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On the lower bound of k-maximal digraphs

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ABSTRACT

For a digraph D, let $\lambda(D)$ be the arc-strong-connectivity of D. For an integer $k > 0$, a simple digraph D with $|V(D)| \geq k+1$ is k-**maximal** if every subdigraph H of D satisfies $\lambda(H) \leq k$ but for adding new arc to D results in a subdigraph H' with $\lambda(H') \geq k+1$. We prove that if D is a simple k-maximal digraph on $n > k+1 \geq 2$ vertices, then

$$|A(D)| \geq \binom{n}{2} + (n-1)k + \left\lfloor \frac{n}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right).$$

This bound is best possible. Furthermore, all extremal digraphs reaching this lower bound are characterized.

1. The problem

We consider finite simple graphs and simple digraphs. We generally use G to denote a graph and D a digraph, and follow [3] and [2] for undefined notation in graphs and in digraphs, respectively. In particular, $\kappa'(G)$ denotes the edge connectivity of a graph G and $\lambda(D)$ denotes the arc-strong-connectivity of a digraph D. If G is a simple graph, then G^c denotes the complement of G. If $X \subseteq E(G^c)$, then $G+X$ is the simple graph with vertex set $V(G)$ and edge set $E(G) \cup X$. We will use $G+e$ for $G+\{e\}$. Likewise, if D is a simple digraph, let D^c denote the complement of D. For $X \subseteq A(D^c)$ and $e \in A(D^c)$, we similarly define the simple digraphs $D+X$ and $D+e$, respectively. If H, K are subdigraphs of D, then $H \cup K$ is the subdigraph of D with vertex set $V(H) \cup V(K)$ and arc set $A(H) \cup A(K)$. Throughout this paper, we use the notation (u, v) to denote an arc oriented from u to v in a digraph. If $W \subseteq V(D)$ or if $W \subseteq A(D)$, then $D[W]$ denotes the subdigraph of D induced by W. For $v \in V(D)$, we use $D - v$ for $D[V(D) - \{v\}]$. For graphs H and G, we denote $H \subseteq G$ when H is a subgraph of G. Similarly, for digraphs H and D, $H \subseteq D$ means H is a subdigraph of D. We write $D \cong D'$ to represent the fact that D and D' are isomorphic digraphs.

Given a graph G, Matula [6–8] first studied the quantity

$$\overline{\kappa}'(G) = \max\{\kappa'(H) : H \subseteq G\}.$$

He called $\overline{\kappa}'(G)$ the **strength** of G. Mader [5] considered an extremal problem related to $\overline{\kappa}'(G)$. For an integer $k > 0$, a simple graph G with $|V(G)| \geq k+1$ is k-**maximal** if $\overline{\kappa}'(G) \leq k$ but for any edge $e \in E(G^c)$, $\overline{\kappa}'(G+e) > k$. In [5], Mader proved the following.

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Theorem 1.1 (*Mader [5]*). *If G is a k-maximal graph on $n > k \geq 1$ vertices, then*

$$|E(G)| \leq (n-k)k + \binom{k}{2}.$$

Furthermore, this bound is best possible.

It has been noted that being a k-maximal graph requires a certain level of edge density. Towards this direction, the following was proved in 1990.

Theorem 1.2 (*Lai, Theorem 2 of [4]*). *If G is a k-maximal graph on $n > k+1 \geq 2$ vertices, then*

$$|E(G)| \geq (n-1)k - \binom{k}{2}\left\lfloor \frac{n}{k+2} \right\rfloor.$$

Furthermore, this bound is best possible.

It is natural to consider extending the theorems above to digraphs. Towards this direction, for a digraph D, we define

$$\overline{\lambda}(D) = \max\{\lambda(H) : H \subseteq D\}.$$

Let $k \geq 0$ be an integer. A simple digraph D with $|V(D)| \geq k+1$ is k-**maximal** if $\overline{\lambda}(D) \leq k$ but for any arc $e \in A(D^c)$, $\overline{\lambda}(D+e) \geq k+1$. Following Matula [6], we may also call $\overline{\lambda}(D)$ the **strength** of digraph D and so a k-maximal digraph is also called a k-maximal strength digraph. For positive integers n and k satisfying $n \geq k+1$, define

$$\mathcal{D}(n,k) = \{D : D \text{ is a simple digraph with } |V(D)| = n \text{ and } D \text{ is } k\text{-maximal}\}.$$

Thus we are to investigate the upper and lower bounds of the set of numbers $\{|A(D)| : D \in \mathcal{D}(n,k)\}$. For notational convenience, if $h < k$, we define $\binom{h}{k} = 0$. The following has been obtained.

Theorem 1.3 (*Anderson et al. Theorem 1.2 of [1]*). *Let n and k be positive integers with $n \geq k+1$. If $D \in \mathcal{D}(n,k)$, then*

$$|A(D)| \leq k(2n-k-1) + \binom{n-k}{2}.$$

Furthermore, the bound is best possible.

In fact, all extremal digraphs in $\mathcal{D}(n,k)$ reaching this upper bound are characterized in [1]. The purpose of this research is to determine the lower bound. The following is the main result.

Theorem 1.4. *Let n and k be positive integers with $n \geq k+1$. If $D \in \mathcal{D}(n,k)$, then*

$$|A(D)| \geq \binom{n}{2} + (n-1)k + \left\lfloor \frac{n}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right).$$

Furthermore, the bound is best possible.

In the next section, we investigate properties of k-maximal digraphs. In Section 3, we present a constructive characterization of a family of k-maximal digraphs $\mathcal{E}'(k)$. In the last section, we will prove Theorem 1.4 and show that the members in the family $\mathcal{E}'(k)$ are precisely the digraphs attaining the upper bound in Theorem 1.4.

2. Properties of k-maximal digraphs

Throughout this section, n and k denote integers with $n > k \geq 0$. We present some properties of k-maximal digraphs to be utilized later. Let $\mathcal{D}(k)$ be the family of all k-maximal digraphs. Thus

$$\mathcal{D}(k) = \cup_{n \geq k+1} \mathcal{D}(n,k).$$

For any integer $n \geq 0$, let K_n^* denote the complete digraph on n vertices. Thus K_n^* is a simple digraph such that for any pair of distinct vertices $u, v \in V(K_n^*)$, both (u,v) and (v,u) are in $A(K_n^*)$. By definition, we observe the following

$$K_{k+1}^* \in \mathcal{D}(k) \quad \text{and} \quad \text{if } H \in \mathcal{D}(k) \quad \text{and} \quad |V(H)| = k+1, \quad \text{then } H \cong K_{k+1}^*. \tag{1}$$

Lemma 2.1 (*Lemma 2.1 of [1]*). *A digraph $D \in \mathcal{D}(0)$ if and only if D is an acyclic tournament.*

Lemma 2.1 indicates that we may exclude the case $k = 0$ in our study. Therefore, we will always assume that $k > 0$ in the rest of this paper. Following [2], if D is a digraph and if $X, Y \subseteq V(D)$, then define

$$(X,Y)_D = \{(x,y) \in A(D) : x \in X, y \in Y\}.$$

We further define that, for $X \subseteq V(D)$,

$$\partial_D^+(X) = (X, V(D) - X)_D \quad \text{and} \quad \partial_D^-(X) = (V(D) - X, X)_D.$$

For each $v \in V(D)$, we define

$$N_D^+(v) = \{u \in V(D) : (v, u) \in A(D)\} \quad \text{and} \quad N_D^-(v) = \{u \in V(D) : (u, v) \in A(D)\}.$$

When the digraph D is understood from the context, we sometimes omit the subscript D in the notations above. By the definition of arc-strong connectivity in [2], a digraph D satisfies $\lambda(D) \geq k$ if and only if for any nonempty proper subset $X \subset V(D)$, $|\partial_D^+(X)| \geq k$.

Definition 2.2. Let $H \in \mathcal{D}(k)$ and let $\{v_1, v_2, \ldots, v_k\} \subset V(H)$ be a subset of k distinct vertices. Let u be a vertex not in $V(H)$. Define a digraph $[H, K_1]_k$ ($[K_1, H]_k$, respectively) as follows:
(i) $V([H, K_1]_k) = V([K_1, H]_k) = V(H) \cup \{u\}$.
(ii) $A([H, K_1]_k) = A(H) \cup \{(v_1, u), (v_2, u), \ldots, (v_k, u)\} \cup \left(\bigcup_{v \in V(H)} \{(u, v)\}\right)$. ($A([K_1, H]_k) = A(H) \cup \{(u, v_1), (u, v_2), \ldots, (u, v_k)\} \cup \left(\bigcup_{v \in V(H)} \{(v, u)\}\right)$, respectively).
Note that each of $[H, K_1]_k$ and $[K_1, H]_k$ represents a family of graphs as the set $\{v_1, v_2, \ldots, v_k\} \subset V(H)$ may vary.

Definition 2.3. Let $H_1, H_2 \in \mathcal{D}(k)$, and let $\{u_1, u_2, \ldots, u_k\} \subset V(H_1)$ be a multiset of $V(H_1)$ and $\{v_1, v_2, \ldots, v_k\} \subset V(H_2)$ be a multiset of $V(H_2)$ such that all the arcs $(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)$ are distinct. Define a digraph $[H_1, H_2]_k$ as follows.
(i) $V([H_1, H_2]_k) = V(H_1) \cup V(H_2)$.
(ii) $A([H_1, H_2]_k) = A(H_1) \cup A(H_2) \cup \{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\} \cup \left(\bigcup_{u \in V(H_1), v \in V(H_2)} \{(v, u)\}\right)$.
Note that $[H_1, H_2]_k$ represents a family of digraphs.

Lemma 2.4 (*Corollary 2.6 of [1]*). *Let $D \in \mathcal{D}(k) - \{K_{k+1}^*\}$ be a digraph. Then there exists a nonempty proper subset $X \subseteq V(D)$ such that one of the following holds.*
(i) *$|X| = 1$, and for some $H \in \mathcal{D}(k)$, $D \in [K_1, H]_k$.*
(ii) *$|V(D) - X| = 1$ and for some $H \in \mathcal{D}(k)$, $D \in [H, K_1]_k$.*
(iii) *For some $H_1, H_2 \in \mathcal{D}(k)$, we have $D[X] = H_1$ and $D \in [H_1, H_2]_k$.*

3. Structure of k-maximal digraphs

Let $H(k, 2)$ be the digraph obtained from K_{k+2}^* by removing an arc from K_{k+2}^*. Note that if $D \cong H(k, 2)$, then D has exactly one vertex (to be denoted $x^-(D)$) of indegree k and exactly one vertex (to be denoted $x^+(D)$) of outdegree k.

Definition 3.1. Let n and k be positive integers. Define $\mathcal{S}(n, k)$ to be the set of all integral sequences $(s_1, s_2, \ldots, s_m)$ satisfying $s_1 + s_2 + \cdots + s_m = n$ such that $s_1 = k+2$, and for $i \geq 2$, $s_i \in \{1, -1, k+2, -(k+2)\}$. For any $\mathbf{s} = (s_1, s_2, \ldots, s_m) \in \mathcal{S}(n, k)$, define digraphs $L(\mathbf{s}) = L(s_1 s_2, \ldots, s_m)$ as follows.
(i) For $i = 1$, define $L_1 \cong H(k, 2)$.
(ii-A) For $i \geq 2$, if $s_i = 1$ ($s_i = -1$, respectively), then define $L_i \in [L_{i-1}, K_1]_k$ ($L_i \in [K_1, L_{i-1}]_k$, respectively).
(ii-B) For $i \geq 2$, if $s_i = k+2$ ($s_i = -(k+2)$, respectively), then define $L_i \in [L_{i-1}, H(k, 2)]_k$ ($L_i \in [H(k, 2), L_{i-1}]_k$, respectively), in such a way that for any $1 \leq t \leq i$ with $|s_t| = k + 2$ and with J_t denoting this $H(k, 2)$, we have $d_{L_i}^-(x^-(J_t)) \geq k + 1$ ($d_{L_i}^+(x^+(J_t)) \geq k + 1$, respectively).
(iii) Define $L(\mathbf{s}) = L_m$. By Definitions 2.2 and 2.3, each $L(\mathbf{s})$ represents a collection of digraphs.
(iv) Given $\mathbf{s} = (s_1, s_2, \ldots, s_m) \in \mathcal{S}(n, k)$, define $J_i = K_1$ if $|s_i| = 1$ and $J_i = H(k, 2)$ if $|s_i| = k + 2$. Then the sequence of digraphs $J_1, J_2, \ldots, J_m$ is called a **construction sequence** of $L(\mathbf{s})$.
(v) Define $\mathcal{E}(n, k) = \{L(\mathbf{s}) : \mathbf{s} \in \mathcal{S}(n, k)\}$ and $\mathcal{E}(k) = \{\cup_{n \geq k+2} (\mathcal{E}(n, k) \cup \{K_{k+1}^*\})\}$.

For a digraph D, an arc subset $W = (X, V(D) - X)_D$ for some proper nonempty subset X is called an **arc-cut**. If $|W| = t$ and W is an arc-cut, then W is called a t-**arc-cut**.

Observation 3.2. *We will make a few observations from Definition 3.1.*
(i) *By definition, $H(k, 2) \in \{L(\mathbf{s})\}$ with $\mathbf{s}$ being the sequence of only one term $k + 2$. Since there is only one arc $a \in A(H(k, 2)^c)$, we have $H(k, 2) + a = K_{k+2}^*$ and so*

$$H(k, 2) \in \mathcal{D}(k). \tag{2}$$

(ii) *Let $D \in \mathcal{E}(k) - \{H(k, 2)\}$. We may assume that $n = |V(D)| > k + 2$ and for some $\mathbf{s} = (s_1, s_2, \ldots, s_m) \in \mathcal{S}(n, k)$, $D \in L(\mathbf{s})$ with construction sequence $J_1, J_2, \ldots, J_m$. Using the notation in Definition 3.1, we let $L_i = D[\cup_{j+1}^i V(J_j)]$. For any k-arc-cut $W = (X, V(D) - X)_D$ of D, there must be an i with $1 \leq i < m$ such that $W = (V(L_i), V(J_{i+1}))_D$ or $W = (V(J_{i+1}), V(L_i))_D$.*

We will justify Observation 3.2(ii). Since $n = |V(D)| > k+2$, we have $m \geq 2$. When $m = 2$, by Definition 3.1(ii-A) and (ii-B), we observe that if W_1 is a k-arc-cut of D, then we must have $W_1 = (V(J_1), V(J_2))_D$ or $W_1 = (V(J_2), V(J_1))_D$. Hence we assume that $m > 2$. Inductively, assume that for any digraph $D' \in \mathcal{E}(k) - \{H(k,2)\}$ with $|V(D')| < |V(D)|$ and with construction sequence $J_1', J_2', \ldots, J_{m'}'$, if W' is a k-arc-cut of D', then there must be an i with $1 \leq i < m'$ such that $W' = (\cup_{j+1}^{i} V(J_j'), V(J_{i+1}'))_{D'}$ or $W' = (V(J_{i+1}'), \cup_{j+1}^{i} V(J_j'))_{D'}$. Let $W = (X, V(D)-X)_D$ be an k-arc-cut of D. If $X \cap V(J_m) = \emptyset$ or if $J_m \subseteq X$, then by Observation 3.2, W is an k-arc-cut of L_{m-1}, and so by induction, there must be an i with $1 \leq i < m-1$ such that Observation 3.2(ii) holds. Hence we must have $X \cap V(J_m) \neq \emptyset$ and $(V(D)-X) \cap J_m \neq \emptyset$. In this case, as $|V(J_m)| \geq 2$, we must have $s_m = k+2$ and $J_m = H(k,2)$. It follows that $H(k,2)$ contains an arc-cut $X \cap A(J_m)$ of size at most k. But by Definition 3.1(ii-B), J_m does not have an arc-cut of size k. This contradiction justifies Observation 3.2(ii).

Observation 3.2(i) can be extended, as shown in Theorem 3.6.

Lemma 3.3. *For any $D \in \mathcal{E}(k)$, we have*

$$\lambda(D) = \overline{\lambda}(D) = k. \tag{3}$$

Proof. By Definition 3.1, it suffices to show that if $D = L(\mathbf{s})$ for some $\mathbf{s} = (s_1, s_2, \ldots, s_m) \in \mathcal{S}(n,k)$, then (3) holds. We argue by induction on m. By (2), (3) holds for $m = 1$. Assume that $m > 1$ and (3) holds for smaller values of m. We adopt the notation in Definition 3.1 and let $J_1, J_2, \ldots, J_m$ be the construction sequence of D. Let $\mathbf{s}' = (s_1, s_2, \ldots, s_{m-1})$ and $D' = D - V(J_m)$. Then $\mathbf{s}' \in \mathcal{S}(n - s_m, k)$, and $D' = L(\mathbf{s}')$. By Definition 3.1, $D \in [D', J_m]_k$. By induction, $\lambda(D') = \overline{\lambda}(D') = k$.

We argue by contradiction to prove that $\lambda(D) \geq k$, and assume that D has a proper nonempty subset $X \subset V(D)$ such that $|\partial_D^+(X)| < k$. If both $X \cap V(D') \neq \emptyset$ and $V(D') - X \neq \emptyset$, then by $\lambda(D') = k$, we have a contradiction $k > |\partial_D^+(X)| \geq |(V(D') \cap X, V(D') - X)_{D'}| \geq k$. Hence either $V(D') \cap X = \emptyset$ or $V(D') \subseteq X$. Similarly, as $J_m \in \{K_1, H(k,2)\}$, if both $X \cap V(J_m) \neq \emptyset$ and $V(J_m) - X \neq \emptyset$, then $J_m = H(k,2)$, and so $k > |\partial_D^+(X)| \geq |(V(J_m) \cap X, V(J_m) - X)_{D'}| \geq \lambda(H(k,2)) = k$, a contradiction. It follows that we must have $X = V(D')$ or $X = V(J_m)$. By Definition 2.2 or 2.3, we have again a contradiction: $k > |\partial_D^+(X)| \geq \min\{|(V(J_m), V(D'))_D|, |(V(D'), V(J_m))_D|\} \geq k$. This proves that $\lambda(D) \geq k$.

We now prove $\overline{\lambda}(D) = k$ by contradiction. Assume that D has a subdigraph H such that $\lambda(H) \geq k+1$. If both $V(H) \cap V(D') \neq \emptyset$ and $V(H) \cap V(J_m) \neq \emptyset$, then $\lambda(H) \leq |(V(H) \cap V(D'), V(H) \cap V(J_m))_H| \leq |(V(D'), V(J_m))_D| = k$, contrary to $\lambda(H) \geq k+1$. Thus since $J_m \in \{K_1, H(k,2)\}$, we must have $H \subseteq D'$. By induction, $\overline{\lambda}(D') = k$, and so $\lambda(H) \leq \overline{\lambda}(D') = k$, contrary to the assumption $\lambda(H) \geq k+1$. This proves the lemma. □

A special class of graphs in $\mathcal{E}(k)$ has been studied in [1]. Let $\mathcal{S}_M(n,k)$ be the subset of $\mathcal{S}(n,k)$ such that $\mathbf{s} = (s_1, s_2, \ldots, s_m) \in \mathcal{S}_M(n,k)$ if and only if $|s_2| = |s_3| = \cdots |s_m| = 1$. Let $\mathcal{M}(k) = \cup_{n \geq k+2}\{L(\mathbf{s}) : \mathbf{s} \in \mathcal{S}_M(n,k)\}$.

Theorem 3.4 (*Anderson et al., Theorem 3.2(ii) of [1]*). $\mathcal{M}(k) \subseteq \mathcal{D}(k)$.

The observations stated in Lemma 3.5 follow immediately from Definition 3.1. For example, in Lemma 3.5(i), if for some $2 \leq t \leq m-1$, (4) holds, then the digraph sequence $J_1, J_2, \ldots, J_{t-1}, J_{t+1}, \ldots, J_m, J_t$ is also a construction sequence of D such that for $\mathbf{s}' = (s_1, \ldots, s_{t-1}, s_{t+1}, \ldots, s_m, s_t)$, we have then $D \in L(\mathbf{s}')$. The justification of Lemma 3.5(ii) is similar and will be omitted.

Lemma 3.5. *Let $D \in L(\mathbf{s})$ for some $\mathbf{s} = (s_1, s_2, \ldots, s_m) \in \mathcal{S}(n,k)$ with a construction sequence $J_1, J_2, \ldots, J_m$. Each of the following holds.*

(i) *If for some t with $2 \leq t \leq m-1$, and for all j with $t+1 \leq j \leq m$,*

$$\text{either } s_j > 0 \quad \text{and} \quad (J_t, J_j)_D = \emptyset, \text{ or } s_j < 0 \quad \text{and} \quad (J_j, J_t)_D = \emptyset, \tag{4}$$

then $D - V(J_t) = L(\mathbf{s}')$, where $\mathbf{s}' = (s_1, \ldots, s_{t-1}, s_{t+1}, \ldots, s_m) \in \mathcal{S}(n - s_t, k)$.

(ii) *Suppose that for some t with $1 \leq t < m$, we have $s_{t+1} = k+2$. If for each j with $t+2 \leq j \leq m$,*

$$\text{either } s_j > 0 \quad \text{and} \quad (J_1 \cup J_2 \cup \cdots \cup J_t, J_j)_D = \emptyset, \text{ or } s_j < 0 \quad \text{and} \quad (J_j, J_1 \cup J_2 \cup \cdots \cup J_t)_D = \emptyset, \tag{5}$$

then $D - V(J_1 \cup J_2 \cup \cdots \cup J_t) = L(\mathbf{s}')$, where $\mathbf{s}' = (s_{t+1}, s_{t+2}, \ldots, s_m) \in \mathcal{S}(n - \sum_{i=1}^{t} s_i, k)$.

Lemma 3.5 can be applied in inductive augments involving digraphs in $\mathcal{E}(k)$. This allows us to prove a generalization of Theorem 3.4, as stated in the theorem below.

Theorem 3.6. *Let $k \geq 1$ be an integer. Then $\mathcal{E}(k) \subseteq \mathcal{D}(k)$.*

Proof. Let $D \in \mathcal{E}(k)$ with $n = |V(D)|$. In the proof arguments below, we shall adopt the notation in Definition 3.1 to use $L_1, L_2, \ldots, L_m$ to denote the graphs in the process to build L_m.

We argue by induction on n to prove the theorem. By Definition 3.1, $n \geq k+2$, and $n = k+2$ if and only if $D = H(k,2)$. By (2), $D = H(k,2) \in \mathcal{D}(k)$. Thus we may assume that $n > k+2$ and for any digraph $D' \in \mathcal{E}(k)$ with $|V(D')| \leq n-1$, $D' \in \mathcal{D}(k)$. We are to show that if $D \in \mathcal{E}(n,k)$, then $D \in \mathcal{D}(k)$.

By contradiction, we assume that $D \in \mathcal{E}(n,k) - \mathcal{D}(k)$, and so for some $a = (u, v) \in A(D^c)$, we have

$$\overline{\lambda}(D+a) \le k. \tag{6}$$

Assume that $D = L(\mathbf{s})$ for some $\mathbf{s} = (s_1, s_2, \ldots, s_m) \in \mathcal{S}(n,k)$ with m minimized and $a = (u, v) \in A(D^c)$; and let $J_1, J_2, \ldots, J_m$ be the corresponding construction sequence of D. Since $n > k+2$, we have $m \ge 2$. By symmetry, we assume that $D \in [L_{m-1}, J_m]_k$. By induction, $L_{m-1} \in \mathcal{D}(k)$. If $u, v \in V(L_{m-1})$, then $\overline{\lambda}(D+a) \ge \overline{\lambda}(L_{m-1}+a) \ge k+1$. Hence we may assume that

$$u \in V(L_{m-1}) \quad \text{and} \quad v \in V(J_m). \tag{7}$$

By (6),

$$\text{there exists a nonempty proper subset } X \subset V(D+a), \text{ such that } |\partial^+_{D+a}(X)| \le k. \tag{8}$$

By Definition 3.1(ii) or (iii), there are k arcs from L_{m-1} to J_m. We assume that $(V(L_{m-1}), V(J_m))_D = \{a_1, a_2, \ldots, a_k\}$. Let $a_i = (v_i, w_i)$, $1 \le i \le k$. By Definition 3.1, $\{v_1, v_2, \ldots, v_k\} \subseteq V(L_{m-1})$ and $w_1, w_2, \ldots, w_k \in V(J_m)$. If there exists a t with $2 \le t \le m-1$, such that $V(J_t) \cap \{v_1, v_2, \ldots, v_k, u\} = \emptyset$, and such that for all $j > t$, either $s_j > 0$ and $(J_t, J_j)_D = \emptyset$, or $s_j < 0$ and $(J_j, J_t)_D = \emptyset$, then by Lemma 3.5(i), $D - V(J_t) = L(\mathbf{s}')$, where $\mathbf{s}' = (s_1, \ldots, s_{t-1}, s_{t+1}, \ldots, s_m) \in \mathcal{S}(n - s_t, k)$. By induction, $D - V(J_t) \in \mathcal{D}(k)$, and so $\overline{\lambda}(D+a) \ge \overline{\lambda}((D - V(J_t)) + a) \ge k+1$, contrary to (6). Hence we may assume that for any t with $1 < t \le m-1$, there exists a $j > t+1$ such that

$$0 < \begin{cases} |V(J_t) \cap \{v_1, v_2, \ldots, v_k, u\}| + |(J_t, J_j)_D| & \text{if } s_j > 0 \\ |V(J_t) \cap \{v_1, v_2, \ldots, v_k, u\}| + |(J_j, J_t)_D| & \text{if } s_j < 0 \end{cases}. \tag{9}$$

Let $X \subset V(D)$ be a subset satisfying (8). Define $I' = \{i : 1 \le i \le m \text{ and } V(J_i) \cap X = \emptyset\}$ and $I'' = \{i : 1 \le i \le m \text{ and } V(J_i) \cap X \ne \emptyset\}$.

Claim 1. For any i with $1 \le i \le m$, if $|V(J_i)| = k+2$, then either $X \cap V(J_i) = \emptyset$ or $V(J_i) - X = \emptyset$. (As $|s_i| \in \{1, k+2\}$, it follows that for any $1 \le i \le m$, either $X \cap V(J_i) = \emptyset$ or $V(J_i) - X = \emptyset$.)

Proof of Claim 1. By contradiction, suppose for some i' with $1 \le i' \le m$ and with $|V(J_{i'})| = k+2$, and both $X \cap V(J_{i'}) \ne \emptyset$ and $V(J_{i'}) - X \ne \emptyset$. If $k \ge |X \cap V(J_{i'})| \ge 2$, then as $J_{i'} = H(k,2)$, we have $\min\{|X \cap V(J_{i'})|, |V(J_{i'}) - X|\} \ge 2$. It follows by the definition of $H(k,2)$ that $|\partial^+_{D+a}(X)| \ge |\partial^+_{J_{i'}}(X \cap V(J_{i'}))| \ge k+1$, contrary to (8). Hence we may assume that $|X \cap V(J_{i'})| \in \{1, k+1\}$, and so $|\partial^+_{J_{i'}}(X \cap V(J_{i'}))| = k$. By (7), $|\{u, v\} \cap V(J_{i'})| \le 1$ and so $\min\{|X \cap V(J_{i'})|, |V(J_{i'}) - X|\} = 1$. It follows that

$$|(X \cap V(J_{i'}), V(J_{i'}) - X)_{D+a}| = |(X \cap V(J_{i'}), V(J_{i'}) - X)_D| = |\partial^+_{J_{i'}}(X \cap V(J_{i'}))| = k. \tag{10}$$

By (10), we must have $\{v_1, \ldots, v_k\} \subseteq X \cap V(J_{i'})$ and $\{w_1, \ldots, w_k\} \subseteq V(J_{i'}) - X$. Also by (10), for any $j \ne i'$, if $X \cap V(J_j) \ne \emptyset$ and $V(J_j) - X \ne \emptyset$, then $\partial^+_{J_j}(X \cap V(J_j)) \ne \emptyset$. This, together with (10), implies $|\partial^+_{D+a}(X)| \ge |\partial^+_{J_{i'}}(X \cap V(J_{i'}))| + |\partial^+_{J_j}(X \cap V(J_j))| \ge k+1$, contrary to (8). Hence we have

$$\text{for any } j \ne i', \text{ if } X \cap V(J_j) \ne \emptyset, \text{ then } V(J_j) \subseteq X. \tag{11}$$

Since $J_{i'} = H(k,2)$, $J_{i'}$ has a unique vertex $x_1 = x^+(J_{i'})$ such that $d^+_{J_{i'}}(x_1) = k$ and a unique vertex $x_2 = x^-(J_{i'})$ such that $d^-_{J_{i'}}(x_2) = k$. It follows by (10) that either $V(J_{i'}) \cap X = \{x_1\}$ or $V(J_{i'}) - X = \{x_2\}$.

Assume first that $i' > 1$ and i_1 is the smallest integer satisfying $1 \le i_1 < i'$ such that $i_1 \in I''$. If $i_1 > 1$, then either $s_{i_1} > 0$, whence by (11), $\cup_{1 \le t \le i_1 - 1} V(J_t) \cap X = \emptyset$, and so by Definition 2.2 or 2.3, $|\partial^+_{D+a}(X)| \ge |V(J_{i_1}), V(J_1))_D| \ge |V(J_1)| = k+2$; or $s_{i_1} < 0$, whence by (10) and by Definition 2.2 or 2.3, $|\partial^+_{D+a}(X)| \ge |(V(J_{i'}) \cap X, V(J_{i'}) - X)_D| + |(V(J_{i_1}), L_{i_1 - 1})_D| \ge k+1$. In either case, a contradiction to (8) is obtained. Therefore we assume that $i_1 = 1$. If there exists an i'' with $1 < i'' < i'$ such that $X \cap V(J_{i''}) = \emptyset$, then assume that i'' is the smallest such integer. By Definition 2.2 or 2.3, $|(V(L_{i''-1}), V(J_{i''}))_D| > 0$. This, together with (10), implies that $|\partial^+_{D+a}(X)| \ge |(V(J_{i'}) \cap X, V(J_{i'}) - X)_D| + |(V(L_{i''-1}), V(J_{i''}))_D| \ge k+1$, contrary to (8). Therefore, no such i'' exists, and so we conclude that $V(L_{i'-1}) \subseteq X$. It follows by Definition 3.1(ii-B) that $|\partial^+_{D+a}(X)| \ge \min\{d^+_{L_{i'}}(x_1), d^-_{L_{i'}}(x_2)\} \ge k+1$, contrary to (8).

Therefore, we may assume that $i' = 1$. If for some t with $1 < t \le m$, $|s_t| = k+2$, then by Definition 3.1(ii-B), we have $|\partial^+_{D+a}(X)| \ge \min\{d^+_{L_t}(x_1), d^-_{L_t}(x_2)\} \ge k+1$, contrary to (8). Hence for all $t > 1$, we have $|s_t| = 1$. It follows by Theorem 3.4 that $D \in \mathcal{D}(k)$, contrary to (6). This justifies Claim 1.

Claim 2. Suppose that $V(J_1) \cap X = \emptyset$. Let $i_1 > 1$ be the smallest integer such that $V(J_{i_1}) \cap X \ne \emptyset$, and $i_2 \le m$ be the largest integer such that for any t with $i_1 \le t \le i_2$, we have $V(J_t) \subseteq X$. Each of the following holds.

(i) For any $i \ge 2$, if $V(J_i) \cap X \ne \emptyset$, then $s_i < 0$.

(ii) $V(J_m) \cap X = \emptyset$.

(iii) $(V(J_{i_1}), V(L_{i_1-1}))_D = \partial^+_{D+a}(X)$ and $|\partial^+_{D+a}(X)| = |(V(J_{i_1}), V(L_{i_1-1}))_D| = k$.
(iv) $u \notin X$.
(v) $\overline{\lambda}(D + a) \geq k + 1$. (Thus a contradiction to (6) is obtained.)

Proof of Claim 2. (i) Suppose that $V(J_1) \cap X = \emptyset$. By Definition 3.1, $|V(J_1)| = s_1 = k+2$. If for some $i \geq 2$ with $V(J_i) \cap X \neq \emptyset$, we have $s_i > 0$, then by Definition 3.1, for each vertex $x \in V(J_i)$ and for each vertex $y \in V(J_1)$, $(x, y) \in A(D)$. It follows by $|V(J_1)| = s_1 = k + 2$ and by Claim 1 that $|\partial^+_{D+a}(X)| \geq |(V(J_i), V(J_1))_D| \geq k + 2$, contrary to (8). This justifies (i).
(ii) Since $D = [L_{m-1}, J_m]_k$, we have $s_m > 0$ and so by Claim 2(i) and by Claim 1, $V(J_m) \cap X = \emptyset$.
(iii) By Claim 2(i), $s_{i_1} < 0$. Thus by Definition 3.1(ii), $|(V(J_{i_1}), V(L_{i_1-1}))_D| = k$. By the definition of i_1, $V(L_{i_1-1}) \cap X = \emptyset$ and $V(J_{i_1}) \subseteq X$. Hence $(V(J_{i_1}), V(L_{i_1-1}))_D \subseteq \partial^+_{D+a}(X)$. By (8), we have $|\partial^+_{D+a}(X)| = |(V(J_{i_1}), V(L_{i_1-1}))_D| = k$, which implies $(V(J_{i_1}), V(L_{i_1-1}))_D = \partial^+_{D+a}(X)$.
(iv) If $u \in X$, then by Claim 2(ii), we have $(u, v) \in \partial^+_D(X)$. This, together with Claim 2(iii), implies that $|\partial^+_{D+a}(X)| \geq k + 1$, contrary to (8).
(v) For any $t > i_2$ with $s_t > 0$, by (8) and Claim 2(iii), we must have

$$(X, V(J_t))_{D+a} = (\cup_{i\in I''} V(J_i), V(J_t))_{D+a} = \emptyset.$$

Let $\mathbf{s}''$ be a subsequence of $\mathbf{s}$ by deleting all terms s_i with $i \in I''$ from $\mathbf{s}$; and let $D'' = D - X$. It follows that $D'' = L(\mathbf{s}'')$ and so $D'' \in \mathcal{S}(n - |X|, k)$. Since $I'' \neq \emptyset$, by induction, $D'' \in \mathcal{E}(k)$. By Claim 2(iv), $u \notin X$ and so both ends u and v are in $V(D'')$. Since $D'' \in \mathcal{E}(k)$, we have $\overline{\lambda}(D + a) \geq \lambda(D'' + a) \geq k + 1$. This completes the proof for Claim 2.

Claim 3. Suppose that $V(J_1) \subseteq X$. Let $i_2 \leq m$ be the largest integer such that for any t with $1 \leq t \leq i_2$, we have $V(J_t) \subseteq X$. Each of the following holds.
(i) For any $i > i_2$, if $V(J_i) \cap X = \emptyset$, then $s_i > 0$.
(ii) $(V(L_{i_2}), V(J_{i_2+1}))_D = \partial^+_{D+a}(X)$ and $|\partial^+_{D+a}(X)| = |(V(L_{i_2}), V(J_{i_2+1}))_D| = k$.
(iii) $m > i_2 + 1$.
(iv) Suppose that $s_{i_2+1} = 1$ and $t > i_2 + 1$. Then $V(J_t) \cap X = \emptyset$ if and only if $s_t > 0$; and $V(J_t) \subseteq X$ if and only if $s_t < 0$. In particular, $V(J_m) \cap X = \emptyset$ and $u \notin X$.
(v) Let $i_3 > 1$ be the largest integer such that $V(J_{i_3}) \subseteq X$. Then $m - 1 > i_3 > i_2$, $V(J_{i_3}) \cap \{v_1, v_2, \ldots, v_k, u\} = \emptyset$, and for any $h > i_3$, $(V(J_{i_3}), V(J_h))_D = \emptyset$.

Proof of Claim 3. (i) Let $i > i_2$ be an index such that $V(J_i) \cap X = \emptyset$. If $s_i < 0$, then by Definition 2.2 or 2.3, for any $x \in V(L_{i_2})$ and for any $y \in V(J_i)$, we have $(x, y) \in A(D)$. It follows that $|\partial^+_{D+a}(X)| \geq |(V(J_1), V(J_i))_D| \geq |s_1| = k + 2$, contrary to (6).
(ii) By Claim 3(i), $s_{i_2+1} > 0$. By Definition 3.1(ii), $|(V(L_{i_2}), V(J_{i_2+1}))_D| = k$. By the definition of i_2, $V(L_{i_2}) \cap X = \emptyset$ and $V(L_{i_2}) \subseteq X$. Hence $(V(L_{i_2}), V(J_{i_2+1}))_D \subseteq \partial^+_{D+a}(X)$. By (8), we have $|\partial^+_{D+a}(X)| = |(V(L_{i_2}), V(J_{i_2+1}))_D| = k$, which implies $(V(L_{i_2}), V(J_{i_2+1}))_D = \partial^+_{D+a}(X)$.
(iii) If $i_2 + 1 = m$, then we must have $u \in V(L_{i_2})$, and so $(u, v) \in (V(L_{i_2}), V(J_{i_2+1}))_{D+a} \subseteq \partial^+_{D+a}(X)$. As $(u, v) \notin (V(L_{i_2}), V(J_{i_2+1}))_D$, this yields a contradiction to $(V(L_{i_2}), V(J_{i_2+1}))_D = \partial^+_{D+a}(X)$.
(iv) Suppose that $s_{i_2+1} = 1$ and fix $t > i_2+1$. Assume that $V(J_t) \cap X = \emptyset$. By Claim 3(ii) and by (6), $(V(J_t), X)_D = \emptyset$. Hence by the definition of $[L_{t-1}, J_t]_k$, we must have $s_t > 0$. Conversely, assume that both $s_t > 0$ and $V(J_t) \subseteq X$, then by the definition of $[L_{t-1}, J_t]_k$, $(V(J_k), V(L_{t-1}))_D \neq \emptyset$, contrary to Claim 3(ii). This proves that $V(J_t) \cap X = \emptyset$ if and only if $s_t > 0$.

Now assume that $V(J_t) \subseteq X$. If $s_t > 0$, then $(V(J_t), V(J_{i_2+1}))_D \neq \emptyset$, by the definition of $[L_{t-1}, J_t]_k$, contrary to Claim 3(ii). Therefore, we must have $s_t < 0$. Conversely, assume that $s_t < 0$ and $V(J_t) \cap X = \emptyset$. By the definition of $[J_t, L_{t-1}]_k$, we have $(V(L_{t-1}), V(J_t))_D \neq \emptyset$, again contrary to Claim 3(ii).

As $D = [L_{m-1}, J_m]_k$, we have $s_m > 0$, and so $V(J_m) \cap X = \emptyset$. By Claim 3(ii) and since $v \in V(J_m)$, we conclude that $u \notin X$. This proves (iv).
(v) By Claim 3(iv), $V(J_m) \cap X = \emptyset$, and so $m > i_3$. We argue by contradiction to assume that $i_3 = i_2$. Then by the definitions of i_2 and i_3, we have $X = \cup_{t=1}^{i_3} V(J_t) = V(L_{i_3})$. For any $j > i_3$, by Claim 5(i), $s_j > 0$. By Claim 3(iv), $u \in X$. If $m = i_3 + 1$, then u must be in X, a contradiction. Hence $m \geq i_3 + 2$. Similarly, by $k \geq |\partial^+_{D+a}(X)|$, $\{v_1, v_2, \ldots, v_k\} \cap X = \emptyset$. By Claim 3(i), $s_{i_3+2} > 0$. Since $(L_{i_3}, J_{i_3+1})_D \cup (L_{i_3}, J_{i_3+2})_D \subseteq \partial^+_{D+a}(X)$ and since $|(L_{i_3}, J_{i_3+1})_D| = k$, it follows by $k \geq |\partial^+_{D+a}(X)|$ that $|(L_{i_3}, J_{i_3+2})_D| = 0$. This, $\{v_1, v_2, \ldots, v_k\} \cap X = \emptyset$, yields a contradiction to (9). This proves that $m > i_3 > i_2$.

We now show the other conclusions of Claim 3(v). By Claim 3(iv), $V(J_m) \cap X = \emptyset$ and $u \notin X$. By Definition 3.1 we have $(J_{i_3}, J_{i_3+1})_D \subseteq (J_{i_3}, J_{i_3+1} \cup J_m)_D \subseteq \partial^+_{D+a}(X)$, which implies that

$$k = |(J_{i_3}, J_{i_3+1})_D| \leq |(J_{i_3}, J_{i_3+1})_D| + |(J_{i_3}, J_m)_D| \leq |\partial^+_{D+a}(X)| \leq k.$$

$|(J_{i_3}, J_m)_D| = 0$. Since $w_1, w_2, \ldots, w_k \in V(J_m)$, it follows that $V(J_{i_3}) \cap \{v_1, v_2, \ldots, v_k, u\} = \emptyset$. By the choice of i_3, for any $h > i_3$, we have $V(J_h) \cap X = \emptyset$, and so $(V(J_{i_3}), V(J_h))_D \subseteq \partial^+_{D+a}(X)$. By Claim 3(ii), we must have $(V(J_{i_3}), V(J_h))_D = \emptyset$. This justifies Claim 3.

We now continue the proof of the theorem. By Claim 2(v), we may assume that $s_1 = -(k+2)$, and so Claim 3 applies. By Claim 3(iv) and with i_3 being defined in Claim 3(v), we conclude that $s_h > 0$, for any $h > i_3$. Therefore, Claim 3(v) presents a contradiction to (9). This proves the theorem. □

To determine the extremal graphs of Theorem 1.4, we need to construct a new family of digraphs.

Definition 3.7. For an integer $k > 0$, define $\mathcal{E}_1(k)$ to be the family consisting of digraphs satisfying each of the following.
(A) $\mathcal{E}(k) \subset \mathcal{E}_1(k)$.
(B) If digraphs H and H' satisfy

$$H, H' \in \mathcal{E}_1(k) \cup \{K_1\} \quad \text{with } |V(H)| + |V(H')| > 2, \tag{12}$$

then $[H, H']_k \subset \mathcal{E}_1(k)$.

Lemma 3.8. *For any $D \in \mathcal{E}_1(k)$.*
(i) $|V(D)| \geq k + 2$.
(ii) $\lambda(D) = k$.
(iii) *For any k-arc-cut W of D, there exist two digraphs H and H' satisfying (12) such that $D \in [H, H']_k$ and $W = (V(H), V(H'))_D$.*

Proof. By Definition 3.7 and by induction on $|V(D)|$ for a digraph $D \in \mathcal{E}_1(k)$, Lemma 3.8(i) and (ii) hold. To prove Lemma 3.8(iii), we assume that D has a k-arc-cut $W = \{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\}$. Thus for some nonempty subsets $X, V(D) - X$, we have $W = (X, V(D) - X)_D$. If $D \in \mathcal{E}(k)$, then by Observation 3.2(ii), Lemma 3.8(iii) must hold. Hence by Definition 3.7, we assume that $D \in [H, H']_k$ for some H, H' satisfying (12); and that Lemma 3.8(iii) holds for digraphs in $\mathcal{E}_1(k)$ with smaller order than D. Let $Z = (V(H), V(H'))_D$.

Case 1. $X \cap V(H') = \emptyset$, or $(V(D) - X) \cap V(H') = \emptyset$.

By symmetry, we assume that $X \cap V(H') = \emptyset$. Then X is a k-arc-cut of H. By induction, there exist digraphs L, L' satisfying (12) such that $H \in [L, L']_k$ and $W = (V(L), V(L'))_H$. As $X \cap V(H') = \emptyset$, we have $V(L) = X$. Since W is an arc-cut of D, $W \cap Z = \emptyset$ and so $D \in [L, L'']_k$ with $W = (V(L), V(L''))_D$, $L'' = D - X \in [L', H']_k$ and $Z = (V(L), V(D) - X)_{L''}$. Since $L', H' \in \mathcal{E}_1(k) \cup \{K_1\}$, it follows by Definition 3.7 that $L'' \in \mathcal{E}_1(k)$. This implies that Lemma 3.8(iii) holds.

Case 2. $X \cap V(H') \neq \emptyset$ and $(V(D) - X) \cap V(H') \neq \emptyset$.

Let $W_1 = (X \cap V(H), V(H) - X)_H$ and $W_2 = (X \cap V(H'), V(H') - X)_{H'}$. Thus $W = W_1 \cup W_2$ and $|W_1| + |W_2| = |W| = k$. If both $H, H' \in \mathcal{E}_1(k)$, then by Lemma 3.8(ii), we must have $|W_1| \geq k$ and $|W_2| \geq k$, contrary to the fact that $|W_1| + |W_2| = |W| = k$. Hence either $H = K_1$ or $H' = K_1$. Suppose that $H = K_1$ with $V(H) = \{v\}$. By the definition of $[H, H']_k$, for any $v' \in X \cap V(H')$, $(v', v) \in A(D)$.

Thus if $v \notin X$, then $X \subset V(H')$ and so $W \subseteq (X, \{v\})_D \cup (X, V(H') - X)_D$. It follows from Lemma 3.8(ii) that $k = |W| = |(X, \{v\})_D| + |(X, V(H') - X)_D| \geq |(X, \{v\})_D| + k$, and so $(X, \{v\})_D = \emptyset$ and $D \in [\{v\}, H']_k$. By induction, there exist digraphs L, L' satisfying (12) such that $H' \in [L, L']_k$ and $W = (V(L), V(L'))_{H'}$. Let $L'' \in [\{v\}, L']_k$. Then $L'' \in \mathcal{E}_1(k)$ and $D \in [L, L'']_k$ with $W = (V(L), V(L''))_D$. Hence Lemma 3.8(iii) holds.

Therefore, we must have $v \in X$, which implies that $(\{v\}, V(H') - X)_D \neq \emptyset$. It follows that $k = |W| = |(\{v\}, V(H') - X)_D| + |(X - \{v\}, V(H') - X)_D| > |(X - \{v\}, V(H') - X)_D|$. This implies that $\lambda(H') \leq |(X - \{v\}, V(H') - X)_D| < k$, contrary to Lemma 3.8(ii). This completes the proof of the lemma. □

Lemma 3.9. *For any integer $k > 1$, we have $\mathcal{E}_1(k) \subseteq \mathcal{D}(k)$.*

Proof. Let $D \in \mathcal{E}_1(k)$. We need to show that $D \in \mathcal{D}(k)$. If $D \in \mathcal{E}(k)$, then by Theorem 3.6, $D \in \mathcal{D}(k)$. Hence we assume that $D \in \mathcal{E}_1(k) - \mathcal{E}(k)$, and Lemma 3.9 holds for graphs in $\mathcal{E}_1(k)$ with smaller order.

For any $e \in A(D^c)$, if $\lambda(D + e) \geq k + 1$, then $D \in \mathcal{D}(k)$. Hence we assume that $\lambda(D + e) \leq k + 1$. Let W be a j-arc-cut of $D + e$ for some $j \leq k$. By Lemma 3.8(ii), $e \notin W$ and so by Lemma 3.8(iii), for some digraphs H, H' satisfying (12), $D \in [H, H]_k$ and $W = (V(H), V(H'))_D$. Let $e = (u, v)$. Since $e \notin W$, we cannot have $u \in V(H)$ and $v \in V(H')$. By the definition of $[H, H']_k$, we cannot have $v \in V(H)$ and $u \in V(H')$. Hence either $u, v \in V(H)$ or $u, v \in V(H')$. Without loss of generality, we assume that $u, v \in V(H)$, and so $e \in A(H^c)$. By (12), $H \in \mathcal{E}_1(l)$ and so by induction, $\overline{\lambda}(H + e) \geq k + 1$. It follows that $\overline{\lambda}(D + e) \geq \overline{\lambda}(H + e) \geq k + 1$, and so by definition, $D \in \mathcal{D}(k)$. □

Definition 3.10. Let n and k be integers with $n > k > 0$ and q, r be nonnegative integers satisfying $n = q(k + 2) + r$ with $0 \leq r \leq k + 1$,
(i) Define $\mathcal{S}'(n, k)$ to be the set of all integral sequences $(s_1, s_2, \ldots, s_{q+r})$ such that $s_1 = k+2$, and for $i \geq 2$, $|s_i| \in \{1, k+2\}$. Note that if $(s_1, s_2, \ldots, s_{q+r}) \in \mathcal{S}'(n, k)$, then as $q(k + 2) + r = n = \sum_{i=1}^{q+r} |s_i|$, there are exactly r of the $|s_i|$'s equaling one and q of the $|s_i|$'s equaling $k + 2$. Define $\mathcal{E}'(n, k) = \{L(\mathbf{s}) \; : \; \mathbf{s} \in \mathcal{S}'(n, k)\}$ and $\mathcal{E}'(k) = \cup_{n \geq k+2} \mathcal{E}'(n, k)$.
(ii) Define $\mathcal{E}'_1(k)$ to be the family consisting of digraphs satisfying each of the following.
(ii-A) $\mathcal{E}'(k) \subset \mathcal{E}'_1(k)$.
(ii-B) For $H, H' \in \mathcal{E}'_1(k) \cup \{K_1\}$ satisfying $|V(H_1)| + |V(H_2)| > 2$ and $\lfloor \frac{n}{k+2} \rfloor = \lfloor \frac{|V(H_1)|}{k+2} \rfloor + \lfloor \frac{|V(H_2)|}{k+2} \rfloor$, $[H, H']_k \subset \mathcal{E}'_1(k)$.

By Definition 3.10, the corollary below follows immediately from Theorem 3.6 and Lemma 3.9.

Corollary 3.11. $\mathcal{E}_1'(k) \subseteq \mathcal{D}(k)$.

Given the structure of digraphs in $\mathcal{E}_1'(k)$, we can compute the size of digraphs in $\mathcal{E}_1'(k)$.

Lemma 3.12. *Let $n > k+1 \geq 2$ be integers. For any digraph $D \in \mathcal{E}_1'(k)$, we have*

$$|A(D)| = \binom{n}{2} + (n-1)k + \left\lfloor \frac{n}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right). \tag{13}$$

Proof. We first assume that $D \in \mathcal{E}'(k)$ with $|V(D)| = n$ and $n > k+1 \geq 2$. If $n = k+2$, then by Definition 3.10, we have $D = H(k, 2)$, and so $|A(D)| = (k+2)(k+1) - 1$. Thus (13) holds. Assume that $n > k+2$ and (13) holds for smaller values of n. Let q, r be nonnegative integers satisfying $n = q(k+2) + r$ with $0 \leq r \leq k+1$. By Definitions 3.1 and 3.10, we have $|s_{q+r}| \in \{1, k+2\}$.

Case 1. $|s_{q+r}| = 1$.

By Definition 3.10, we may assume that $s_{q+r} = 1$ and $D \in [H, K_1]_k$ for some $H \in \mathcal{E}'(k)$. Denote $V(K_1) = \{v\}$. Since $s_{q+r} = 1$, we have $r \geq 1$, and so $n - 1 = q(k+2) + r - 1$, which implies $\lfloor \frac{n}{k+2} \rfloor = \lfloor \frac{n-1}{k+2} \rfloor$. By induction, we have

$$\begin{aligned} |A(D)| &= |A(H)| + k + (n-1) \\ &= \binom{n-1}{2} + (n-2)k + \left\lfloor \frac{n-1}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right) + k + (n-1) \\ &= \binom{n}{2} + (n-1)k + \left\lfloor \frac{n}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right). \end{aligned}$$

Case 2. $|s_{q+r}| = k+2$.

By Definition 3.10, we may assume that $s_{q+r} = k+2$ and $D = [H, H(k, 2)]_k$ for some $H \in \mathcal{E}'(k)$. Since $s_1 = k+2$ and $s_{q+r} = k+2$, we have $q \geq 2$, and so $n - (k+2) = (q-1)(k+2) + r$, which implies $\lfloor \frac{n}{k+2} \rfloor = \lfloor \frac{n-(k+2)}{k+2} \rfloor + 1$. By induction, we have

$$\begin{aligned} |A(D)| &= |A(H)| + k + (n-(k+2))(k+2) + |A(H(k,2))| \\ &= \binom{n-(k+2)}{2} + (n-(k+2)-1)k + \left\lfloor \frac{n-(k+2)}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right) \\ &\quad + k + [n-(k+2)](k+2) + (k+2)(k+1) - 1 \\ &= \binom{n}{2} + (n-1)k + \left\lfloor \frac{n}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right). \end{aligned}$$

Thus, (13) holds for any $D \in \mathcal{E}'(k)$. Next, we assume that $D \in \mathcal{E}_1'(k) - \mathcal{E}'(k)$. By Definition 3.10, there exist $H, H' \in \mathcal{E}_1'(k)$ satisfying Definition 3.10(ii-B). Let $n_1 = |V(H)|$ and $n_2 = |V(H')|$. Thus $n = n_1 + n_2$ and $\lfloor \frac{n}{k+2} \rfloor = \lfloor \frac{n_1}{k+2} \rfloor + \lfloor \frac{n_2}{k+2} \rfloor$. By induction, we have

$$\begin{aligned} |A(D)| &= |A(H)| + k + n_1 n_2 + |A(H')| \binom{n_1}{2} + (n_1-1)k + \left\lfloor \frac{n_1}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right) + k + n_1 n_2 \\ &\quad + \binom{n_2}{2} + (n_2-1)k + \left\lfloor \frac{n_2}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right) \\ &= \binom{n_1}{2} + \binom{n_2}{2} + n_1 n_2 + (n-1)k + \left\lfloor \frac{n}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right) \\ &= \binom{n}{2} + (n-1)k + \left\lfloor \frac{n}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right). \end{aligned}$$

By induction, (13) holds for any $D \in \mathcal{E}_1'(k)$. □

The following lemma gives us more information on the structure of digraphs in $\mathcal{D}(k)$.

Lemma 3.13. *Let $k \geq 2$ be an integer. If $D \in \mathcal{D}(k)$ and if for some $H_1, H_2 \in \mathcal{D}(k)$, we have $D \in [H_1, H_2]_k$, then for each $i \in \{1, 2\}$, $H_i \neq K_{k+1}^*$.*

Proof. By contradiction, we assume that $H_2 \cong K^*_{k+1}$ and $D \in [H_1, H_2]_k$, and so $D \in \mathcal{D}(k)$. Let $V(H_2) = \{v_1, v_2, \ldots, v_{k+1}\}$. By Definition 2.2, we may assume that $|(H_1, H_2)_D| = k$, and so we may assume that $N^+_D(V(H_1), V(H_2)) \subseteq \{v_1, v_2, \ldots, v_k\}$. Since $H_1, H_2 \in \mathcal{D}(k)$, both $|V(H_1)| \geq k+1$ and $|V(H_2)| \geq k+1$. Thus there must be a vertex $u \in V(H_1)$ and an integer i with $1 \leq i \leq k$, such that $a = (u, v_i) \notin A(D)$. Since $D \in \mathcal{D}(k)$, $D + a$ has a subdigraph D' with $\lambda(D') \geq k+1$. Note that $d^-_{D+a}(v_{k+1}) = d^-_D(v_{k+1}) = k$, $v_{k+1} \notin V(D')$. Since, for each j with $1 \leq j \leq k$ and $j \neq i$, $d^-_{D+a-v_{k+1}}(v_j) \leq d^-_{D-v_{k+1}}(v_j) + 1 \leq k$, it follows that $v_j \notin V(D')$ for each j with $1 \leq j \leq k$ and $j \neq i$. Since $k \geq 2$, $d^-_{D+a-v_{k+1}}(v_i) \leq k$, and $v_i \notin V(D')$ as well. This implies that $a \notin A(D')$, and so $D' \subseteq D$. Contrary to the assumption that $D \in \mathcal{D}(k)$. This proves the lemma. □

4. The extremal function

The main result of this section is Theorem 4.1, which clearly implies Theorem 1.4.

Theorem 4.1. *Let n, k be integers with $n > k + 1 \geq 2$. Then for any $D \in \mathcal{D}(n, k)$, we have*

$$|A(D)| \geq \binom{n}{2} + (n-1)k + \left\lfloor \frac{n}{k+2} \right\rfloor \left(1 + 2k - \binom{k+2}{2}\right). \tag{14}$$

Furthermore, equality holds in (14) if and only if $D \in \mathcal{E}'_1(k)$.

Proof. We argue by induction to prove (14) on $n = |V(D)|$. If $n = k+2$, then $D = H(k, 2)$. Thus we have $|A(D)| = (k+2)(k+1) - 1$, and so (14) holds. Assume that $n > k+2$ and (14) holds for smaller values of n. Let $q, r \geq 0$ be integers satisfying $n = q(k+2) + r$ with $0 \leq r \leq k+1$.

As $n > k+2$, $D \ncong K^*_{k+2}$. By Lemma 2.4, one of the three conclusions of Lemma 2.4 must hold.

Claim 1. If Lemma 2.4(i) or (ii) holds, then (14) holds as well. Moreover, if $r = 0$, then (14) holds with strict inequality.

Without loss of generality, we assume that $D \in [H, K_1]_k$ for some $H \in \mathcal{D}(k)$ with $V(K_1) = \{v\}$. As $|V(D)| = n-1$, by Definition 2.2, $|\partial^+_D(v)| = n-1$ and $|\partial^-_D(v)| = k$.

Case 1: $r = 0$.

Then $q - 1 = \lfloor \frac{n-1}{k+2} \rfloor$. By induction, we have

$$\begin{aligned}
|A(D)| &= |A(H)| + k + (n-1) \\
&\geq \binom{n-1}{2} + (n-2)k + (q-1)\left(1 + 2k - \binom{k+2}{2}\right) + k + (n-1) \\
&= \binom{n}{2} + (n-1)k + (q-1)\left(1 + 2k - \binom{k+2}{2}\right) \\
&> \binom{n}{2} + (n-1)k + q\left(1 + 2k - \binom{k+2}{2}\right).
\end{aligned}$$

Thus (14) holds with strict inequality in this case.

Case 2: $r > 0$.

Then $q = \lfloor \frac{n-1}{k+1} \rfloor$. By induction,

$$\begin{aligned}
|A(D)| &= |A(H)| + k + (n-1) \\
&\geq \binom{n-1}{2} + (n-2)k + q\left(1 + 2k - \binom{k+2}{2}\right) + k + (n-1) \\
&= \binom{n}{2} + (n-1)k + q\left(1 + 2k - \binom{k+2}{2}\right).
\end{aligned} \tag{15}$$

Thus (14) holds in this case as well, and so Claim 1 follows.

By Claim 1, we may assume that Lemma 2.4(iii) holds. Thus $D \in \{[H_1, H_2]_k, [H_2, H_1]_k\}$ for some $H_1, H_2 \in \mathcal{D}(k)$. Let $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. Then $n = n_1 + n_2$. Without loss of generality, we assume that $n_1 \geq n_2$. By Lemma 3.13, $n_2 \geq k+2$. Let $q_1, q_2 \geq 1$, r_1, r_2 be integers satisfying $n_1 = q_1(k+2) + r_1$, $0 \leq r_1 \leq k+1$, and $n_2 = q_2(k+2) + r_2$, $0 \leq r_2 \leq k+1$. Thus $q_1 = \lfloor \frac{n_1}{k+2} \rfloor$ and $q_2 = \lfloor \frac{n_2}{k+2} \rfloor$.

Claim 2. If Lemma 2.4(iii) holds, then (14) holds. Moreover, if $r_1 + r_2 \geq k+2$, then (14) holds with strict inequality.

Since $n = n_1 + n_2 = (q_1 + q_2)(k + 2) + (r_1 + r_2)$, we observe that $r_1 + r_2 \leq k + 1$ if and only if $q_1 + q_2 = q$, and if and only if $r = r_1 + r_2$. With this observation, we consider the following two cases. Note that if $n_1 \geq 2$ and $n_2 \geq 2$, then $\binom{n_1}{2} + \binom{n_2}{2} + n_1n_2 = \binom{n}{2}$.

Case 1: $r_1 + r_2 \leq k + 1$.

Then $q_1 + q_2 = q$. By Induction,

$$\begin{aligned} |A(D)| &= |A(H_1)| + k + n_1n_2 + |A(H_2)| \\ &\geq \binom{n_1}{2} + (n_1 - 1)k + q_1\left(1 + 2k - \binom{k+2}{2}\right) + k + n_1n_2 + \binom{n_2}{2} \\ &\quad + (n_2 - 1)k + q_2\left(1 + 2k - \binom{k+2}{2}\right) \\ &= \binom{n}{2} + (n-1)k + q\left(1 + 2k - \binom{k+2}{2}\right). \end{aligned} \tag{16}$$

Hence (14) holds in this case.

Case 2: $r_1 + r_2 \geq k + 2$.

Then $q_1 + q_2 = q - 1$ and $r = r_1 + r_2 - (k + 2)$. Observe that for any $k \geq 1$, $1 + 2k < \binom{k+2}{2}$, and so by induction,

$$\begin{aligned} |A(D)| &= |A(H_1)| + k + n_1n_2 + |A(H_2)| \\ &\geq \binom{n_1}{2} + (n_1 - 1)k + q_1\left(1 + 2k - \binom{k+2}{2}\right) + k + n_1n_2 + \binom{n_2}{2} \\ &\quad + (n_2 - 1)k + q_2\left(1 + 2k - \binom{k+2}{2}\right) \\ &= \binom{n}{2} + (n-1)k + (q-1)\left(1 + 2k - \binom{k+2}{2}\right) \\ &> \binom{n}{2} + (n-1)k + q\left(1 + 2k - \binom{k+2}{2}\right). \end{aligned}$$

Thus (14) holds with strict inequality in this case, and so Claim 2 is justified.

Claim 3. If equality holds in (14) for a digraph $D \in \mathcal{D}(k, n)$, then $D \in \mathcal{E}_1'(k)$.

Let $D \in \mathcal{D}(k, n)$ be a digraph satisfying equality in (14). We argue by induction on $n = |V(D)| \geq k + 2$. If $n = k + 2$, then $D = H(k, 2) \in \mathcal{E}'(k)$. Assume that $n > k + 2$ and that Claim 3 holds for smaller values of n. Since $n > k + 2$, by Lemma 2.4, one of the conclusions of Lemma 2.4 must hold.

If D satisfies Lemma 2.4(i) or (ii), without loss of generality, we assume that $D \in [H, K_1]_k$ for some $H \in \mathcal{E}'(k)$ with $V(K_1) = v$. By Claim 1, if equality holds in (14), then $r > 0$, which implies that $n - 1 = q(k + 2) + (r - 1)$, with $0 \leq r - 1 \leq k$. Since equality in (14) holds, it follows by (15) that $|A(H)| = \binom{n-1}{2} + (n-2)k + (q-1)\left(1 + 2k - \binom{k+2}{2}\right)$. By induction, $H \in \mathcal{E}_1'(n - 1, k)$. By Definition 3.10, $D \in \mathcal{E}'(n, k)$, and so $D \in \mathcal{E}'(k)$ in this case.

Hence we may assume that D satisfies Lemma 2.4(iii), and so $D \in [H_1, H_2]_k$ for some $H_1, H_2 \in \mathcal{E}'(k)$. Again, let $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$; and let $q_1, q_2 \geq 1, r_1, r_2$ be integers satisfying $n_1 = q_1(k + 2) + r_1, 0 \leq r_1 \leq k + 1$, and $n_2 = q_2(k+2)+r_2, 0 \leq r_2 \leq k+1$. By Claim 2, if equality holds in (14), then $r_1+r_2 \leq k+1$, which implies that $q = q_1+q_2$ and $r = r_1 + r_2$. Since equality in (14) holds, it follows by (15) that both $|A(H_1)| = \binom{n_1}{2} + (n_1 - 1)k + q_1\left(1 + 2k - \binom{k+2}{2}\right)$ and $|A(H_2)| = \binom{n_2}{2} + (n_2 - 1)k + q_2\left(1 + 2k - \binom{k+2}{2}\right)$. Therefore by induction, $H_1, H_2 \in \mathcal{E}_1'(k)$. By Definition 3.10, $D \in [H_1, H_2]_k$, which is in $\mathcal{E}_1'(n, k)$, and so $D \in \mathcal{E}_1'(k)$. This induction argument justifies the claim.

Now Theorem 4.1 follows from Lemma 3.12 and Claims 1–3. □

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