Supereulerian Digraphs with Large Arc-Strong Connectivity

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Abstract: Let D be a digraph and let $\lambda(D)$ be the arc-strong connectivity of D, and $\alpha'(D)$ be the size of a maximum matching of D. We proved that if $\lambda(D) \geq \alpha'(D) > 0$, then D has a spanning eulerian subdigraph. J. Graph Theory 81: 393–402, 2016

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1. INTRODUCTION

We consider finite graphs and finite and simple digraphs. Usually, we use G to denote a graph and D a digraph. Undefined terms and notations will follow [6] for graphs and [2] for digraphs. In particular, $\kappa(G)$, $\kappa'(G)$, $\alpha(G)$, and $\alpha'(G)$ denote the connectivity, the edge connectivity, the independence number, and the matching number of a graph G; and

$\kappa(D)$ and $\lambda(D)$ denotes the vertex-strong connectivity and the arc-strong connectivity of a digraph D, respectively. As it is implied by Corollary 5.4.3 of [2], we have $\lambda(D) \geq \kappa(D)$. Throughout this article, we use the notation (u, v) to denote an arc oriented from u to v in a digraph; and use $[u, v]$ to denote either (u, v) or (v, u). When $[u, v] \in A(D)$, we say that u and v are adjacent. If two arcs of D have a common vertex, we say that these two arcs are adjacent in D.

If D is a digraph, we often use $G(D)$ to denote the underlying undirected graph of D, the graph obtained from D by erasing all orientation on the arcs of D. The independence number and the matching number of a digraph D are defined as

$$\alpha(D) = \alpha(G(D)) \text{ and } \alpha'(D) = \alpha'(G(D)),$$

respectively.

For graphs H and G, by $H \subseteq G$ we mean that H is a subgraph of G. Similarly, for digraphs H and D, by $H \subseteq D$ we mean that H is a subdigraph of D. Following [2], for a digraph D with $X, Y \subseteq V(D)$, define

$$(X, Y)_D = \{(x, y) \in A(D) \ : \ x \in X, y \in Y\}.$$

When $Y = V(D) - X$, we define

$$\partial_D^+(X) = (X, V(D) - X)_D \text{ and } \partial_D^-(X) = (V(D) - X, X)_D.$$

For a vertex $v \in V(D)$, $d_D^+(v) = |\partial_D^+(\{v\})|$ and $d_D^-(v) = |\partial_D^-(\{v\})|$ are the **out-degree** and the **in-degree** of v in D, respectively. Finally, we define the following notations: $\delta^+(D) = \min\{d_D^+(v) \ : \ v \in V(D)\}$ and $\delta^-(D) = \min\{d_D^-(v) \ : \ v \in V(D)\}$. For any vertex $v \in V(D)$, define

$$\partial_D(v) = \partial_D^+(v) \cup \partial_D^-(v), \text{ and } d_D(v) = d_D^+(v) + d_D^-(v).$$

When the digraph D is understood from the context, we often omit the subscript D. By the definition of $\lambda(D)$ in [2], we note that for any integer $k \geq 0$ and a digraph D,

$$\lambda(D) \geq k \text{ if and only if for any nonempty proper subset } X \subset V(D), |\partial_D^+(X)| \geq k. \quad (1)$$

Motivated by the Chinese Postman Problem, Boesch et al. [5] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning Eulerian subgraphs, and they ([5]) indicated that this problem would be very difficult. Pulleyblank [17] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been lots of researches on this topic. Catlin [7] in 1992 presented the first survey on supereulerian graphs. Later Chen et al. [8] gave an update in 1995, specifically on the reduction method associated with the supereulerian problem. A recent survey on supereulerian graphs is given in [14].

It is a natural to consider the supereulerian problem in digraphs. A strong digraph D is **eulerian** if for any $v \in V(D)$, $d_D^+(v) = d_D^-(v)$. A strong-connected digraph D is **supereulerian** if D contains a spanning eulerian subdigraph. The main problem is to determine supereulerian digraphs. Several efforts have been made. The earlier studies were done by Gutin ([10, 11]). Recently, the following have been obtained.

Theorem 1.1 (Hong et al. [13]). *Let D be a strong simple digraph on n vertices. If $\delta^+(D) + \delta^-(D) \geq n - 4$, then either D is supereulerian, or D belongs to a class of well-characterized digraphs.*

Theorem 1.2 (J. Bang-Jensen and A. Maddaloni [3]). *Let D be a strong simple digraph on n vertices. If $d(x) + d(y) \geq 2n - 3$ for any pair of non adjacent vertices x and y, then D is supereulerian.*

Theorem 1.3 (J. Bang-Jensen and A. Maddaloni [3]). *Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D has a spanning subdigraph H such that for any $v \in V(H)$, $d_H^+(v) = d_H^-(v) > 0$.*

A well-known theorem of Chvátal and Erdös states that if $|V(G)| \geq 3$ and if $\kappa(G) \geq \alpha(G)$, then G is hamiltonian. Thomassen [18] gave an infinite family of nonhamiltonian (but supereulerian) digraphs such that $\kappa(D) = \alpha(D) = 2$, showing that the the Chvátal-Erdös Theorem does not extend to digraphs. This motivates Bang-Jensen and Thommassé (2011, unpublished, see [3]) to make the following conjecture.

Conjecture 1.4. *Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

Theorem 1.3 is an effort towards this conjecture. In [3], Conjecture 1.4 has been verified in several families of digraphs. There have been investigations on supereulerian properties of a graph G with given inequality constraints on $\kappa'(G)$, $\alpha(G)$, and $\alpha'(G)$, as seen in [1, 12, 15, 19], and [20], among others. In [3], Bang-Jensen and Maddaloni also proved that if $\kappa'(G) \geq \alpha(G)$ for a graph G, then G is supereulerian. The main result of this article is the following.

Theorem 1.5. *Let D be a strong digraph. If $\lambda(D) \geq \alpha'(D)$, then D is supereulerian.*

The following corollary is immediate.

Corollary 1.6. *Let D be a strong digraph. If $\kappa(D) \geq \alpha'(D)$, then D is supereulerian.*

In the next section, we present some of the former theorems and develop a few lemmas that will be used in our arguments. The proof of the main result will be given in the last section.

2. TOOLS

In this section, we present some tools needed in our arguments. Let M be a matching in a graph G. We use $V(M)$ to denote the set of vertices in G that are incident with an edge in M. (Similarly, we define $V(M)$ if M is a matching in a digraph D). A path P is an M-**augmenting path** if the edges of P are alternately in M and in $E(G) - M$, and if both end vertices of P are not in $V(M)$. The following theorem is fundamental.

Theorem 2.1 (Berge, [4]). *A matching M in G is a maximum matching if and only if G does not have M-augmenting paths.*

The next theorem is on hamiltonian digraphs, will also be needed in our proofs. Note that hamiltonian digraphs are also supereulerian digraphs.

Theorem 2.2 (Meyniel [16]). *A strong digraph D on n vertices satisfying $d(x) + d(y) \geq 2n - 1$ for all pairs of nonadjacent vertices x, y is hamiltonian.*

Two more lemmas will also be needed. Throughout the rest of this section, D always denotes a digraph.

Lemma 2.3. *Let $k > 0$ be an integer and D be a digraph with a matching M such that $|M| = k$. Suppose that $V(D) - V(M)$ has a subset X with $|X| \geq 2$ such that for any $v \in X$, $d(v) \geq 2k - 1$. If X has at least one vertex u such that $d(u) \geq 2k + 1$, then there exists a matching M' in D such that $|M| < |M'|$.*

Proof. By contradiction, we assume that M is a maximum matching in D. By Theorem 2.1, D has no M-augmenting path. Let $u, v \in X$ be distinct vertices such that $d(u) \geq 2k + 1$. Since M is maximum, u and v are not adjacent in D, and so any vertices adjacent to u or v must be in $V(M)$. Since D has no M-augmenting path, we have the following observations.

(A) For each arc $e = [x, y] \in M$, exactly one in $\{[u, x], [v, y]\}$ can be in $A(D)$, and exactly one in $\{[u, y], [v, x]\}$ can be in $A(D)$. If not, by symmetry, we may assume that $[u, x], [v, y] \in A(D)$, and so $\{[u, x], [x, y], [v, y]\}$ induces an M-augmenting path in D.

(B) Since $d(u) \geq 2k + 1$, we may assume that M has an arc $e' = [x', y']$ such that $[u, x'], [u, y'] \in A(D)$.

(C) From Observation (A), for each arc $e = [x, y] \in M$, if $[u, x] \in A(D)$ and $[u, y] \in A(D)$, then $[v, x], [v, y] \notin A(D)$, and so $|(\{u, v\}, \{x, y\})_D \cup (\{x, y\}, \{u, v\})_D| \leq 4$.

(D) From Observation (A), for each arc $e = [x, y] \in M$, if $[u, x] \in A(D)$ and $[v, x] \in A(D)$, then $[u, y], [v, y] \notin A(D)$, and so $|(\{u, v\}, \{x, y\})_D \cup (\{x, y\}, \{u, v\})_D| \leq 4$.

(E) From Observation (A), for each arc $e = [x, y] \in M$, if $[v, x] \in A(D)$ and $[v, y] \in A(D)$, then $[u, x], [u, y] \notin A(D)$, and so $|(\{u, v\}, \{x, y\})_D \cup (\{x, y\}, \{u, v\})_D| \leq 4$.

It follows from Observations (C), (D), and (E) that $|\partial(u) \cup \partial(v)| \leq 4|M|$. As $d(v) \geq 2k - 1$ and $d(u) \geq 2k + 1$, we have

$$4k = (2k - 1) + (2k + 1) \leq |\partial(u) \cup \partial(v)| \leq 4|M| = 4k.$$

This implies that every arc in M is adjacent to exactly 4 arcs in $\partial_D(u) \cup \partial_D(v)$. From Observation (B) and by the fact that D has no M-augmenting path, we must have $[v, x'], [v, y'] \notin A(D)$, and so v can only be adjacent to $V(M) - \{x', y'\}$. As $d(v) \geq 2k - 1 = 2(k - 1) + 1$, there must be an arc $[x'', y''] \in M$ such that $[v, x''], [v, y''] \in A(D)$. Define

$$M_u = \{[x, y] \in M : [u, x], [u, y] \in A(D)\} \text{ and } M_v = \{[x, y] \in M : [v, x], [v, y] \in A(D)\}.$$

Since D has no M-augmenting path, $M_u \cap M_v = \emptyset$. Let $M' = M - (M_u \cup M_v)$. Again by the fact that D has no M-augmenting path, for each arc $e = [x, y] \in M'$, at most one end of e is adjacent to vertices in $\{u, v\}$, and so

$$|(\{u\}, \{x, y\})_D \cup (\{x, y\}, \{u\})_D| \leq 2 \text{ and } |(\{v\}, \{x, y\})_D \cup (\{x, y\}, \{v\})_D| \leq 2.$$

It follows that $4|M_u| + 2|M'| \geq d(u) \geq 2k + 1$ and $4|M_v| + 2|M'| \geq d(v) \geq 2k - 1$. Since $4|M_u| + 2|M'|$ is even, we must have $4|M_u| + 2|M'| \geq 2k + 2$. It follows that

$$4k = 4|M| = (4|M_u| + 2|M'|) + (4|M_v| + 2|M'|) \geq (2k + 2) + (2k - 1) = 4k + 1,$$

a contradiction. This proves the lemma. ■

Corollary 2.4. *For every digraph D, $\lambda(D) \leq 2\alpha'(D)$.*

Proof. Let $\alpha'(D) = k$. By contradiction, we assume that $\lambda(D) \geq 2k + 1$. Hence $|V(D)| \geq 2k + 2$. Let M denote a maximum matching of D. Then $|V(D) - V(M)| \geq$

$2k+2-2k=2$. Since $\lambda(D) \geq 2k+1$, for every vertex $u \in V(D)-V(M)$, $d(u) \geq 2k+1$. It follows by Lemma 2.3 that M is not a maximum matching of D, and so a contradiction obtains. ■

Lemma 2.5. *Let $k > 0$ be an integer, D be a digraph on $n \geq 2k+2$ vertices and M be a maximum matching of D with $|M| = k$. Suppose that*

$$\min\{d^+(v), d^-(v)\} \geq k, \text{ for every vertex } v \in V(D). \tag{2}$$

Let $X = V(D) - V(M)$. Then

$$\text{for every } x \in X, \text{ we have } d^+(x) = d^-(x) = k. \tag{3}$$

Moreover, if $n \geq 2k+3$ or if $n = 2k+2$ and D is strong, then for every $e = [u, v] \in M$ and for every $x \in X$, we have the following conclusions.

(i) *There exists exactly one $v(e) \in \{u, v\}$ such that both $(v(e), x)$ and $(x, v(e))$ are in $A(D)$, and the vertex $u(e) \in \{u, v\} - \{v(e)\}$ is not adjacent to any vertex in X.*
(ii) *The set $\{u(e) : e \in M\}$ is an independent set in D such that $d^+(u(e)) = d^-(u(e)) = k$ for any $e \in M$ and such that for any $e, e' \in M$, $(u(e), v(e'))$, $(v(e'), u(e)) \in A(D)$.*

Proof. Let $N(X)$ denote the set of vertices in D that is adjacent to a vertex in X. Since M is a maximum matching, by Theorem 2.1

$$D \text{ does not have an } M\text{-augmenting path.} \tag{4}$$

By (4), we have $N(X) \subseteq V(M)$.

Since $n \geq 2k+2$ and $|M| = k$, we have $|X| \geq 2$. Since M is a maximum matching, it follows by (2) and by Lemma 2.3 that (3) must hold. In the rest of the proof for this lemma, we consider two cases.

Case 1. D is a digraph with $n = |V(D)| \geq 2k+3$. Let $m = |X|$. Then $m \geq 3$. By (3),

$$|(X, V(M))_D| = mk = |(V(M), X)_D|. \tag{5}$$

We further observed that, by (4),

$$\text{for any } e \in M, \text{ the vertices of } e \text{ are incident with at most } 2m \text{ arcs in } (X, V(M))_D \cup (V(M), X)_D. \tag{6}$$

Let $e = [u, v] \in M$. If $u, v \in N(X)$, then by (4), there must be a unique $x \in X$ such that x is adjacent to both u and v; and for any $x' \in X - \{x\}$, x' is not adjacent to either u nor v. By (3), x is adjacent to $2k$ vertices in $V(M)$. As $|(\{x\}, \{u, v\})_D \cup (\{u, v\}, \{x\})_D| \leq 4$, it follows by (5) and $m \geq 3$ that

$$\begin{aligned}|(X, V(M) - \{u, v\})_D| &+ |(V(M) - \{u, v\}, X)_D| \\ &\geq 2k(m-1) + (2k-4) = 2km - 4 \\ &\geq 2km - 2m + 2 = 2m(k-1) + 2 > 2m(k-1) + 1.\end{aligned}$$

If $k > 1$, then there must be an arc $a \in M - \{e\}$ such that the vertices of a are incident with at least $2m+1$ edges in $(X, V(M))_D \cup (V(M), X)_D$, contrary to (6). Hence in this case we must have $k = 1$ and $M = \{e\}$. As $|X| \geq 3$, it follows from (5) with $m \geq 3$ that D must have an M-augmenting path, contrary to (4). Hence for any $e = (u, v) \in M$, exactly one of u or v is adjacent to vertices in X.

Thus for each $e \in M$, let $v(e)$ denote the unique vertex of e that is adjacent to vertices in X and $u(e)$ the other vertex of e which is not adjacent to any vertex in X. Then as $|X| = m$, $|(v(e), X)_D \cup (X, v(e))_D| \le 2m$. It follows from (5) that

$$2km \ge \sum_{e \in M} |(v(e), X)_D \cup (X, v(e))_D| = |(X, V(M))_D| + |(V(M), X)_D| = 2mk.$$

This implies that for each $x \in X$ and for each $e \in M$, both $(x, v(e))$ and $(v(e), x)$ are in $A(D)$. This proves (i) for Case 1.

Let $Y = \{u(e) : e \in M\}$ and we shall show that Y is an independent set. In fact, if for some $e_1, e_2 \in M$, $[u(e_1), u(e_2)] \in A(D)$, then for any distinct $x_1, x_2 \in X$, $\{[x_1, v(e_1)], e_1, [u(e_1), u(e_2)], e_2, [x_2, v(e_2)]\}$ induces an M-augmenting path, contrary to (4). Thus each $u(e) \in Y$ can only be adjacent to vertices in $\{v(e) : e \in M\}$. As $|\{v(e) : e \in M\}| = k$, by (2), we conclude that for each $e \in M$, $d^+(u(e)) = d^-(u(e))$, and for any $e, e' \in M$, $(u(e), v(e'))$, $(v(e'), u(e)) \in A(D)$. This proves (ii) for Case 1.

Case 2. D is strong and $|V(D)| = 2k + 2$.

Then $X = \{w, z\}$. Let $M = \{e_1, \dots, e_k\}$. Let $M_w \subseteq M$ denote the arcs in M each of which has a vertex adjacent to w. We define M_z similarly. By (3), $|M_w| \ge \frac{k}{2}$ and $|M_z| \ge \frac{k}{2}$.

Subcase 1. $|M_w| = \frac{k}{2}$ or $|M_z| = \frac{k}{2}$.

Note that in this case, k must be even. We assume, without loss of generality, that $M_w = \{e_1, \dots, e_{\frac{k}{2}}\}$. By (4), we must have $M_z = \{e_{\frac{k}{2}+1}, \dots, e_k\}$. Again by (4), for each $x \in V(M_w)$ and $y \in V(M_z)$, we conclude that $[x, y] \notin A(D)$. Thus

$$(V(M_w) \cup \{w\}, V(M_z) \cup \{z\})_D \cup (V(M_z) \cup \{z\}, V(M_w) \cup \{w\})_D = \emptyset,$$

contrary to the assumption that D is strong. This shows that Subcase 1 cannot occur.

Subcase 2. $|M_w| > \frac{k}{2}$ and $|M_z| > \frac{k}{2}$.

Therefore, $M_w \cap M_z \ne \emptyset$. Note that by (4), if an arc $e \in M$ whose vertices are adjacent to both w and z, then exactly one vertex of e can be adjacent to both w and z. Let $M' = M_w \cap M_z = \{e'_i = [x_i, y_i], (i = 1, ..., d;\ 1 \le d \le k)\} \subseteq M$. Without lose of generality, we assume that each e'_i has a unique vertex x_i with $[x_i, w], [x_i, z] \in A(D)$. Let $M'' = M - M' = \{e''_j = [r_j, s_j], (j = 1, ..., k - d)\}$. We justify the following observations.

(A) By (4), for each y_i, $[y_i, w], [y_i, z] \notin A(D)$.

(B) The set $\{y_1, y_2, \dots, y_d\}$ must be an independent set. This is warranted by (4).

(C) Suppose that Lemma 2.5(i) or (ii) does not hold. Then $d \le k - 1$.

In fact, if $d = k$, then $M = M'$ and each x_i is adjacent to both w and z. By (3), for each $e_i = [x_i, y_i] \in M$, we must have (x_i, w), (x_i, z), (w, x_i), $(z, x_i) \in A(D)$. Hence Lemma 2.5(i) must hold. Furthermore, by Observations (A) and (B), each y_i can only be adjacent to $\{x_1, x_2, \dots, x_d\}$. By (2), for any i, we must have $d^+(y_i) = d^-(y_i) = k$, and for any $1 \le i, i' \le k$, we must have $(x_i, y_{i'}), (y_{i'}, x_i) \in A(D)$. Hence Lemma 2.5(ii) holds as well.

(D) From Observation (A), for each $e = [x_i, y_i] \in M'$, $|(\{x_i, y_i\}, \{w, z\})_D \cup (\{w, z\}, \{x_i, y_i\})_D| \le 4$.

(E) For each j with $1 \leq j \leq k-d$, there exists exactly one vertex in $\{w,z\}$ that is adjacent to both r_j and s_j.
By the definition of M'', for each j with $1 \leq j \leq k-d$, there exists at most one vertex in $\{w,z\}$ that is adjacent to both r_j and s_j. By contradiction, we assume that $[r_1,s_1] \in M_w - M'$ with $[w,r_1] \in A(D)$ and $[w,s_1] \notin A(D)$. Then $|(\{w,z\},\{r_1,s_1\})_D \cup (\{r_1,s_1\},\{w,z\})_D| \leq 2$. For any other $[r_j,s_j] \in M''$ with $j \geq 2$, we have $|(\{w,z\},\{r_j,s_j\})_D \cup (\{r_j,s_j\},\{w,z\})_D| \leq 4$. It follows from (3) and Observation (D) that

$$\begin{aligned}4k &= |(\{w,z\},V(M))_D \cup (V(M),\{w,z\})_D| \\ &= |(\{w,z\},V(M'))_D \cup (V(M'),\{w,z\})_D| \\ &\quad + |(\{w,z\},\{r_1,s_1\})_D \cup (\{r_1,s_1\},\{w,z\})_D| \\ &\quad + \sum_{j=2}^{k-d} |(\{w,z\},\{r_j,s_j\})_D \cup (\{r_j,s_j\},\{w,z\})_D| \leq 4(k-1)+2 < 4k,\end{aligned}$$

a contradiction. This justifies Observation (E).

(F) For any $[x_i,y_i] \in M'$ and for any $[r_j,s_j] \in M''$, $[y_i,r_j], [y_i,s_j] \notin A(D)$. In fact, if $[y_i,r_j] \in A(D)$, then by Observation (E), we may assume that $[r_j,s_j] \in M_w$, and so $\{[w,s_j],[r_j,s_j],[s_j,y_i],[x_i,y_i],[z,x_i]\}$ will induce an M-augmenting path, contrary to (4). This justifies (F).

We argue by contradiction to prove (i) and (ii). As $M' = M_w \cap M_z \neq \emptyset$, $d \geq 1$ and so y_1 exists. By Observations (A), (B), and (F), y_1 can only be adjacent to $\{x_1,x_2,\ldots,x_d\}$. Hence $d(y_1) \leq 2|\{x_1,x_2,\ldots,x_d\}| = 2d$. By (2) and by Observation (C), that $2k \leq d(y_1) \leq 2d \leq 2(k-1)$, a contradiction. This implies that we must have $d=k$, and so by Observation (C), this proves the lemma for Case 2. ■

3. PROOF OF THE MAIN RESULT

Throughout this section, D denotes a digraph and $k \geq 1$ be an integer. In this section, we shall prove a slightly stronger version than Theorem 1.5, stated as Theorem 3.1 below. By (1), Theorem 1.5 follows immediately from Theorem 3.1(i). As a byproduct in the argument, we also prove that if for every vertex $v \in V(D)$, $\min\{d^+(v), d^-(v)\} \geq \alpha'(D) > 0$, then $\lambda(D) \geq \alpha'(D)$, as stated in Theorem 3.1(ii) below.

Theorem 3.1. *Let $k > 0$ be an integer, D be a digraph on $n \geq 2k$ vertices with $\alpha'(D) = k$. Suppose that if $n \leq 2k+2$, then D is strong. If*

$$\min\{d^+(v), d^-(v)\} \geq k, \text{ for every vertex } v \in V(D). \tag{7}$$

then each of the following holds.

(i) *D is supereulerian.*
(ii) $\lambda(D) \geq k$.

Proof. Let M be a matching of maximum size of D. We proceed our proof in the following cases.

Case 1. $2k \leq n \leq 2k + 1$.

If $n = 2k$, then by theorem 2.2, D is hamiltonian, and so D is also supereulerian. If $n = 2k + 1$, then by Theorem 1.2, D is also supereulerian. It remains to prove that $\lambda(D) \geq k$.

Let X be an arbitrary nonempty proper subset of $V(D)$, and let $Y = V(D) - X$. Since $|X| + |Y| = n \leq 2k + 1$, either $1 \leq |X| \leq k$ or $1 \leq |Y| \leq k$. By symmetry, we may assume that $1 \leq |X| = m \leq k$. By (7), for each $x \in X$. $|(\{x\}, Y)_D| \geq k - (m - 1)$. Thus $|\partial_D^+(X)| \geq m(k - (m - 1)) = -m^2 + m(k + 1)$. As this is a quadratic function with $1 \leq m \leq k$, it follows that $|\partial_D^+(X)| \geq -m^2 + m(k + 1) \geq k$, and so by (1), $\lambda(D) \geq k$. This proves Case 1.

Case 2. $n \geq 2k + 2$.

(i). Since $k \geq 1$, by (7), D must has an eulerian subdigraph. Let S be an eulerian subdigraph of D with

$$|V(S)| \text{ is maximized among all eulerian subdigraphs of } D. \tag{8}$$

Let $s = |V(S)|$. If $s = n$, then S is a spanning eulerian subdigraph of D and we are done. By contradiction, we assume that $n > s \geq 1$. Hence $V(D) - V(S) \neq \emptyset$. We are to prove (i) in the following two subcases.

Subcase 2.1. $(V(D) - V(S)) - V(M) \neq \emptyset$.

Pick $v \in (V(D) - V(S)) - V(M)$. Since $A(S) \neq \emptyset$, we pick an arc $e = [x, y] \in A(S)$. Since M is a maximum matching of D, $V(M) \cap \{x, y\} \neq \emptyset$, and so we may assume that $x \in V(M)$. Therefore, there exists an arc $a = [x, z] \in M$. If $x, z \in V(S)$, then by Lemma 2.5(i), there exists a vertex $v(a) \in \{x, z\}$ such that $(v, v(a)), (v(a), v) \in A(D)$. It follows that $A(S) \cup \{(v, v(a)), (v(a), v)\}$ induces an eulerian subdigraph S_1 with $|V(S_1)| > |V(S)|$, contrary to (8). Hence we may assume that $z \notin V(S)$. By Lemma 2.5(ii), we have $(x, z), (z, x) \in A(D)$, and so $A(S) \cup \{(x, z), (z, x)\}$ induces an eulerian subdigraph S_1 with $|V(S_1)| > |V(S)|$, contrary to (8) also. This completes the proof for Subcase 2.1.

Subcase 2.2. $V(D) - V(S) \subseteq V(M)$.

Pick $v \in V(D) - V(S)$. Since $v \in V(M)$, there must be an arc $a = [u, v] \in M$. If $u \in V(S)$, then by Lemma 2.5(ii), both $(u, v), (v, u) \in A(D)$. Hence $A(S) \cup \{(u, v), (v, u)\}$ induces an eulerian subdigraph S_6 of D with $|V(S_6)| > |V(S)|$, contrary to (8). Therefore, we must have $u \notin V(S)$.

Since $n \geq 2k + 2 = |V(M)| + 2$, there must be a vertex $w \in V(D) - V(M)$. Since $V(D) - V(S) \subseteq V(M)$, $w \in V(S)$. By Lemma 2.5(i), there must be a $v(a) \in \{u, v\}$ such that $(w, v(a)), (v(a), w) \in A(D)$. It follows that $A(S) \cup \{(w, v(a)), (v(a), w)\}$ induces an eulerian subdigraph S_7 of D with $|V(S_7)| > |V(S)|$, contrary to (8). This completes the proof of Theorem 3.1(i).

(ii). Let X satisfying $\emptyset \neq X \subset V(D)$ be an arbitrary nonempty proper vertex subset. We are to prove $|\partial_D^+(X)| \geq k$.

Let $Z = V(D) - V(M)$. By Lemma 2.5 (i), for any $e = [u, v] \in M$, there exists a unique $v(e) \in \{u, v\}$ such that for any $z \in Z$, $(z, v(e)), (v(e), z) \in A(D)$. Let $M_v = \{v(e) : e \in M\}$, and $M_u = V(M) - M_v$. Let $m \geq 2$ be the integer satisfying $n = 2k + m$. By Lemma 2.5, for each $v \in M_v$, and for any $u \in Z \cup M_u$, $(v, u), (u, v) \in$

$A(D)$. It follows by Lemma 2.5 that

$$\min\{d^+(v), d^-(v)\} \geq k+m, \text{ for any } v \in M_v; \text{ and } d^+(z) = d^-(z) = k \text{ for any } z \in M_u \cup Z. \quad (9)$$

We consider the following cases.

Case 1. $(M_u \cup Z) \subseteq X$ (or $(M_u \cup Z) \cap X = \emptyset$). We assume that $M_u \cup Z \subseteq X$ as by symmetry, the proof for $M_u \cup Z \cap X = \emptyset$ is similar. As $V(D) = M_u \cup M_v \cup Z$, there exists a $y \in M_v - X \subset V(D) - X$. By Lemma 2.5, $|\partial_D^+(X)| \geq |(M_u \cup Z, \{y\})_D| = |Z| + |M_u| = k+m > k$.

Case 2. $M_v \subseteq X$ (or $M_v \subseteq V(D) - X$). We assume that $M_v \subseteq X$, as by symmetry, the proof for $M_v \subseteq V(D) - X$ is similar. Then $M_u \cup Z - X \neq \emptyset$. Pick $y \in M_u \cup Z - X$. Then by Lemma 2.5, $|\partial_D^+(X)| \geq |(M_v, \{y\})_D| = |M_v| = k$.

Case 3. Both $M_u \cup Z - X \neq \emptyset$ and $X \cap (M_u \cup Z) \neq \emptyset$, and both $M_v - X \neq \emptyset$ and $X \cap M_v \neq \emptyset$.

Pick $x \in X \cap (M_u \cup Z)$ and $y \in (M_u \cup Z) - X$. Then by Lemma 2.5,

$$|\partial_D^+(X)| \geq |(\{x\}, M_v - X)_D| + |(M_v \cap X, \{y\})_D| = |M_v - X| + |M_v \cap X| = |M_v| = k.$$

It follows that we always have $|\partial_D^+(X)| \geq k$, and so $\lambda(D) \geq k$. This proves (ii). ■

REFERENCES

[1] M. An and L. Xiong, Supereulerian graphs, collapsible graphs and matchings, Acta Mathematicae Applicatae Sinia (English Series), to appear.

[2] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, 2nd edn., Springer-Verlag, London, 2009.

[3] J. Bang-Jensen and A. Maddaloni, Sufficient conditions for a digraph to be supereulerian, J Graph Theory, to appear.

[4] C. Berge, Two theorems in graph theory, Proc Nat Acad Sci USA 43 (1957), 842–844.

[5] F. T. Boesch, C. Suffel, and R. Tindell, The spanning subgraphs of eulerian graphs, J Graph Theory 1 (1977), 79–84.

[6] J. A. Bondy and U. S. R. Murty, Graph Theory, Springer, New York, 2008.

[7] P. A. Catlin, Supereulerian graphs: a survey, J Graph Theory 16 (1992), 177–196.

[8] Z. H. Chen H.-J. Lai, Reduction techniques for super-Eulerian graphs and related topics-a survey, Combinatorics and graph theory' 95, Vol. 1 (Hefei), World Sci. Publishing, River Edge, NJ, 1995, pp. 53–69.

[9] V. Chvátal and P. Erdös, A note on Hamiltonian circuits, Discrete Math 2 (1972), 111–113.

[10] G. Gutin, Cycles and paths in directed graphs. PhD thesis, School of Mathematics, Tel Aviv University, 1993.

[11] G. Gutin, Connected (g,f)-factors and supereulerian digraphs, Ars Combin 54 (2000), 311–317.

[12] L. Han, H.-J. Lai, L. Xiong and H. Yan, The Chvátal-Erdös condition for supereulerian graphs and the Hamiltonian index, Discrete Math, 310 (2010), 2082–2090.

[13] Y. Hong, H.-J. Lai and Q. Liu, Supereulerian digraphs, Discrete Math 330 (2014), 87–95.

[14] H.-J. Lai, Y. Shao and H. Yan, An update on supereulerian graphs, WSEAS Trans Math 12 (2013), 926–940.

[15] H.-J. Lai and H. Y. Yan, Supereulerian graphs and matchings, Appl Math Lett 24 (2011), 1867–1869.

[16] H. Meyniel, Une condition suffisante d'existence d'un circuit hamiltonien dans un graphe orienté, J Combin Theory Ser B 14 (1973), 137–147.

[17] W. R. Pulleyblank, A note on graphs spanned by Eulerian graphs, J Graph Theory 3 (1979), 309–310.

[18] C. Thomassen. Long cycles in digraphs, Proc London Math Soc 42(3) (1981), 231–251.

[19] R. Tian and L. Xiong, The Chvátal-Erdös condition for a graph to have a spanning trail, Graphs Combin, to appear.

[20] J. Xu, P. Li, Z. Miao, K. Wang and H.-J. Lai, Supereulerian graphs with small matching number and 2-connected hamiltonian claw-free graphs, Int J Comp Math 91 (2014), 1662–1672.