Let G be a connected multigraph and let G' be the reduction of G. Then each of the following holds:*

(a) $G \in \mathcal{CL}$ if and only if $G' = K_1$. In particular, $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$.
(b) G has a dominating Eulerian subgraph if and only if G' has a dominating Eulerian subgraph containing all the contracted vertices of G'.
(c) If G is a reduced graph, then G is simple, K_3-free with $\delta(G) \le 3$, and either $G \in \{K_1, K_2, K_{2,t}(t \ge 2)\}$ or $|E(G)| \le 2|V(G)| - 5$.

Theorem K. *Let G be a 3-edge-connected multigraph of order n. Let G' be the reduction of G. Then each of the following holds:*

(a) (Chen [4]) If $n \le 14$, then either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$.
(b) (Chen [4]) If $n = 15$, $G \notin \mathcal{SL}$ and $G' \notin \{P, P_{14}\}$, then G is 2-connected, 3-edge-connected and essentially 4-edge-connected reduced graph with girth at least 5 and $V(G) = D_3(G) \cup D_4(G)$ where $|D_4(G)| = 3$ and $D_4(G)$ is an independent set.
(c) (Chen et al. [6] or Zhang [23]) Let $\alpha'(G')$ be the size of a maximum matching in G'. Then $\alpha'(G') \ge \min\{\frac{|V(G')|-1}{2}, \frac{|V(G')|+5}{3}\}$.
(d) (Chen [4]) If $G \ne K_1$ is a 2-edge-connected reduced graph of order $n \le 7$, then $|D_2(G)| \ge 3$; and if $|D_2(G)| = 3$, then $G \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\}$.

Throughout the rest of this section, G denotes an essentially 3-edge-connected simple graph. Then $D_1(G) \cup D_2(G)$ is an independent set. Let E_1 be the set of pendant edges in G. For each $x \in D_2(G)$, there are two edges e_x^1 and e_x^2 incident with x. Let $X_2(G) = \{e_x^1 \mid x \in D_2(G)\}$. Thus $|X_2(G)| = |D_2(G)|$. Define

$$G_0 = G/(E_1 \cup X_2(G)) = (G - D_1(G))/X_2(G).$$

Note that even if G is a K_3-free simple graph, G_0 may contain multiple edges. But by Theorem J(c), its reduction G_0' is a K_3-free simple graph.

Following [21], the graph G_0 is called the **core** of G. A vertex in G_0 is G_0-nontrivial if it is adjacent to a vertex in $D_1(G) \cup D_2(G)$ in G. Thus, if $x \in D_2(G)$ and $N_G(x) = \{u, v\}$, and if u_x is a vertex in G_0 obtained by contracting the edge xu, then both u_x and v are G_0-nontrivial (although u_x is a contracted vertex and v is not a contracted vertex in G_0).

Let G_0' be the reduction of G_0. For a vertex $v \in V(G_0')$, let $\Gamma_0(v)$ be the preimage of v in G_0 and let $\Gamma(v)$ be the preimage of v in G. Then $\Gamma(v)$ is the subgraph formed by $\Gamma_0(v)$ and some edges in $E_1 \cup X_2(G)$. A vertex $v \in V(G_0')$ is *nontrivial* in G_0' (or G_0'*-nontrivial*) if $|V(\Gamma(v))| > 1$ or $\Gamma(v) = K_1$ and v is adjacent to a vertex in $D_2(G)$. Thus, a nontrivial vertex in G_0' is either a contracted vertex or adjacent to a vertex in $D_2(G)$ in G.

The following theorem will be needed.

Theorem L *(Shao, Section 1.4 of [21]). Let G be an essentially 3-edge-connected simple graph and $L(G)$ is not complete. Let G_0 be the core of graph G, and let G_0' be the reduction of G_0, then each of the following holds:*

(a) G_0 is well defined, nontrivial and $\delta(G_0) = \kappa'(G_0) \geq 3$ and so $\kappa'(G_0') \geq \kappa'(G_0) \geq 3$.
(b) $L(G)$ is Hamiltonian if and only if G_0' has a dominating Eulerian subgraph containing all the nontrivial vertices in G_0'.

Note that Theorem L(b) would not hold if "the nontrivial vertices" of G_0' is replaced by "contracted vertices". For instance, let G be the graph obtained from the Petersen graph P by subdividing five edges of a perfect matching of P. Then G does not have a dominating Eulerian subgraph and $G_0' = G_0 = P$; all 10 vertices in G_0' are nontrivial vertices and 5 of them are contracted vertices. Obviously, G_0' does not have a dominating Eulerian subgraph containing all the G_0'-nontrivial vertices but has a dominating Eulerian subgraph containing the 5 contracted vertices.

Let G_T be a contraction graph of G. Let v be a vertex in G_T and let $G(v)$ be the preimage of v in G. For a vertex x in $V(G(v))$, let $i(x)$ be the number of edges in $E(G_T)$ that are incident with x in G. Then for any $x \in V(G(v))$,

$$d_G(x) = i(x) + |N_{G(v)}(x)| \quad \text{and} \quad i(x) \leq \sum_{w \in V(G(v))} i(w) = d_{G_T}(v). \tag{2}$$

When $G_T = G_0'$, $G(v) = \Gamma(v)$ and $i(x) \leq \sum_{w \in V(\Gamma(v))} i(w) = d_{G_0'}(v)$.

Lemma 2.1. *Let G be a connected K_3-free simple graph. Let G_T be a contraction graph of G. For a vertex $v \in V(G_T)$, let $G(v)$ be the preimage of v in G. If $G(v) \neq K_1$, then $|V(G(v))| \geq \bar{\sigma}_2(G) - d_{G_T}(v)$.*

Proof. Since $G(v)$ is a nontrivial connected graph, $G(v)$ has an edge xy. As a subgraph of G, $G(v)$ is a K_3-free simple graph. Then $N_{G(v)}(x) \cap N_{G(v)}(y) = \emptyset$ and

$$|N_{G(v)}(x)| + |N_{G(v)}(y)| = |N_{G(v)}(x) \cup N_{G(v)}(y)| \leq |V(G(v))|. \tag{3}$$

By (1), (2) and (3),

$$\begin{aligned}\bar{\sigma}_2(G) &\le d_G(x) + d_G(y) = i(x) + |N_{G(v)}(x)| + i(y) + |N_{G(v)}(y)| \\ &\le (i(x) + i(y)) + |V(G(v))| \le d_{G_T}(v) + |V(G(v))|.\end{aligned}$$

Thus, $|V(G(v))| \ge \bar{\sigma}_2(G) - d_{G_T}(v)$. $\square$

3. An associated theorem

Let G be a simple graph and let $H = L(G)$ be the line graph of G. By the definition of a line graph, if $H = L(G)$ is k-connected and is not complete, then G is essentially k-edge-connected, and that $|V(H)| = |E(G)|$ and $\delta(H) = \min\{d_G(x) + d_G(y) - 2 \mid xy \in E(G)\}$. Thus, $\delta(H) = \bar{\sigma}_2(G) - 2$.

Throughout the rest of this paper, we always assume that $p > 0$ is an integer and ϵ is a real number. We prove the following theorem first.

Theorem 3.1. *Let G be an essentially 3-edge-connected simple graph with size $n = |E(G)|$ and*

$$\bar{\sigma}_2(G) \ge \frac{n+\epsilon}{p} + 2. \tag{4}$$

Let G_0 be the core of G and let G_0' be the reduction of G_0. If $n > 4p^2 - 11p - p\epsilon - \epsilon$, then exactly one of the following holds.

(a) $G_0 \in \mathcal{CL}$;
(b) G_0' is not in $\mathcal{CL}$ with $|V(G_0')| \le \max\{12, 3p-5\}$ and $\kappa'(G_0') \ge 3$.

Proof. It is clear that (a) and (b) are mutually exclusive. Suppose $G_0 \notin \mathcal{CL}$. By Theorem J(a), $G_0' \ne K_1$. Let $c = |V(G_0')|$. By Theorem L(a), $\kappa'(G_0') \ge \kappa'(G_0) \ge 3$. By Theorem J(c), G_0' is a K_3-free simple graph and

$$|E(G_0')| \le 2|V(G_0')| - 5 = 2c - 5. \tag{5}$$

Let $M(G_0')$ be a maximum matching in G_0' and let $m = |M(G_0')|$. If $c \le 12$, then Theorem 3.1(b) holds. Thus, in the following, we assume that $c \ge 13$. Then $\frac{c-1}{2} \ge \frac{c+5}{3}$. By Theorem K(c), $m \ge \frac{c+5}{3}$ and so

$$c \le 3m - 5. \tag{6}$$

Let $M(G_0') = \{u_1'v_1', u_2'v_2', \cdots, u_m'v_m'\}$. Let $S' = \{u_1', u_2', \cdots, u_m', v_1', v_2', \cdots, v_m'\}$. Then $S' \subseteq V(G_0')$. Let $\Gamma(u_i')$ and $\Gamma(v_i')$ be the preimages of u_i' and v_i' in G, respectively. Since each edge $e = u'v'$ in $E(G_0')$ is an edge in $E(G)$, there is a vertex u in $\Gamma(u')$ and there is a vertex v in $\Gamma(v')$ such that $e = uv$ is an edge in $E(G)$. Thus, G has a matching

$M(G) = \{u_1v_1, u_2v_2, \cdots, u_mv_m\}$ corresponding to $M(G'_0)$ in G'_0. For each $v' \in S'$, let v be the corresponding vertex in $V(\Gamma(v'))$. Then

$$d_G(v) \le i(v) + |V(\Gamma(v'))| - 1. \tag{7}$$

Since $\Gamma(v')$ is a connected graph, $|E(\Gamma(v'))| \ge |V(\Gamma(v'))| - 1$. By (7), for each $u_jv_j \in M(G)$ with the corresponding edge $u'_jv'_j \in M(G'_0)$ $(1 \le j \le m)$,

$$\begin{aligned} d_G(u_j) &\le i(u_j) + |V(\Gamma(u'_j))| - 1 \le i(u_j) + |E(\Gamma(u'_j))|; \\ d_G(v_j) &\le i(v_j) + |V(\Gamma(v'_j))| - 1 \le i(v_j) + |E(\Gamma(v'_j))|. \end{aligned} \tag{8}$$

Then by (8),

$$\sum_{j=1}^{m}(d_G(u_j) + d_G(v_j)) \le \sum_{j=1}^{m}(i(u_j) + i(v_j)) + \sum_{j=1}^{m}(|E(\Gamma(u'_j))| + E(\Gamma(v'_j))|). \tag{9}$$

By (2), $i(v) \le d_{G'_0}(v')$ and so

$$\sum_{j=1}^{m}(i(u_j) + i(v_j)) \le \sum_{v' \in S'} d_{G'_0}(v') \le \sum_{v' \in V(G'_0)} d_{G'_0}(v') = 2|E(G'_0)|. \tag{10}$$

Since $E(G) = (\bigcup_{j=1}^{m} E(\Gamma(u'_j))) \cup (\bigcup_{j=1}^{m} E(\Gamma(v'_j))) \cup E(G'_0) \cup (\bigcup_{v' \in V(G'_0) - S'} E(\Gamma(v')))$

$$|E(G)| \ge \sum_{j=1}^{m} |E(\Gamma(u'_j))| + \sum_{j=1}^{m} |E(\Gamma(v'_j))| + |E(G'_0)|. \tag{11}$$

By (9), (10) and (11),

$$\sum_{j=1}^{m}(d_G(u_j) + d_G(v_j)) \le |E(G'_0)| + |E(G)|. \tag{12}$$

By (4), (5), (6), (12) and $n = |E(G)|$,

$$m(\frac{n+\epsilon}{p} + 2) \le m\bar{\sigma}_2(G) \le \sum_{j=1}^{m}(d_G(u_j) + d_G(v_j)) \le (2c-5) + n \le 2(3m-5) - 5 + n,$$

which yields

$$m \le \frac{(n-15)p}{n+\epsilon-4p} = p + \frac{(4p-\epsilon-15)p}{n+\epsilon-4p}.$$

Since m is an integer, when $n > 4p^2 - 11p - \epsilon p - \epsilon$, $m \le p$. By (6), $|V(G'_0)| = c \le 3p-5$. □

4. Proof of Theorem 1.1

To deal with claw-free graphs $H = L(G)$, by Theorem B, we only need to focus on the properties of K_3-free simple graphs G with $\bar{\sigma}_2(G) \geq \frac{n+\epsilon}{p} + 2$. Note that even when G is a K_3-free simple graph, its core G_0 may contain multiple edges. But by Theorem J(c), the reduction G_0' of G_0 is simple and K_3-free. For convenience, we define the following:

- $S_0 = \{v \in V(G_0') \mid v \text{ is a nontrivial vertex in } G_0'\}$;
- $S_1 = \{v \in V(G_0') \mid v \text{ is a contracted vertex in } G_0'\}$;
- $Y = V(G_0') - S_1$.

By definition, S_1 is a subset of S_0. However, for the graphs satisfying the assumptions of Theorem 3.1, $S_1 = S_0$ as shown in Proposition 4.1(b) below.

Proposition 4.1. *Let G be an essentially 3-edge-connected K_3-free simple graph with size $n = |E(G)|$ and with $\bar{\sigma}_2(G) \geq \frac{n+\epsilon}{p} + 2$, where $p > 0$ is a given integer and ϵ is a given number. Suppose that G_0, the core of G, is not in $\mathcal{CL}$. Let G_0' be the reduction of G_0. Let S_0, S_1 and Y be the sets defined above. If $n > N(p, \epsilon) = 6p^2 + (p + 1)|\epsilon|$, then each of the following holds:*

(a) For each $v \in S_1$, let $\Gamma(v)$ be the preimage of v in G. Then $|V(\Gamma(v))| \geq \bar{\sigma}_2(G) - d_{G_0'}(v)$.
(b) $S_1 = S_0$.
(c) Y is an independent set and $N_{G_0'}(y) \subseteq S_1$ for any $y \in Y$.
(d) $|S_0| \leq p$. Furthermore, if $|S_0| = p$, then
(di) $|E(G_0')| \geq \epsilon + p + \sum_{y \in Y} d_{G_0'}(y) \geq \epsilon + p + 3|Y|$;
(dii) $|V(G_0')| \leq 2p - 5 - \epsilon$.

Proof. Let $c = |V(G_0')|$. Since $G_0 \notin \mathcal{CL}$, $c > 1$. As the assumptions of Proposition 4.1 imply the assumptions of Theorem 3.1, it follows from Theorem 3.1 that $1 < c \leq \max\{12, 3p - 5\}$.

Let v be a vertex in $V(G_0')$, let $\Gamma_0(v)$ be the preimage of v in G_0, and let $\Gamma(v)$ be the preimage of v in G. By Theorem J(c), $\kappa'(G_0') \geq 3$ and G_0' is simple and K_3-free, and so

$$d_{G_0'}(v) \leq |V(G_0')| - 2 \leq \max\{10, 3p - 7\}. \tag{13}$$

(a) For each $v \in S_1$, v is a contracted vertex in G_0' and so $\Gamma(v)$ is nontrivial. It follows from the fact that G_0' satisfies Lemma 2.1 (where $G_T = G_0'$ and $G(v) = \Gamma(v)$).

(b) By way of contradiction, suppose that $S_1 \neq S_0$. Let v be a vertex in $S_0 - S_1$. Then v is not a contracted vertex and v is nontrivial. Thus, $d_G(v) = d_{G_0'}(v)$ and v is adjacent to a vertex $u \in D_2(G)$. By (1) and (13),

$$\frac{n+\epsilon}{p} + 2 \leq \bar{\sigma}_2(G) \leq d_G(v) + d_G(u) = d_{G_0'}(v) + 2 \leq |V(G_0')| \leq \max\{12, 3p - 5\},$$

contrary to the fact that $n > N(p,\epsilon) \geq \max\{10p-\epsilon, 3p^2-7p-\epsilon\}$ and so (b) is proved.

(c) To the contrary, suppose that there are two vertices y_1 and y_2 in Y such that $y_1y_2 \in E(G_0')$. Since $y_i \in Y = V(G_0') - S_1$ $(i=1,2)$, $d_G(y_i) = d_{G_0'}(y_i)$. By (1), (4) and (13),

$$\frac{n+\epsilon}{p} + 2 \leq \bar{\sigma}_2(G) \leq d_G(y_1) + d_G(y_2) \leq 2\max\{10, 3p-7\},$$

contrary to the fact that $n > N(p,\epsilon) \geq \max\{18p-\epsilon, (6p-16)p-\epsilon\}$, and so (c) is proved.

(d) By way of contradiction, suppose that $s = |S_0| > p$. Let $S_0 = \{v_1, v_2, \cdots, v_s\}$. By (b) above, $S_0 = S_1$. Then by (a), $|E(\Gamma(v_j))| \geq |V(\Gamma(v_j))| - 1 \geq \bar{\sigma}_2(G) - d_{G_0'}(v_j) - 1$. Since $\left(\bigcup_{j=1}^{s} E(\Gamma(v_j))\right) \cup E(G_0') \subseteq E(G)$ and $n = |E(G)|$,

$$|E(G)| \geq \sum_{j=1}^{s} |E(\Gamma(v_j))| + |E(G_0')| \geq \sum_{j=1}^{s} (\bar{\sigma}_2(G) - d_{G_0'}(v_j) - 1) + |E(G_0')|;$$

$$n \geq s\bar{\sigma}_2(G) - \sum_{j=1}^{s} d_{G_0'}(v_j) - s + |E(G_0')|. \tag{14}$$

Since $\sum_{j=1}^{s} d_{G_0'}(v_j) \leq \sum_{v\in V(G_0')} d_{G_0'}(v) = 2|E(G_0')|$, by (14), we have

$$n \geq s\bar{\sigma}_2(G) - |E(G_0')| - s \geq s(\frac{n+\epsilon}{p} + 2) - |E(G_0')| - s;$$

$$n + |E(G_0')| - s \geq \frac{s(n+\epsilon)}{p}. \tag{15}$$

By Theorem J(c) and $c \leq \max\{12, 3p-5\}$, $|E(G_0')| \leq 2c-5 \leq \max\{19, 6p-15\}$. Since $s \geq p+1$, it follows by (15) that

$$n + \max\{19, 6p-15\} - (p+1) \geq n + |E(G_0')| - s \geq \frac{s(n+\epsilon)}{p};$$

$$\max\{18p-p^2, (5p-16)p\} - (p+1)\epsilon \geq (s-p)n \geq n,$$

contrary to the fact that $n > N(p,\epsilon) \geq \max\{18p-p^2, (5p-16)p\} - (p+1)\epsilon$. Thus, $|S_0| \leq p$.

Suppose that $|S_0| = p$. By putting $s = p$ and $\bar{\sigma}_2(G) \geq \frac{n+\epsilon}{p} + 2$ in (14), we obtain

$$n \geq p(\frac{n+\epsilon}{p} + 2) - \sum_{j=1}^{p} d_{G_0'}(v_j) - p + |E(G_0')|,$$

which yields

$$\sum_{j=1}^{p} d_{G'_0}(v_j) \geq \epsilon + p + |E(G'_0)|. \tag{16}$$

Since $2|E(G'_0)| = \sum_{v\in V(G'_0)} d_{G'_0}(v) = \sum_{v\in S_0} d_{G'_0}(v) + \sum_{y\in Y} d_{G'_0}(y)$,

$$\sum_{j=1}^{p} d_{G'_0}(v_j) = \sum_{v\in S_0} d_{G'_0}(v) = 2|E(G'_0)| - \sum_{y\in Y} d_{G'_0}(y). \tag{17}$$

By (16), (17) and $d_{G'_0}(y) \geq \delta(G'_0) \geq 3$ for any $y \in Y$, we have

$$2|E(G'_0)| - \sum_{y\in Y} d_{G'_0}(y) \geq \epsilon + p + |E(G'_0)|;$$

$$|E(G'_0)| \geq \epsilon + p + \sum_{y\in Y} d_{G'_0}(y) \geq \epsilon + p + 3|Y|. \tag{18}$$

The proof of (di) is completed.

Since $d_{G'_0}(y) \geq \delta(G'_0) \geq 3$, by (18) and $|E(G'_0)| \leq 2|V(G'_0)| - 5$,

$$|Y| \leq \frac{|E(G'_0)| - \epsilon - p}{3} \leq \frac{2|V(G'_0)| - 5 - \epsilon - p}{3}. \tag{19}$$

Since $|V(G'_0)| = |S_0| + |Y|$ and $|S_0| = p$, by (19)

$$|V(G'_0)| = p + |Y| \leq p + \frac{2|V(G'_0)| - 5 - \epsilon - p}{3}.$$

Solving the inequality above, we have $|V(G'_0)| \leq 2p - 5 - \epsilon$. The proof is completed. $\square$

Proof of Theorem 1.1. By Theorem B, graph H is Hamiltonian if and only if its closure $cl(H)$ is Hamiltonian. Since $|V(H)| = |V(cl(H))|$, $\delta(cl(H)) \geq \delta(H)$ and $\kappa(cl(H)) \geq \kappa(H) \geq 3$, the graph $cl(H)$ satisfies the same hypotheses as H. Thus, we may assume that H is closed claw-free graph. By Theorem B, there is a K_3-free simple graph G such that $H = cl(H) = L(G)$. Since H is 3-connected with $\delta(H) \geq \frac{n+\epsilon}{p}$, by the definition of line graphs and by (2), G is essentially 3-edge-connected with size $|E(G)| = n = |V(H)|$ and $\bar{\sigma}_2(G) \geq \frac{n+\epsilon}{p} + 2$. Let G_0 be the core of G, i.e., $G_0 = G/(E_1 \cup X_2(G))$.

If $G'_0 = K_1$, then by Theorem J(a), G_0 has a spanning Eulerian subgraph. By Theorem L(b), $cl(H) = L(G)$ is Hamiltonian. Theorem 1.1 is proved for this case.

Next, suppose that $L(G)$ is not Hamiltonian. Then $G'_0 \neq K_1$. By Theorem L(b), G_0 does not have a dominating Eulerian subgraph containing all the nontrivial vertices. In particular, G_0 is not supereulerian. Hence, G_0 is not collapsible. By Theorem 3.1, G'_0 has order at most $\max\{12, 3p - 5\}$ and $\kappa'(G'_0) \geq 3$. Moreover, G'_0 has at most p contracted vertices by Proposition 4.1. Thus G'_0 is the graph G_p stated in Theorem 1.1. $\square$

Remark. One can check from the proofs above that Theorem 1.1 and Proposition 4.1 are still valid if the expression $N(p, \epsilon) = 6p^2 + (p + 1)|\epsilon|$ is replaced by

$$N(p, \epsilon) = \max\{4p^2 - 11p - p\epsilon - \epsilon, 3p^2 - 7p - \epsilon, 18p - \epsilon, 6p^2 - 16p - \epsilon,$$
$$18p - p^2 - p\epsilon - \epsilon, 5p^2 - 16p - p\epsilon - \epsilon\}.$$

However, even with this new expression, it may not be best possible. We did not make efforts to find best possible lower bound for n to avoid tedious case by case analysis.

5. Applications of Theorem 1.1

With Theorem 1.1, many prior results for Hamiltonicity of 3-connected claw-free simple graphs involved minimum degrees can be improved. In the following, we first prove a theorem for the case $p = 13$ and $\epsilon = 6$ of Theorem 1.1. The best possible minimum degree conditions for Theorems G and H are given in Corollary 5.2. With the proofs of these results, we construct some graphs as depicted in Fig. 5.2 to show that these are best possible results. One of the graphs (Fig. 5.2(b)) also shows that Conjecture A is false.

We will need the following theorem:

Theorem M *(Chen et al. [5]). Let G be a 3-edge-connected graph and let $S \subseteq V(G)$ be a vertex subset such that $|S| \leq 12$. Then either G has an Eulerian subgraph C such that $S \subseteq V(C)$, or G can be contracted to the Petersen graph P in such a way that the preimage of each vertex of P contains at least one vertex in S.*

Let Φ_1 and Φ_2 be two graphs with $|D_2(\Phi_1)| = |D_2(\Phi_2)| = 3$. Define $P(\Phi_1)$ to be a graph obtained from the Petersen graph P by replacing a vertex v of P by Φ_1 in the way that the three edges incident with v in P are incident with the three vertices in $D_2(\Phi_1)$, respectively. For instance, $P(K_{2,3}) = P_{14}$ (see Fig. 2.1). Define $P(\Phi_1, \Phi_2)$ to be the graph obtained from P by replacing two nonadjacent vertices v_i of P $(i = 1, 2)$ by Φ_i in the way that the three edges incident with v_i are incident with the three vertices in $D_2(\Phi_i)$, respectively. (See Fig. 5.1.)

Theorem 5.1. *Let H be a 3-connected claw-free simple graph of order n. If $\delta(H) \geq \frac{n+6}{13}$ and n is sufficiently large, then either H is Hamiltonian or $cl(H) = L(G)$ where G*

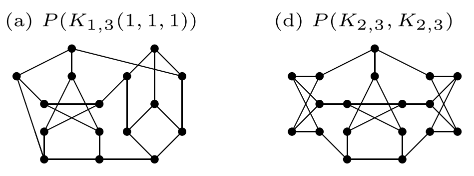

Fig. 5.1. Graphs of $P(K_{1,3}(1, 1, 1))$ and $P(K_{2,3}, K_{2,3})$.

is an essentially 3-edge-connected K_3-free simple graph with size $|E(G)| = n$ and G can be contracted to the Petersen graph P in such a way that each vertex v of P is obtained by contracting a nontrivial connected subgraph $G(v)$ with order $|V(G(v))| \geq \frac{n-7}{13}$ and size $s_v = |E(G(v))| \geq \frac{n-20}{13}$ and that $n = 15 + \sum_{v\in V(P)} s_v$. Furthermore, $G_0' \in \{P, P(K_{2,3}), P(K_{1,3}(1,1,1)), P(K_{2,3}, K_{2,3})\}$.

Proof. This is the special case of Theorem 1.1 with $p = 13$ and $\epsilon = 6$. We may assume that H is non-Hamiltonian. Let $cl(H) = L(G)$. By Theorem B, G is an essentially 3-edge-connected K_3-free simple graph with size $|E(G)| = n$ and $\bar{\sigma}_2(G) = \delta(cl(H))+2 \geq \delta(H) + 2 \geq \frac{n+32}{13}$. Let G_0 be the core of G and let G_0' be the reduction of G_0. By Theorem L, $\kappa'(G_0') \geq \kappa'(G_0) \geq 3$.

Let S_0, S_1 and Y be the subsets of $V(G_0')$ defined in Section 4. By Proposition 4.1, $S_0 = S_1$, $|S_0| = |S_1| \leq p = 13$ and for each $y \in Y$, $N_{G_0'}(y) \subseteq S_1$. If G_0' has an Eulerian subgraph Φ_0 containing S_0, then Φ_0 is a dominating Eulerian subgraph containing all the nontrivial vertices of G_0'. Then by Theorem L(b), $L(G)$ is Hamiltonian, a contradiction. Thus, we assume that G_0' does not have any Eulerian subgraph containing S_0 (and S_1).

Claim 1. $|S_1| \leq 12$.

To the contrary, suppose that $|S_1| \geq 13$. Then $|S_1| = 13$. By Proposition 4.1(d) with $p = 13$ and $\epsilon = 6$, $13 \leq |V(G_0')| \leq 2p - 5 - \epsilon = 15$ and

$$|E(G_0')| \geq 19 + 3|Y|. \tag{20}$$

If $13 \leq |V(G_0')| \leq 14$, then by Theorem K(a), $G_0' = P_{14}$. Then $|Y| = 1$ and $|E(G_0')| = |E(P_{14})| = 21$, contrary to (20) that $|E(G_0')| \geq 19 + 3|Y| = 22$.

If $|V(G_0')| = 15$, then $|Y| = 2$. By Theorem K(b) $V(G_0') = D_3(G_0') \cup D_4(G_0')$ with $|D_4(G_0')| = 3$, and so $|E(G_0')| = 24$, contrary to (20) again that $|E(G_0')| \geq 19+3|Y| = 25$. Thus, $|S_0| = 13$ is impossible. The claim is proved.

By Theorem M and $|S_1| \leq 12$, since G_0' has no Eulerian subgraph containing S_1, G_0' can be contracted to the Petersen graph P in such a way that the preimage of each vertex of P contains at least one vertex in S_1.

For each vertex $v \in V(P)$, let $G_0(v)$ be the subgraph of G_0' that is the preimage of v in G_0'. Then $V(G_0(v)) \cap S_1 \neq \emptyset$. Let $G(v)$ be the preimage of v in G. Since each vertex in S_1 is a contracted vertex, $G(v) \neq K_1$ for each $v \in V(P)$.

By Lemma 2.1 with $G_T = P$ and $d_P(v) = 3$, $|V(G(v))| \geq \bar{\sigma}_2(G) - d_P(v) \geq \frac{n-7}{13}$. Since $G(v)$ is connected, $s_v = |E(G(v))| \geq |V(G(v))| - 1 \geq \frac{n-7}{13} - 1 = \frac{n-20}{13}$. Since $E(G) = E(P) \cup \left(\bigcup_{v\in V(P)} E(G(v))\right)$, $n = |E(G)| = 15 + \sum_{v\in V(P)} |E(G(v))| = 15 + \sum_{v\in V(P)} s_v$.

Next we will show that $G_0' \in \{P, P(K_{2,3}), P(K_{1,3}(1,1,1)), P(K_{2,3}, K_{2,3})\}$.

If for any $v \in V(P)$, $G_0(v) = K_1$, then $G_0' = P$. We are done.

In the following, we assume that there is a vertex $v \in V(P)$ such that $G_0(v) \neq K_1$.

Since P is a 3-regular graph, only three edges outside of $G_0(v)$ are incident with some vertices in $G_0(v)$. For each $v \in V(P)$, let $S_1^v = V(G_0(v)) \cap S_1$ and let $Y_v = V(G_0(v)) - S_1^v$. Then $|V(G_0(v))| = |S_1^v| + |Y_v|$.

If $G_0(v) \neq K_1$, then since $\kappa'(G_0') \geq 3$, $G_0(v)$ has the following properties:

(i) $G_0(v)$ is 2-edge-connected and so $d_{G_0(v)}(x) \geq 2$ for any $x \in V(G_0(v))$.
(ii) $|D_2(G_0(v))| \leq 3$, i.e., the number of vertices of degree 2 in $G_0(v)$ is at most 3.
(iii) $|S_1^v| = |V(G_0(v)) \cap S_1| \leq 3$ since $|S_1| \leq 12$ and $V(G_0(v)) \cap S_1 \neq \emptyset$ for any $v \in V(P)$.
(iv) Y_v is an independent set, by Proposition 4.1(c) and since $Y_v \subseteq Y$.

Claim 2. *For any $v \in V(P)$, $|Y_v| \leq 4$ and $|V(G_0(v))| \leq 7$.*

If $G_0(v) = K_{2,t}$, then by (ii) above, $t = |D_2(G_0(v))| \leq 3$. Since Y_v is an independent set in this $K_{2,t}$, we have $|Y_v| \leq t \leq 3$, and so $|V(G_0(v))| = 5$. We are done for this case.

Next, we assume that $G_0(v) \neq K_{2,t}$.

Note that Y_v is an independent set and $E(G_0(v))$ contains edges joining Y_v and S_1^v. It follows by $|D_2(G_0(v))| \leq 3$ that

$$\begin{aligned}|E(G_0(v))| &\geq 2|D_2(G_0(v)) \cap Y_v| + 3(|Y_v| - |D_2(G_0(v)) \cap Y_v|)\\ &\geq 3|Y_v| - |D_2(G_0(v))| \geq 3|Y_v| - 3.\end{aligned}$$

By Theorem J(c) and $G_0(v) \notin \{K_1, K_2, K_{2,t}\}$, $|E(G_0(v))| \leq 2|V(G_0(v))| - 5$. We have

$$3|Y_v| - 3 \leq |E(G_0(v))| \leq 2|V(G_0(v))| - 5 = 2|Y_v| + 2|S_1^v| - 5;$$

which yields $|Y_v| \leq 2|S_1^v| - 2$. Since $|S_1^v| \leq 3$, $|Y_v| \leq 4$, and so $|V(G_0(v))| = |S_1^v| + |Y_v| \leq 7$. Claim 2 is proved.

Since $G_0(v)$ is 2-edge-connected reduced graph of order at most 7, by Theorem K(d), $G_0(v) \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\}$. Since $|S_1^v| \leq 3$, by Claim 2, if $|V(G_0(v))| = 7$, $|Y_v| = 4$ and $|S_1^v| = 3$. Note that Y_v is an independent set and $J'(1,1)$ has the independent number 3. Thus, $G_0(v) \neq J'(1,1)$ and so $G_0(v) \in \{K_{2,3}, K_{1,3}(1,1,1)\}$ if $G_0(v) \neq K_1$.

Since $|S_1| \leq 12$ and $V(G_0(v)) \cap S_1 \neq \emptyset$ for any $v \in V(P)$, we have the following fact:

Fact 1. If there is a $G_0(v)$ with $|S_1^v| = 3$, then $G_0(u) = K_1$ for every $u \in V(P) - \{v\}$.

We complete our proof by checking on each of the following cases.

Case 1. $G_0(v_1) = K_{1,3}(1,1,1)$ for some $v_1 \in V(P)$.

Then $|S_1^v| = 3$. By Fact 1, only this $G_0(v_1) \neq K_1$. Thus, $G_0' = P(K_{1,3}(1,1,1))$.

Case 2. $G_0(v_1) = K_{2,3}$ for some $v_1 \in V(P)$.

Since $|S_1^v| \leq 3$ and $|Y_v| \leq 3$ in $K_{2,3}$, either $|Y_v| = 2$ and $|S_1^v| = 3$ or $|Y_v| = 3$ and $|S_1^v| = 2$.

Subcase 2.a. $|Y_v| = 2$ and $|S_1^v| = 3$.

By Fact 1 again, only $G_0(v_1) \neq K_1$. Thus, $G'_0 = P(K_{2,3}) = P_{14}$.

Subcase 2.b. $|Y_v| = 3$ and $|S_1^v| = 2$.

Then the three vertices of degree 2 in $G_0(v_1) = K_{2,3}$ are the vertices in $Y_v \subseteq Y$.

If only $G_0(v_1) \neq K_1$, then $|S_1| = 11$ and we still have $G'_0 = P(K_{2,3}) = P_{14}$.

If there is another vertex v_2 in P such that $G_0(v_2) \neq K_1$, then $G_0(v_2) = K_{2,3}$ and so $|S_1| = 12$. Like $G_0(v_1)$, the three vertices of degree 2 in $G_0(v_2) = K_{2,3}$ are all in Y.

Since Y is an independent set in G'_0, none of the vertices of degree 2 in $G_0(v_1)$ is adjacent to a vertex of degree 2 in $G_0(v_2)$. Thus, v_1 is not adjacent to v_2 in P. Hence $G'_0 = P(K_{2,3}, K_{2,3})$. The proof is completed. □

Corollary 5.2. *Let H be a 3-connected claw-free simple graph of order n and n is sufficiently large.*

(a) If $\delta(H) \geq \frac{n+7}{11}$, then either H is Hamiltonian or $H \in \mathcal{F}_1$;
(b) If $\delta(H) \geq \frac{n+10}{12}$, then either H is Hamiltonian or $H \in \mathcal{F}_1 \cup \mathcal{F}_2$.

Proof. Suppose that H is not Hamiltonian. Let G be the preimage of $cl(H) = L(G)$. Then G is an essentially 3-edge-connected K_3-free simple graph with size $n = |E(G)|$. Since $\frac{n+7}{11} \geq \frac{n+10}{12} \geq \frac{n+6}{13}$, both (a) and (b) satisfy the assumptions of Theorem 5.1. Thus, $G'_0 \in \{P, P(K_{2,3}), P(K_{1,3}(1,1,1)), P(K_{2,3}, K_{2,3})\}$, and so G'_0 is a 3-regular graph with $|S_1| \leq 12$. For each $v \in S_1$, let $\Gamma(v)$ be the preimage of v in G. Let $\{e_1, e_2, e_3\}$ be the three edges between $\Gamma(v)$ and $G - V(\Gamma(v))$.

For each $x \in V(G)$, let $E_G(x)$ be the set of edges in G incident with x. Then $d_G(x) = |E_G(x)|$. Since $E(G) \supseteq E(G'_0) \cup (\bigcup_{v \in S_1} E(\Gamma(v)))$,

$$n = |E(G)| \geq |E(G'_0)| + \sum_{v \in S_1} |E(\Gamma(v))|. \tag{21}$$

Claim 1. *Let $M = \{y_1z_1, y_2z_2, \cdots, y_tz_t\}$ be a matching of size t in $\Gamma(v)$ ($t \leq \alpha'(\Gamma(v))$). Then $|E(\Gamma(v))| \geq t\bar{\sigma}_2(G) - t^2 - 3$.*

As an induced subgraph of G, $G[\{y_1, y_2, \cdots, y_t, z_1, z_2, \cdots, z_t\}]$ is a K_3-free simple graph. By Turán's Theorem, it has at most t^2 edges. Then

$$\begin{aligned}\sum_{i=1}^{t}(d_G(y_i) + d_G(z_i)) - t^2 &= \sum_{i=1}^{t}(|E_G(y_i)| + |E_G(z_i)|) - t^2 \\ &\leq |\bigcup_{i=1}^{t}(E_G(y_i) \cup E_G(z_i))|.\end{aligned}$$

Since $\bigcup_{i=1}^{t}(E_G(y_i)\cup E_G(z_i)) \subseteq \{e_1,e_2,e_3\}\cup E(\Gamma(v))$ and since $d_G(y_i)+d_G(z_i) \geq \bar{\sigma}_2(G)$,

$$\begin{aligned} t\bar{\sigma}_2(G) - t^2 &\leq \sum_{i=1}^{t}(|E_G(y_i)| + |E_G(z_i)|) - t^2 \\ &\leq |\bigcup_{i=1}^{t}(E_G(y_i)\cup E_G(z_i))| \leq 3 + |E(\Gamma(v))|. \end{aligned}$$

Thus, $|E(\Gamma(v))| \geq t\bar{\sigma}_2(G) - t^2 - 3$. Claim 1 is proved.

For each vertex $v \in S_1$, since $\Gamma(v) \neq K_1$, $\alpha'(\Gamma(v)) \geq 1$. By Claim 1 with $t = 1$,

$$|E(\Gamma(v))| \geq \bar{\sigma}_2(G) - 4. \tag{22}$$

If $\Gamma(v)$ is a tree, since G is essentially 3-edge-connected, $\Gamma(v) = K_{1,r}$ where $r = |E(\Gamma(v))|$.

If $\Gamma(v)$ is not a tree, then since G is K_3-free and simple, $\alpha'(\Gamma(v)) \geq 2$. By Claim 1,

$$|E(\Gamma(v))| \geq 2\bar{\sigma}_2(G) - 7. \tag{23}$$

(a) This is the special case of Theorem 1.1 with $p = 11$ and $\epsilon = 7$.

If $|S_1| = 11$, then by Proposition 4.1(d), $11 = |S_1| \leq |V(G_0')| \leq 22 - 5 - 7 = 10$, a contradiction. Thus, $|S_1| \leq 10$. From the proof of Theorem 5.1, we can see that $G_0' = P$ and $|S_1| = |V(G_0')| = 10$.

Now, we only need to show that $\Gamma(v)$ is a tree for any $v \in S_1 = V(P)$.

To the contrary, suppose that at least one $\Gamma(v)$ is not a tree. By (21), (22) and (23),

$$n \geq |E(P)| + \sum_{u\in S_1-\{v\}} |E(\Gamma(u))| + |E(\Gamma(v))| \geq 15 + 9(\bar{\sigma}_2(G) - 4) + (2\bar{\sigma}_2(G) - 7),$$

which yields $\bar{\sigma}_2(G) \leq \frac{n+28}{11}$, contrary to that $\bar{\sigma}_2(G) = \delta(cl(H)) + 2 \geq \delta(H) + 2 \geq \frac{n+29}{11}$.

Thus, for all $v \in V(P)$, $\Gamma(v) = K_{1,r}$. Thus, $H \in \mathcal{F}_1$. Corollary 5.2(a) is proved.

(b) This is the special case of Theorem 1.1 with $p = 12$ and $\epsilon = 10$.

If $|S_1| = 12$, then by Proposition 4.1(d), $12 = |S_1| \leq |V(G_0')| \leq 24 - 5 - 10 = 9$, a contradiction. Thus, $|S_1| \leq 11$. It follows from the proof of Theorem 5.1 that $G_0' \in \{P, P(K_{2,3})\}$.

Case 1. $G_0' = P(K_{2,3})$.

From the proof of Subcase 2.b of Theorem 5.1, $|S_1| = 11$ and the two vertices of degree 3 in $K_{2,3}$ part are in S_1. Again, we just need to show that for all $v \in S_1$, $\Gamma(v)$ is a tree.

To the contrary, suppose that at least one $\Gamma(v)$ is not a tree. By (21), (22) and (23),

$$\begin{aligned} n &\geq |E(P(K_{2,3}))| + \sum_{u\in S_1-\{v\}} |E(\Gamma(u))| + |E(\Gamma(v))| \\ &\geq 21 + 10(\bar{\sigma}_2(G) - 4) + (2\bar{\sigma}_2(G) - 7), \end{aligned}$$

which yields $\bar{\sigma}_2(G) \le \frac{n+26}{12}$, contrary to that $\bar{\sigma}_2(G) = \delta(cl(H)) + 2 \ge \delta(H) + 2 \ge \frac{n+34}{12}$.

Thus, $\Gamma(v) = K_{1,r}$ where $r = |E(\Gamma(v))| \ge \bar{\sigma}_2(G) - 4$, and so $G \in \mathcal{F}_2$. Case 1 is proved.

Case 2. $G_0' = P$.

Then $|S_1| = 10$. If for any $v \in S_1 = V(P)$, $\Gamma(v) = K_{1,r}$, then $G \in \mathcal{F}_1$. We are done.

Next, we assume that $\Gamma(v_1)$ contains a cycle for some $v_1 \in S_1$.

Claim 2. $\Gamma(v)$ *is a tree for any* $v \in S_1 - \{v_1\}$.

To the contrary, suppose that, for some $v_2 \in S_1 - \{v_1\}$, $\Gamma(v_2)$ is not a tree. By (21), (22) and (23),

$$\begin{aligned} n &\ge |E(P)| + \sum_{u \in S_1 - \{v_1, v_2\}} |E(\Gamma(u))| + |E(\Gamma(v_1))| + |E(\Gamma(v_2))| \\ &\ge 15 + 8(\bar{\sigma}_2(G) - 4) + 2(2\bar{\sigma}_2(G) - 7), \end{aligned}$$

which yields $\bar{\sigma}_2(G) \le \frac{n+31}{12}$, contrary to that $\bar{\sigma}_2(G) = \delta(cl(H)) + 2 \ge \delta(H) + 2 \ge \frac{n+34}{12}$. Claim 2 is proved.

Next we show that $\Gamma(v_1) = \overline{K}_{2,t}$, a connected spanning subgraph of $K_{2,t}$.

Claim 3. $\alpha'(\Gamma(v_1)) = 2$.

To the contrary, suppose that $\alpha'(\Gamma(v_1)) \ge 3$. By Claim 1 with $t = 3$, $|E(\Gamma(v_1))| \ge 3\bar{\sigma}_2(G) - 12$. Then by (21) and (22),

$$n \ge |E(P)| + \sum_{u \in S_1 - \{v_1\}} |E(\Gamma(u))| + |E(\Gamma(v_1))| \ge 15 + 9(\bar{\sigma}_2(G) - 4) + 3\bar{\sigma}_2(G) - 12,$$

which yields $\bar{\sigma}_2(G) \le \frac{n+33}{12}$, contrary to that $\bar{\sigma}_2(G) = \delta(cl(H)) + 2 \ge \delta(H) + 2 \ge \frac{n+34}{12}$. Claim 3 is proved.

Let $C_t = x_1x_2x_3x_4 \cdots x_tx_1$ be a shortest cycle in $\Gamma(v_1)$. Claim 3 and the fact that G is K_3-free imply that $\min\{d_{\Gamma(v_1)}(x_i) \mid x_i \in \{x_1, x_2, x_3\}\} = 2$, otherwise, $\Gamma(v_1)$ would have a matching of size at least 3.

Without lost of generality, we assume that $d_{\Gamma(v_1)}(x_2) = 2$. Then $N_{\Gamma(v_1)}(x_2) = \{x_1, x_3\}$ and for each $x_i \in N_{\Gamma(v_1)}(x_2)$ $(i = 1, 3)$, $|E_G(x_i)| \ge \bar{\sigma}_2(G) - 2 > 3$. Combining this with the fact that G is K_3-free and $\alpha'(\Gamma(v)) = 2$, we have that for each $z \in N_{\Gamma(v_1)}(x_1) \cup N_{\Gamma(v_1)}(x_3)$, either $d_{\Gamma(v_1)}(z) = 1$ or $N_{\Gamma(v_1)}(z) = \{x_1, x_3\}$, which implies that $C_t = x_1x_2x_3x_4x_1$.

Let $X = \{x_1, x_3\}$ and let $Y = N_{\Gamma(v_1)}(x_1) \cup N_{\Gamma(v_1)}(x_3)$. Then $\Gamma(v_1)$ is a bipartite graph with bipartition $X \cup Y$. Thus, $\Gamma(v_1) = \overline{K}_{2,t}$ and so $G \in \mathcal{F}_2$. The proof is completed. $\square$

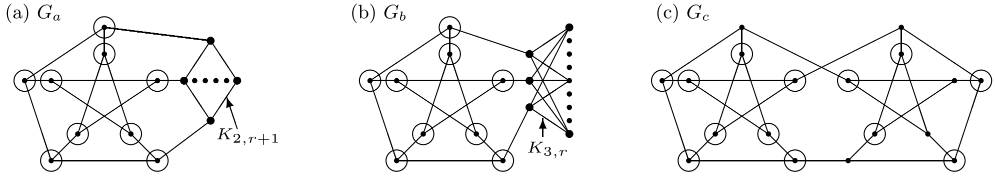

Fig. 5.2. Graphs to show that Corollary 5.2 and Theorem 5.1 are best possible.

Remark. Corollary 5.2 and Theorem 5.1 are best possible as explained in the following examples. In the graphs depicted in Fig. 5.2, we assume that the degree of each vertex marked with $\odot$ is $r+3$, where $r > 0$ is the number of pendant edges incident with the vertex. Then none of the graphs in Fig. 5.2 has a dominating Eulerian subgraph. Also graph G_b is a counterexample to Conjecture A.

(a) For graph G_a, $n = |E(G_a)| = 11r + 17$ and $\bar{\sigma}_2(G_a) \geq r + 4 = \frac{n+28}{11}$. Let $H_a = L(G_a)$. Then $n = |V(H_a)|$ and $\delta(H_a) = \bar{\sigma}_2(G_a) - 2 \geq \frac{n+6}{11}$. But H_a is non-Hamiltonian and $H_a \notin \mathcal{F}_1$. Thus, the condition $\delta(H) \geq \frac{n+7}{11}$ in Corollary 5.2(a) is best possible.

(b) For graph G_b, $n = |E(G_b)| = 12r + 15$ and $\bar{\sigma}_2(G_b) \geq r + 4 = \frac{n+33}{12}$. Let $H_b = L(G_b)$. Then $n = |V(H_b)|$ and $\delta(H_b) = \bar{\sigma}_2(G_b) - 2 \geq \frac{n+9}{12}$. But H_b is non-Hamiltonian and $H_b \notin \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Thus, the condition $\delta(H) \geq \frac{n+10}{12}$ in Corollary 5.2(b) is best possible, and contrary to what the author [17] guessed, the bound on $\delta(H)$ in Theorem H can not be relaxed to $\delta(H) \geq (n+6)/12$ and Conjecture A is false.

(c) For graph G_c, $n = |E(G_c)| = 13r + 27$ and $\bar{\sigma}_2(G_c) \geq r + 4 = \frac{n+25}{13}$. Let $H_c = L(G_c)$. Then $n = |V(H_c)|$. For a given ϵ, let $r \geq \epsilon - 1$. Then $n > 14 + 13\epsilon$ and $\delta(H_c) = \bar{\sigma}_2(G_c) - 2 \geq \frac{n-1}{13} \geq \frac{n+\epsilon}{14}$. But H_c is non-Hamiltonian and the preimage G_c of $H_c = L(G_c)$ cannot be contracted to the Petersen graph in the way stated in Theorem 5.1 (since one of the vertices in P is not a contracted vertex). Thus, $p = 13$ in Theorem 5.1 cannot be replaced by $p = 14$.

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