Discrete Applied Mathematics 213 (2016) 219–223

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

Fractional spanning tree packing, forest covering and eigenvalues

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ARTICLE INFO

Article history:
Received 30 January 2015
Received in revised form 18 April 2016
Accepted 21 April 2016
Available online 7 July 2016

Keywords:
Eigenvalue
Algebraic connectivity
Strength
Arboricity
Fractional arboricity

ABSTRACT

We investigate the relationship between the eigenvalues of a graph G and fractional spanning tree packing and coverings of G. Let $\omega(G)$ denote the number of components of a graph G. The strength $\eta(G)$ and the fractional arboricity $\gamma(G)$ are defined by

$$\eta(G) = \min \frac{|X|}{\omega(G-X) - \omega(G)}, \quad \text{and} \quad \gamma(G) = \max \frac{|E(H)|}{|V(H)| - 1},$$

where the optima are taken over all edge subsets X whenever the denominator is non-zero. The well known spanning tree packing theorem by Nash-Williams and Tutte indicates that a graph G has k edge-disjoint spanning tree if and only if $\eta(G) \geq k$; and Nash-Williams proved that a graph G can be covered by at most k forests if and only if $\gamma(G) \leq k$. Let $\lambda_i(G)$ ($\mu_i(G)$, $q_i(G)$, respectively) denote the ith largest adjacency (Laplacian, signless Laplacian, respectively) eigenvalue of G. In this paper, we prove the following.

(1) Let G be a graph with $\delta \geq 2s/t$. Then $\eta(G) \geq s/t$ if $\mu_{n-1}(G) > \frac{2s-1}{t(\delta+1)}$, or if $\lambda_2(G) < \delta - \frac{2s-1}{t(\delta+1)}$, or if $q_2(G) < 2\delta - \frac{2s-1}{t(\delta+1)}$.

(2) Suppose that G is a graph with nonincreasing degree sequence $d_1, d_2, \ldots, d_n$ and $n \geq \lfloor \frac{2s}{t} \rfloor + 1$. Let $\beta = \frac{2s}{t} - \frac{1}{\lfloor \frac{2s}{t} \rfloor + 1} \sum_{i=1}^{\lfloor \frac{2s}{t} \rfloor + 1} d_i$. Then $\gamma(G) \leq s/t$, if $\beta \geq 1$, or if $0 < \beta < 1$, $n > \lfloor 2s/t \rfloor + 1 + \frac{2s-2}{t\beta}$ and

$$\mu_{n-1}(G) > \frac{n(2s/t - 2/t - \beta(\lfloor 2s/t \rfloor + 1))}{(\lfloor 2s/t \rfloor + 1)(n - \lfloor 2s/t \rfloor - 1)}.$$

Our result proves a stronger version of a conjecture by Cioabă and Wong on the relationship between eigenvalues and spanning tree packing, and sharpens former results in this area.

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http://dx.doi.org/10.1016/j.dam.2016.04.027

1. Introduction

In this paper, we consider finite undirected simple graphs. Throughout the paper, k, s, t denote positive integers and G denotes a simple graph. We follow the notations of Bondy and Murty [1], unless otherwise defined. However, we use $\omega(G)$ to denote the number of components of G, which differs from [1].

Let G be an undirected simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The **adjacency matrix** of G is an n by n matrix $A(G)$ with entry $a_{ij} = 1$ if there is an edge between v_i and v_j and $a_{ij} = 0$ otherwise, for $1 \le i, j \le n$. We use $\lambda_i(G)$ to denote the ith largest eigenvalue of G; when the graph G is understood from the context, we often use λ_i for $\lambda_i(G)$. With these notations, we always have $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Let $D(G) = (d_{ij})$ be the **degree matrix** of G, that is, the n by n diagonal matrix with d_{ii} being the degree of vertex v_i in G for $1 \le i \le n$. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the **Laplacian matrix** and the **signless Laplacian matrix** of G, respectively. We use $\mu_i(G)$ and $q_i(G)$ to denote the ith largest eigenvalue of $L(G)$ and $Q(G)$, respectively. It is not difficult to see that $\mu_n(G) = 0$. The second smallest eigenvalue of $L(G)$, $\mu_{n-1}(G)$, is known as the **algebraic connectivity** of G.

For a connected graph G, the **spanning tree packing number**, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in G. The **arboricity** $a(G)$ is the minimum number of edge-disjoint forests whose union equals $E(G)$. Fundamental theorems characterizing graphs G with $\tau(G) \ge k$ and with $a(G) \le k$ have been obtained by Nash-Williams and Tutte, and by Nash-Williams, respectively.

Theorem 1.1. *Let G be a connected graph with $E(G) \ne \emptyset$. Each of the following holds.*

(i) *(Nash-Williams [12] and Tutte [15]). $\tau(G) \ge k$ if and only if for any $X \subseteq E(G)$, $|X| \ge k(\omega(G - X) - 1)$.*

(ii) *(Nash-Williams [13]). $a(G) \le k$ if and only if for any subgraph H of G, $|E(H)| \le k(|V(H)| - 1)$.*

Following the terminology in [3,14], we define the **strength** $\eta(G)$ and the **fractional arboricity** $\gamma(G)$ of a graph G respectively by

$$\eta(G) = \min \frac{|X|}{\omega(G - X) - \omega(G)}, \quad \text{and} \quad \gamma(G) = \max \frac{|E(H)|}{|V(H)| - 1},$$

where the optima are taken over all edge subsets X whenever the denominator is non-zero. Theorem 1.1 indicates that for a connected graph G, $\tau(G) \ge k$ if and only if $\eta(G) \ge k$, and, $a(G) \le k$ if and only if $\gamma(G) \le k$. Since $\eta(G)$ and $\gamma(G)$ are possibly fractional, we have $\tau(G) = \lfloor \eta(G) \rfloor$ and $a(G) = \lceil \gamma(G) \rceil$. Thus, $\eta(G)$ is also referred to as the **fractional spanning tree packing number** of G.

Cioabă and Wong [4] investigated the relationship between the second largest adjacency eigenvalue and $\tau(G)$ for a regular graph G, and made Conjecture 1.1(i). Utilizing Theorem 1.1, Cioabă and Wong proved Conjecture 1.1(i) for $k \in \{2, 3\}$.

Conjecture 1.1(i) was then extended to Conjecture 1.1(ii) for any simple graph G (not necessarily regular). See [5–7,9,11] for the conjecture and some partial results. Recently, Conjecture 1.1 was settled in [10].

Conjecture 1.1. (i) *([4]) Let k and d be two integers with $d \ge 2k \ge 4$. If G is a d-regular graph with $\lambda_2(G) < d - \frac{2k-1}{d+1}$, then $\tau(G) \ge k$.*

(ii) *([5,7,9,11]) Let k be an integer with $k \ge 2$ and G be a graph with minimum degree $\delta \ge 2k$. If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \ge k$.*

Motivated by the above conjecture and the corresponding results, we investigate the relationship between $\eta(G)$ and eigenvalues of G. We also consider the relationship between the fractional arboricity $\gamma(G)$ and algebraic connectivity $\mu_{n-1}(G)$. Theorems 1.2 and 1.3 are the main results.

Theorem 1.2. *Let G be a graph with $\delta \ge 2s/t$.*

(i) *If $\mu_{n-1}(G) > \frac{2s-1}{t(\delta+1)}$, then $\eta(G) \ge s/t$.*

(ii) *If $\lambda_2(G) < \delta - \frac{2s-1}{t(\delta+1)}$, then $\eta(G) \ge s/t$.*

(iii) *If $q_2(G) < 2\delta - \frac{2s-1}{t(\delta+1)}$, then $\eta(G) \ge s/t$.*

Remark 1. Theorem 1.2 indicates that, for a graph G with $\delta \ge 2k$, if $\mu_{n-1}(G) > \frac{2k-1}{\delta+1}$, or $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, or $q_2(G) < 2\delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \ge k$. This was proved in [10] and settled Conjecture 1.1.

Theorem 1.3. *Suppose that G is a graph with nonincreasing degree sequence $d_1, d_2, \ldots, d_n$ and $n \ge \lfloor \frac{2s}{t} \rfloor + 1$. Let $\beta = \frac{2s}{t} - \frac{1}{\lfloor \frac{2s}{t} \rfloor + 1} \sum_{i=1}^{\lfloor \frac{2s}{t} \rfloor + 1} d_i$.*

(i) *If* $\beta \geq 1$, *then* $\gamma(G) \leq s/t$.
(ii) *If* $0 < \beta < 1$, $n > \lfloor 2s/t \rfloor + 1 + \frac{2s-2}{t\beta}$ *and*

$$\mu_{n-1}(G) > \frac{n(2s/t - 2/t - \beta(\lfloor 2s/t \rfloor + 1))}{(\lfloor 2s/t \rfloor + 1)(n - \lfloor 2s/t \rfloor - 1)}, \quad (1)$$

then $\gamma(G) \leq s/t$.

Remark 2. By Theorem 1.1(ii), it is not hard to see for a graph G with n vertices, if $n \leq 2k$ then $a(G) \leq k$. When $n \geq 2k+1$, under the same conditions of Theorem 1.3, we also have $a(G) \leq k$.

Corollary 1.4. *Let* $k > 0$ *be an integer, and* G *be a* d*-regular graph. Then* $a(G) > k$ *if and only if* $d \geq 2k$.

In Section 2, we display some preliminaries and mechanisms, including eigenvalue interlacing properties and the quotient matrix, which will be applied in the proofs of the main results, to be presented in Section 3.

2. Preliminaries

In this section, we present some of the preliminaries to be used in our arguments. For a square matrix A, $tr(A)$ denotes the trace of A. For a graph G, we use $\bar{d}(G)$ to denote the average degree of G. Let $U \subseteq V(G)$, $\bar{d}_G(U)$ or simply $\bar{d}(U)$ denotes the average degree of all vertices of U in G. Thus $\bar{d}(G[U])$ and $\bar{d}(U)$ are different. The former means the average degree of the induced subgraph $G[U]$, while the latter is the average degree of all vertices of U in G. The following theorem is commonly referred to as the Weyl Inequalities. See also page 29 of [2] for the Courant–Weyl inequalities.

Theorem 2.1 (*Weyl Inequalities*)**.** *Let* B *and* C *be Hermitian matrices of order* n. *Then for* $1 \leq i, j \leq n$,
(i) $\lambda_i(B) + \lambda_j(C) \leq \lambda_{i+j-n}(B+C)$ *if* $i + j \geq n+1$.
(ii) $\lambda_i(B) + \lambda_j(C) \geq \lambda_{i+j-1}(B+C)$ *if* $i + j \leq n+1$.

Corollary 2.2. *Let* δ *be the minimum degree of a graph* G. *Then for* $i = 1, 2, \ldots, n$,
(i) $\mu_{n-i+1} + \lambda_i \geq \delta$.
(ii) $\delta + \lambda_i \leq q_i$.

Proof. (i) By the definition, $L(G) = D(G) - A(G)$. Then $L(G) + A(G) = D(G)$. By Theorem 2.1(ii), $\lambda_{n-i+1}(L(G)) + \lambda_i(A(G)) \geq \lambda_n(D(G))$, i.e., $\mu_{n-i+1} + \lambda_i \geq \delta$.
(i) By the definition, $Q(G) = D(G) + A(G)$. By Theorem 2.1(i), $\lambda_n(D(G)) + \lambda_i(A(G)) \leq \lambda_i(Q(G))$, i.e., $\delta + \lambda_i \leq q_i$. □

Given two real sequences $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $n > m$, the second sequence is said to **interlace** the first one if $\theta_i \geq \eta_i \geq \theta_{n-m+i}$, for $i = 1, 2, \ldots, m$. When we say the eigenvalues of a matrix B interlace the eigenvalues of a matrix A, it means the non-increasing eigenvalue sequence of B interlaces that of A.

Theorem 2.3 (*Cauchy Interlacing*)**.** *Let* A *be a real symmetric matrix and* B *be a principal submatrix of* A. *Then the eigenvalues of* B *interlace the eigenvalues of* A.

Given a partition $\pi = \{X_1, X_2, \ldots, X_s\}$ of the set $\{1, 2, \ldots, n\}$ and a matrix A whose rows and columns are labeled with elements in $\{1, 2, \ldots, n\}$, A can be expressed as the following partitioned matrix

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1s} \\ \vdots & \vdots & \vdots \\ A_{s1} & \cdots & A_{ss} \end{bmatrix}$$

with respect to π. The **quotient matrix** A_π of A with respect to π is an s by s matrix (b_{ij}) such that each entry b_{ij} is the average row sum of A_{ij}.

Theorem 2.4 (*Haemers, Corollary 2.3 in [8]*)**.** *The eigenvalues of any quotient matrix of a real symmetric matrix* A *interlace the eigenvalues of* A.

Corollary 2.5. *Suppose that* G *is a simple graph and* π *is a partition of* $V(G)$ *with* $|\pi| = s$. *Let* L_π *be the quotient matrix of* $L(G)$ *with respect to* π. *Then* $\mu_{n-1} \leq \lambda_{s-1}(L_\pi)$.

Proof. By Theorem 2.4, $\lambda_{s-1}(L_\pi) \geq \lambda_{n-s+(s-1)}(L(G))$, i.e., $\mu_{n-1} \leq \lambda_{s-1}(L_\pi)$. □

For any subset $U \subseteq V(G)$, $\partial(U)$ denotes the set of edges each of which has one end in U and the other end in $V(G) \setminus U$.

Lemma 2.6 (*Lemma 3.2 of [10]*)**.** *Suppose that* $X, Y \subset V(G)$ *with* $X \cap Y = \emptyset$. *If* $\mu_{n-1}(G) \geq \max\{\frac{|\partial(X)|}{|X|}, \frac{|\partial(Y)|}{|Y|}\}$, *then* $[e(X,Y)]^2 \geq |X| \cdot |Y| (\mu_{n-1} - \frac{|\partial(X)|}{|X|})(\mu_{n-1} - \frac{|\partial(Y)|}{|Y|})$.

3. The proofs of main results

In this section, we present the proofs of the main results. We begin with a useful lemma.

Lemma 3.1. *Let G be a graph and U be a non-empty proper subset of $V(G)$. Suppose that $\bar{d}(U)$ is the average degree of U in G. If $|\partial(U)| < \bar{d}(U)$, then $|U| > \bar{d}(U)$.*

Proof. By contradiction, we assume that $|U| \le \bar{d}(U)$. Then $|U|(|U|-1)+|\partial(U)| \ge |U|\bar{d}(U)$ by counting the incidences in U in two ways. But $|U|(|U|-1)+|\partial(U)| < \bar{d}(U)(|U|-1)+\bar{d}(U) = |U|\bar{d}(U)$, contrary to the fact that $|U|(|U|-1)+|\partial(U)| \ge |U|\bar{d}(U)$. Thus $|U| > \bar{d}(U)$. □

Proof of Theorem 1.2. By the definition of $\eta(G)$, it suffices to show for any $X \subseteq E(G)$,

$$t|X| \ge s(\omega(G-X)-\omega(G)).$$

Without loss of generality, we may assume that G is connected and so $\omega(G) = 1$. Let $\omega = \omega(G-X)$ and V_i be the vertex set of each component of $G-X$ for $1 \le i \le \omega$, respectively. Without loss of generality, we may assume that $|\partial(V_1)| \le |\partial(V_2)| \le \cdots \le |\partial(V_\omega)|$. If $t|\partial(V_2)| \ge 2s$, then $t|X| \ge t\sum_{1\le i\le\omega}|\partial(V_i)|/2 \ge s(\omega-1)$, done. Thus, we assume that $t|\partial(V_2)| \le 2s-1$.

Let q be the largest index such that $t|\partial(V_q)| \le 2s-1$. Then $2 \le q \le \omega$. By Lemma 3.1, $|V_i| \ge \delta+1$ for $1 \le i \le q$. By Lemma 2.6, for $2 \le i \le q$,

$$\begin{aligned}[e(V_1,V_i)]^2 &\ge |V_1||V_i|\left(\mu_{n-1}-\frac{|\partial(V_1)|}{|V_1|}\right)\left(\mu_{n-1}-\frac{|\partial(V_i)|}{|V_i|}\right)\\ &> \left(\frac{2s-1}{t}-|\partial(V_1)|\right)\left(\frac{2s-1}{t}-|\partial(V_i)|\right)\\ &\ge \left(\frac{2s-1}{t}-|\partial(V_i)|\right)^2.\end{aligned}$$

Thus $e(V_1,V_i) > \frac{2s-1}{t}-|\partial(V_i)|$, which implies that $t\cdot e(V_1,V_i) > 2s-1-t|\partial(V_i)|$, and so $t\cdot e(V_1,V_i) \ge 2s-t|\partial(V_i)|$. Then $t|\partial(V_1)| \ge t\sum_{2\le i\le q} e(V_1,V_i) \ge \sum_{2\le i\le q}(2s-t|\partial(V_i)|)$. Hence $t\sum_{1\le i\le q}|\partial(V_i)| \ge 2s(q-1)$. Thus

$$\begin{aligned}t|X| &\ge t\sum_{1\le i\le\omega}|\partial(V_i)|/2 = \frac{1}{2}\left(t\sum_{1\le i\le q}|\partial(V_i)| + t\sum_{q+1\le i\le\omega}|\partial(V_i)|\right)\\ &\ge \frac{1}{2}[2s(q-1)+2s(\omega-q)]\\ &= s(\omega-1),\end{aligned}$$

which finishes the proof of (i). By Corollary 2.2, (ii) and (iii) follows from (i). □

Proof of Theorem 1.3. (i) Assume that $\gamma(G) > s/t$. By the definition of $\gamma(G)$, G has a nontrivial subgraph H with $|E(H)| > (|V(H)|-1)s/t$. Since H is simple, $|V(H)|(|V(H)|-1) \ge 2|E(H)| > 2(|V(H)|-1)s/t$, and so $|V(H)| > 2s/t$. Thus $|V(H)| \ge \lfloor 2s/t\rfloor+1$. Let $\bar{d}_H$ be the average degree of the subgraph H. Then $\frac{1}{\lfloor 2s/t\rfloor+1}\sum_{i=1}^{\lfloor 2s/t\rfloor+1} d_i \ge \bar{d}_H = \frac{2|E(H)|}{|V(H)|} > \frac{2(|V(H)|-1)s/t}{|V(H)|} > \frac{2s}{t}-1$. It follows that $\beta = \frac{2s}{t} - \frac{1}{\lfloor 2s/t\rfloor+1}\sum_{i=1}^{\lfloor 2s/t\rfloor+1} d_i < 1$, contrary to $\beta \ge 1$.

(ii)We argue by contradiction and assume that $\gamma(G) > s/t$. By the definition of $\gamma(G)$, G has a nontrivial subgraph H with $|E(H)| > (|V(H)|-1)s/t$. It implies that $t|E(H)| > (|V(H)|-1)s$, or in other words, $t|E(H)| \ge (|V(H)|-1)s+1$. Thus

$$|E(H)| \ge (|V(H)|-1)s/t + 1/t. \tag{2}$$

Since H is simple, $|V(H)|(|V(H)|-1) \ge 2|E(H)| > 2(|V(H)|-1)s/t$, and so $|V(H)| > 2s/t$. Thus

$$|V(H)| \ge \lfloor 2s/t\rfloor + 1. \tag{3}$$

Let $V_1 = V(H)$ and $\bar{d}_1$ be the average degree of V_1 in G. Since $|V_1| \ge \lfloor 2s/t\rfloor+1$,

$$\bar{d}_1 \le \frac{1}{\lfloor 2s/t\rfloor+1}\sum_{i=1}^{\lfloor 2s/t\rfloor+1} d_i \le 2s/t-\beta. \tag{4}$$

By (2) and counting the incidences of vertices of V_1 in G, we have

$$|\partial(V_1)| = |V_1|\bar{d}_1 - 2|E(H)| \le |V_1|\bar{d}_1 - 2(|V_1|-1)s/t - 2/t. \tag{5}$$

By (4), (5) and by the definition of β,

$$|\partial(V_1)| \le |V_1|(2s/t-\beta) - 2(|V_1|-1)s/t - 2/t = 2s/t - 2/t - \beta|V_1| \\ \le 2s/t - 2/t - \beta(\lfloor 2s/t \rfloor + 1). \tag{6}$$

By (5), $|V_1|\bar{d}_1 - 2(|V_1|-1)s/t - 2/t \ge 0$. It follows that $|V_1| \le (2s/t - 2/t)/(2s/t - \bar{d}_1) \le \frac{2s/t-2/t}{\beta}$, and so

$$\text{both } |V_1| \le \frac{2s-2}{t\beta} \quad \text{and} \quad |V \setminus V_1| \ge n - \frac{2s-2}{t\beta}. \tag{7}$$

Moreover, by (5),

$$|\partial(V_1)| \le |V_1|\bar{d}_1 - 2(|V_1|-1)s/t - 2/t < |V_1|\bar{d}_1 - \bar{d}_1(|V_1|-1) - 2/t < \bar{d}_1. \tag{8}$$

By (8) and Lemma 3.1,

$$|V_1| \ge d_n + 1. \tag{9}$$

Let $r = |\partial(V_1)|$ and $V' = V \backslash V_1$. The quotient matrix of the Laplacian matrix $L(G)$ with respect to the partition $\pi = (V_1, V')$ is

$$A_\pi = \begin{bmatrix} \frac{r}{|V_1|} & -\frac{r}{|V_1|} \\ -\frac{r}{|V'|} & \frac{r}{|V'|} \end{bmatrix}.$$

By Corollary 2.5,

$$\mu_{n-1}(G) \le \lambda_1(A_\pi) \le tr(A_\pi) = \frac{r}{|V_1|} + \frac{r}{|V'|} = \frac{(|V_1|+|V'|)r}{|V_1||V'|} = \frac{nr}{|V_1||V'|}. \tag{10}$$

By (3) and (9), we have $|V_1| \ge \max\{\lfloor 2s/t \rfloor + 1, d_n + 1\} = \lfloor 2s/t \rfloor + 1$. By (7), $|V'| \ge n - \frac{2s-2}{t\beta}$. As $n > \lfloor 2s/t \rfloor + 1 + \frac{2s-2}{t\beta}$, it follows that $|V_1| \cdot |V'|$ is minimized when $|V_1| = \lfloor 2s/t \rfloor + 1$. Thus by (6) and (10),

$$\mu_{n-1}(G) \le \frac{n(2s/t - 2/t - \beta(\lfloor 2s/t \rfloor + 1))}{(\lfloor 2s/t \rfloor + 1)(n - \lfloor 2s/t \rfloor - 1)},$$

contrary to (1). This proves (ii). □

Acknowledgments

Y. Hong is partially supported by NSFC (11401103). X. Gu is partially supported by UWG COSM FRG. Q. Liu is partially supported by NSFC (11301086).

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