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Algorithm for constraint partial inverse matroid problem with weight increase forbidden

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ABSTRACT

In a partial inverse matroid problem, given a matroid $M = (S, \mathcal{I})$, a real valued weight function w on S, and an independent set $I_0 \in \mathcal{I}$, the goal is to modify the weight w as small as possible to a new weight $\bar{w}$ such that there exists a $\bar{w}$-maximum base containing I_0. In this paper, we study a constraint version of the partial inverse matroid problem in which the weight can only be decreased. A polynomial time algorithm is presented under l_∞-norm.

1. Introduction

In an *inverse problem*, one is given an optimization problem with a weight function w, as well as a feasible solution X_0 which might not be optimal with respect to w, the goal is to modify w to a new weight function $\bar{w}$ as small as possible such that X_0 becomes optimal with respect to $\bar{w}$. An inverse problem can be viewed as inferring parameters of a coarse model from an observed solution. For example, suppose one has a model to predict earthquake, in which parameters are not accurate. It is desirable to adjust the parameters according to an observed actual propagation of earthquake, such that the model can produce the observed solution and the adjustment is as small as possible. For other applications of inverse combinatorial problems, the readers may refer to [9] and references therein.

In some circumstances, because of limitation of conditions, only a partial solution is known. In a *partial inverse problem*, given a partial solution X_0, the goal is to modify w to $\bar{w}$ as small as possible such that there exists an optimal solution with respect to $\bar{w}$ which contains X_0. In this paper, we study a constraint partial inverse matroid problem. To define this problem, we first introduce some terminologies.

Suppose S is a finite set and $\mathcal{I} \subseteq 2^S$ is a family of subsets of S. If $M = (S, \mathcal{I})$ satisfies the following two properties, then it is a *matroid*: (i) (*hereditary property*) for any $I \in \mathcal{I}$, any subset of I is also in $\mathcal{I}$; (ii) (*augmenting property*) for any $I, I' \in \mathcal{I}$ with $|I| < |I'|$, there exists an element $x \in I' \setminus I$ such that $I \cup \{x\} \in \mathcal{I}$. Subsets in $\mathcal{I}$ are called *independent sets* and subsets of S which are not in $\mathcal{I}$ are called *dependent*. Maximal independent sets are called *bases*. It is well-known that all bases of a matroid have the same cardinality, which is called the *rank* of the matroid, denoted as r_M. Given a weight function

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$w : S \mapsto \mathbb{R}^+$, the weight of a subset $X \subseteq S$ is $w(X) = \sum_{x \in X} w(x)$. A base with the maximum weight under w is called *w-maximum*.

Notice that a weight function on S can be viewed as a vector of dimension $|S|$. For a vector $x = (x_1, \ldots, x_n)$, use $\|x\|$ to denote the *norm* of x. Some widely used norms include the l_p-norm $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ for $1 \le p < \infty$, l_∞-norm $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$, and Hamming distance $\|x\|_H$ which is the number of nonzero elements of x. For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we write $x \ge y$ if for each $1 \le i \le n$, $x_i \ge y_i$. A norm $\|\cdot\|$ is nondecreasing if $\|x\| \ge \|y\|$ whenever $x \ge y$. Norms $\|\cdot\|_p$ for $1 \le p \le \infty$ and $\|\cdot\|_H$ are all nondecreasing. In this paper, we always assume that norms used are nondecreasing.

The *constraint partial inverse matroid problem* (CPIM) studied in this paper is formally defined as follows.

Definition 1.1 *(CPIM$^-$).* Given a matroid $M = (S, \mathcal{I})$, a nonnegative weight function w, a nonnegative constraint function b, and an independent set $I_0 \in \mathcal{I}$, the goal is to decrease weight function w to a new weight function $\bar{w}$ such that

(i) $0 \le w(x) - \bar{w}(x) \le b(x)$ for every $x \in S$,
(ii) I_0 is contained in a $\bar{w}$-maximum base of M,
(iii) $\|\bar{w} - w\|$ is minimized.

1.1. Related works

Burton and Toint [2] were the first to study inverse combinatorial problem. Since then, there have been a large quantities of studies on various inverse combinatorial problems. For a comprehensive study on this topic, the readers may refer to the surveys of Heuberger [9] and Demange and Monnot [5]. In the following, we only present some results which are closely related to this paper, namely studies on inverse matroid problem and partial inverse problem.

It has long been known that the inverse matroid problem can be solved in polynomial time by solving a linear program [1]. Making use of structural properties of the problem, DellAmico et al. [4] proposed a faster algorithm which runs in time $O(nr_M + r_M^3 + n\varphi)$, where $n = |S|$ and φ is the time complexity of finding the unique circuit when an element is added to a base.

Partial inverse problem has been incorporated into several classical combinatorial optimization problems including partial inverse linear program [8], partial inverse minimum cut problem [7], partial inverse assignment problem [13,14], partial inverse sorting problem [15], and partial inverse most unbalanced spanning tree problem [12]. Lai and Orlin [10] studied a special partial inverse optimization problem in which the partial solution contains only one element. Studies in this track usually follow two lines: either prove that the problem is NP-hard, or find out a polynomial time algorithm for it.

This paper is mainly inspired by [3] in which Cai et al. gave an efficient algorithm for the partial inverse minimum spanning tree problem when weight increase is forbidden. In fact, this result can be generalized to the constraint partial inverse matroid problem in which weight decrease is forbidden (CPIM$^+$) [16]. A natural question is: what if weight increase is forbidden? It turns out that such a small change of statement greatly increases the difficulty. One reason is that the work for CPIM$^+$ is based on the observation that an optimal solution can be obtained by increasing weights only on those elements in I_0 and an explicit expression for the optimal solution can be obtained. While for CPIM$^-$, there are generally a lot more edges outside of I_0 than in I_0, the changes of whose weights closely inter-relate with each other such that it is difficult to obtain an explicit expression for an optimal solution.

1.2. Our contribution

In this paper, we study the CPIM$^-$ problem. First, we show that if the partial solution I_0 is a base (that is, I_0 is a full solution), then the problem can be solved in time $O((n - r_M)(r_M + \varphi))$, where φ is the time to find a fundamental circuit. Making use of this result, we give an $O(k \log k + k\varphi_1 + (n - r_M)(r_M + \varphi))$ time algorithm to find the optimal solution to the CPIM$^-$ problem under an l_∞-norm, where $k = n - |I_0|$ and φ_1 is the time to determine whether a set is independent. An example is given to show that such an algorithm is no longer valid under an l_p-norm with $1 \le p < \infty$ or under the Hamming distance $\|\cdot\|_H$.

The paper is organized as follows. In Section 2, we introduce some terminologies and properties which will be used in the algorithm. Polynomial time algorithm for CIMP$^-$ under l_∞-norm is presented in Section 3. Some discussions are given in Section 4.

2. Preliminaries and properties

In this section, we introduce some symbols and results which will be used in the algorithms as well as their analysis. We refer to [11] for the concepts of matroid. The proofs of the following lemmas can be found in [16].

A *circuit* in a matroid $M = (S, \mathcal{I})$ is a minimal dependent set. For a base B and any element $x \in S \setminus B$, there is a unique circuit in $B + x$, which is called the *fundamental circuit* with respect to B and x, and denoted as $C(B, x)$. By the uniqueness of $C(B, x)$, an element y is in $C(B, x)$ if and only if $B + x - y$ is a base.

For a base B and an element $x \in B$, call

$$K(B, x) = \{y : B - x + y \text{ is a base}\}$$

the *fundamental cut* with respect to B and x. Notice that

$$K(B, x) \cap B = \{x\}. \tag{1}$$

It should be noticed that the above $K(B, x)$ is equivalent with *fundamental cocircuit* defined in Exercise 10 on page 78 of [11].

For an element $x \in S$, let $\mathcal{C}_x$ be the set of circuits containing x. Denote by $\mathcal{P}_x = \{x\} \cup \{C - x : C \in \mathcal{C}_x\}$.

Lemma 2.1. *For any base B, any element $x \in B$, and any $P \in \mathcal{P}_x$, $K(B, x) \cap P \neq \emptyset$.*

We call a base B to be a *w-maximum extension of I_0* if B contains the given independent set I_0 and $w(B \setminus I_0)$ is maximized. Notice that a w-maximum extension of I_0 is not the same as a w-maximum base. For example, let $S = \{x, y, z\}$, $\mathcal{I} = \{I \subseteq S : |I| \leq 2\}$, $w(x) = w(y) = 2$ and $w(z) = 1$. If $I_0 = \{z\}$, then a w-maximum extension of I_0 is $\{x, z\}$ with weight 3. However, the unique w-maximum base is $\{x, y\}$ with weight 4. So, there is no w-maximum base containing I_0. If we modify w to w' by letting $w'(x) = 1$ and keeping the weights of y and z, then $\{y, z\}$ is a w'-maximum base containing I_0. The following lemma characterizes w-maximum extensions.

Lemma 2.2. *For a matroid $M = (S, \mathcal{I})$, an independent set I_0, and a base B containing I_0, B is a w-maximum extension of I_0 if and only if one of the following conditions holds:*

(i) *for any $x \in B \setminus I_0$, $w(x) = \max\{w(y) : y \in K(B, x)\}$;*
(ii) *for any $x \in S \setminus B$, $w(x) = \min\{w(y) : y \in C(B, x) \setminus I_0\}$.*

In particular, taking $I_0 = \emptyset$ in Lemma 2.2, we have the following characterization of w-maximum base. The second characterization is a classic result of the Greedy Algorithm Theorem [6].

Corollary 2.3. *For a matroid $M = (S, \mathcal{I})$, a base B is w-maximum if and only if one of the following conditions holds:*

(i) *for any $x \in B$, $w(x) = \max\{w(y) : y \in K(B, x)\}$;*
(ii) *for any $x \in S \setminus B$, $w(x) = \min\{w(y) : y \in C(B, x)\}$.*

Notice that a w-maximum extension of I_0 can be found in polynomial time using greedy strategy which is similar to finding a w-maximum base of a matroid. The time complexity is $O(k \log k + k\varphi_1)$, where $k = n - |I_0|$ and φ_1 is the time to determine whether a set is in $\mathcal{I}$.

3. Efficient algorithm for CPIM$^-$ under $\|\cdot\|_\infty$-norm

In this section, we first consider the case when I_0 is a base, which is in fact a constraint *full* inverse matroid problem, we denote it as CFIM$^-$.

3.1. Optimal solution to CFIM$^-$

Suppose B_0 is a base of $M = (S, \mathcal{I})$ which is not necessarily w-maximum. Let

$$\bar{w}(x) = \begin{cases} w(x), & \text{if } x \in B_0, \\ \min\{w(y) : y \in C(B_0, x)\}, & \text{if } x \in S \setminus B_0. \end{cases} \tag{2}$$

Notice that x is the only element of $C(B_0, x)$ which does not belong to B_0. Hence, since elements in B_0 do not change weights, the above formula is well-defined.

Theorem 3.1. *An instance of the CFIM$^-$ problem is feasible if and only if $w(x) - \bar{w}(x) \leq b(x)$ holds for any $x \in S \setminus B_0$. Furthermore, when the instance is feasible, $\bar{w}$ defined in (2) is an optimal solution, which can be determined in time $O((n - r_M)(r_M + \varphi))$, where φ is the time to find the unique circuit when an element is added into a base.*

Proof. Notice that $\bar{w}(x) \leq w(x)$ since $x \in C(B_0, x)$.

We first show that any feasible solution w' to the CFIM$^-$ problem has $w'(x) \leq \bar{w}(x)$ for any $x \in S$. If $x \in B_0$, then $w'(x) \leq w(x) = \bar{w}(x)$. If there exists an element $x \in S \setminus B_0$ with $w'(x) > \bar{w}(x)$, by the definition of $\bar{w}$ in (2), there exists an element $y \in C(B_0, x)$ such that $w'(x) > w(y)$. Let $B' = B_0 + x - y$. Then B' is a base with $w'(B') = w'(B_0) + w'(x) - w'(y) \geq w'(B_0) + w'(x) - w(y) > w'(B_0)$, contradicting that B_0 is a w'-maximum base.

Algorithm 1 Algorithm for CPIM$^-$ under $\|\cdot\|_\infty$-norm.

Input: A matroid $M=(S,\mathcal{I})$, a weight function $w: S \mapsto \mathbb{R}^+$, a bound function $b: S \mapsto \mathbb{R}^+$, an independent set I_0.
Output: Either claim that the instance for the CPIM$^-$ problem is infeasible, or output an optimal solution $\bar{w}$ under the $\|\cdot\|_\infty$ norm.

Find a w-maximum extension B_0 of I_0.
Set $\bar{w}(x)=w(x)$ for each $x \in B_0$ and set $\bar{w}(x)=\min\{w(y): y \in C(B_0,x)\}$ for each $x \in S \setminus B_0$.
if there is an $x \in S \setminus B_0$ with $\bar{w}(x) < w(x) - b(x)$ **then**
    Output "infeasible"
else
    Output $\bar{w}$.
end if

Next, we show that B_0 is a $\bar{w}$-maximum base. By Corollary 2.3 (*ii*), it suffices to show that for any $x \in S \setminus B_0$,

$$\bar{w}(x) \leq \min\{\bar{w}(y): y \in C(B_0,x) \setminus \{x\}\},$$

which is equivalent to showing that

$$\bar{w}(x) \leq \min\{w(y): y \in C(B_0,x) \setminus \{x\}\} \tag{3}$$

because $\bar{w}(y)=w(y)$ holds for any $y \in C(B_0,x) \setminus \{x\} \subseteq B_0$. However, (3) clearly follows from (2).

Then, by the monotonicity of the norm $\|\cdot\|$, the theorem follows. The time complexity is obvious. $\square$

3.2. Algorithm for CPIM$^-$ under $\|\cdot\|_\infty$-norm

The algorithm for CPIM$^-$ under $\|\cdot\|_\infty$-norm is described in Algorithm 1.

Theorem 3.2. *Algorithm 1 solves the CPIM$^-$ problem under $\|\cdot\|_\infty$-norm in time $O(k\log k + k\varphi_1 + (n-r_M)(r_M+\varphi))$, where $k = n-|I_0|$, φ_1 is the time to determine whether a set is independent and φ is the time to find the unique circuit when an element is added into a base.*

Proof. Suppose w' is an optimal solution to the CPIM$^-$ problem under $\|\cdot\|_\infty$-norm and B' is a w'-maximum base containing I_0. Assume that the bound constraint b is satisfied by $\bar{w}$. We are to show that $\|w-\bar{w}\|_\infty = \|w-w'\|_\infty$ (and thus $\bar{w}$ is an optimal solution).

Let x be an element with $w(x)-\bar{w}(x) = \|w-\bar{w}\|_\infty$. We may assume that B_0 is not a w-maximum base (otherwise w itself is an optimal solution and we need not do anything more). So, $w(x)-\bar{w}(x) > 0$. It follows that $x \notin B_0$ since otherwise $w(x)-\bar{w}(x)=0$ by the definition of $\bar{w}$.

By Theorem 3.1, B_0 is a $\bar{w}$-maximum base. If $\bar{w}$ is not an optimal solution, then $\|w-\bar{w}\|_\infty > \|w-w'\|_\infty$. It follows that $w(x)-w'(x) \leq \|w-w'\|_\infty < \|w-\bar{w}\|_\infty = w(x)-\bar{w}(x)$, and thus

$$w'(x) > \bar{w}(x). \tag{4}$$

Because B_0 is a w-maximum extension of I_0 and $x \notin B_0$, by Lemma 2.2 (*ii*),

$$w(x) = \min\{w(y): y \in C(B_0,x) \setminus I_0\}. \tag{5}$$

Since B_0 is a $\bar{w}$-maximum base, by Corollary 2.3 (*ii*) and the fact $x \notin B_0$,

$$\bar{w}(x) = \min\{\bar{w}(y): y \in C(B_0,x)\},$$

which is equivalent to

$$\bar{w}(x) = \min\{w(y): y \in C(B_0,x)\},$$

since elements in B_0 do not change their weights. If $\bar{w}(x) = \min\{w(y): y \in C(B_0,x) \setminus I_0\}$, then by (5), we have $\bar{w}(x) = w(x) \geq w'(x)$, contradicting (4). So,

$$\bar{w}(x) = \min\{w(y): y \in C(B_0,x) \cap I_0\}. \tag{6}$$

Let x' be the element with $w(x') = \min\{w(y): y \in C(B_0,x) \cap I_0\}$. Then

$$\bar{w}(x) = w(x'). \tag{7}$$

Since $x' \in I_0 \subseteq B'$, the fundamental cut $K(B',x')$ exists. Since $x' \in C(B_0,x)$, the set $P = C(B_0,x) - x'$ is in $\mathcal{P}_{x'}$. By Lemma 2.1, $K(B',x') \cap P \neq \emptyset$. Let x'' be an element in $K(B',x') \cap P$. If $x'' \in I_0$, then $x'' \in K(B',x') \cap I_0 \subseteq K(B',x') \cap B' = \{x'\}$ (see (1)), and thus $x'' = x'$. But $x'' = x'$ is impossible because $x' \notin P$. Hence $x'' \notin I_0$. Combining this with $x'' \in P \subseteq C(B_0,x)$, we have $x'' \in C(B_0,x) \setminus I_0$. Then by (5),

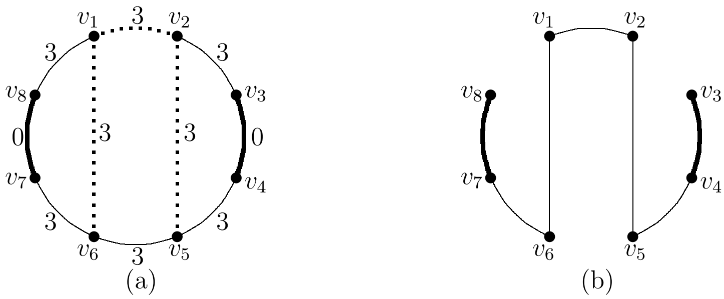

Fig. 1. An instance showing that Algorithm 1 does not work for the $\|\cdot\|_p$-norm with $1 \le p < \infty$ and Hamming distance. In (*a*), numbers indicate weights of corresponding edges, blackened edges are in I_0. Solid edges form a spanning tree T which does not have maximum weight. The algorithm modifies weights on edges v_1v_2, v_1v_6, v_2v_5 from 3 to 0. The $\|\cdot\|_p$ modification is $3^{1/p}\cdot 3$ and the $\|\cdot\|_H$ modification is 3. In (*b*), solid edges form another spanning tree T' containing I_0. In order to make T' to be of maximum weight, it suffices to modify weights on v_1v_8, v_2v_3 from 3 to 0 while the weight on v_5v_6 need not be modified. The $\|\cdot\|_p$ modification is $2^{1/p}\cdot 3$ and the $\|\cdot\|_H$ modification is 2.

$$w(x'') \ge w(x). \tag{8}$$

Because B' is a w'-maximum base and $x' \in B'$, by Corollary 2.3 (*i*),

$$w'(x') = \max\{w'(y) \colon y \in K(B', x')\}.$$

Hence $w'(x') \ge w'(x'')$. Combining this with $w'(x') \le w(x')$ and (7), we have

$$\bar{w}(x) \ge w'(x''). \tag{9}$$

Combining inequalities (8) and (9),

$$\|w - w'\|_\infty \ge w(x'') - w'(x'') \ge w(x) - \bar{w}(x) = \|w - \bar{w}\|_\infty,$$

contradicting our assumption that $\|w - \bar{w}\|_\infty > \|w - w'\|_\infty$. So, $\bar{w}$ is an optimal solution.

For the time complexity, notice that $O(k\log k + k\varphi_1)$ is the time to find B_0 and $O((n - r_M)(r_M + \varphi))$ is the time to determine $\bar{w}$ as in Subsection 3.1. □

4. Conclusion and discussion

In this paper, we studied the partial inverse matroid problem under the constraint that weights can only be decreased (CPIM$^-$). An efficient algorithm is given for CPIM$^-$ under $\|\cdot\|_\infty$-norm.

The example in Fig. 1 shows that Algorithm 1 does not work for the $\|\cdot\|_p$-norm with $1 \le p < \infty$ and Hamming distance $\|\cdot\|_H$. In this example, it is required to decrease weights as small as possible in terms of $\|\cdot\|_p$-norm or $\|\cdot\|_H$-norm such that there is a maximum weight spanning tree containing those blackened edges. Recall that maximum weight spanning tree is a special matroid optimization problem.

It is interesting to further explore whether efficient algorithm exists for CPIM$^-$ under other norms different from $\|\cdot\|_\infty$.

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