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Ore-type degree condition of supereulerian digraphs[☆]

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ABSTRACT

A digraph D is supereulerian if D has a spanning directed eulerian subdigraph. Hong et al. proved that $\delta^+(D) + \delta^-(D) \geq |V(D)| - 4$ implies D is supereulerian except some well-characterized digraph classes if the minimum degree is large enough. In this paper, we characterize the digraphs D which are not supereulerian under the condition $d_D^+(u) + d_D^-(v) \geq |V(D)| - 4$ for any pair of vertices u and v with $uv \notin A(D)$ without the minimum degree constraint.

1. Introduction

We consider finite simple digraphs that do not have loops nor parallel arcs (bi-direction edges are allowed). For undefined terms and notations, refer to [4] for graphs and [1] for digraphs. To avoid possible confusion, we use ditrails, dipaths and dicycles to mean directed trails, paths, and cycles, while trails, paths and cycles refer to undirected graph terminology.

Let D be a digraph. We use uv to denote an arc oriented from a vertex u to a vertex v. For a vertex u of D, the *out-degree* $d_D^+(u)$ (*in-degree* $d_D^-(u)$) is the number of arcs leaving from u (coming to u). If X and Y are disjoint subsets of $V(D)$, then $\lambda_D(X, Y)$ denotes the maximum number of arc-disjoint dipaths from X to Y in D. As in [1], $A(D)$ denotes the set of arcs in D, and $\delta^+(D)$, $\delta^-(D)$ denote the minimum out-degree and the minimum in-degree of D.

Motivated by the Chinese Postman Problem, Boesch, Suffel, and Tindell [3] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning eulerian subgraphs, and they indicated that this problem would be very difficult. Pulleyblank [9] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been lots of researches on this topic. Catlin [5] in 1992 presented the first survey on supereulerian graphs. Later Chen et al. [6] gave an update in 1995, specifically on the reduction method associated with the supereulerian problem. A recent survey on supereulerian graphs is given in [8].

It is natural to investigate supereulerian digraphs. A digraph D is said to be *eulerian* if D is strongly connected and every vertex has a same in-degree and out-degree. If a digraph contains a spanning eulerian subdigraph, then D is said to be *supereulerian*. In [7], Hong et al. proved that for any strong digraph D with $\min\{\delta^+(D), \delta^-(D)\} \geq 4$, if $\delta^+(D)+\delta^-(D) > n-4$ then D is supereulerian and characterize the counterexample when the equality holds.

Later, Bang-Jensen and Maddaloni [2] gave some sufficient Ore-type conditions to be supereulerian. Let D be a digraph on n vertices. A pair of vertices (u, v) of D is said to be *dominating* (*dominated*) if there exists a vertex w such that $uw, vw \in A(D)$ ($wu, uv \in A(D)$). In [2], Bang-Jensen and Maddaloni proved that a strong digraph D is supereulerian if $d_D^+(x)+d_D^-(y) \geq n-1$

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for any ordered pair (x, y) of dominated or dominating non-adjacent vertices. Also, in [2], they proved that a strong D is supereulerian if $d_D^+(x) + d_D^-(x) + d_D^+(y) + d_D^-(y) \geq 2n - 3$ for any pair of non-adjacent vertices. In this paper, we investigate the Ore-type sufficient condition of supereulerian digraphs and obtain the following theorem.

Theorem 1.1. *Let D be a strong digraph of order $n \geq 11$. If*

$$d_D^+(x) + d_D^-(y) \geq n - 4 \text{ for any pair of vertices } (x, y) \text{ with } xy \notin A(D), \tag{1.1}$$

then D is supereulerian if and only if it does not belong to a well characterized family of exceptional digraphs.

The proof arguments take a different approach from that in [7]. The family of exceptional graphs are also different from that in [7]. For simplicity of the statement, we give some terminologies used in this paper first. For a vertex set $X \subset V(D)$, denote by $N_D^+(X)$ the set of vertices in $V(D) - X$ which has an in-neighbor in X and by $N_D^-(X)$ the set of vertices in $V(D) - X$ which has an out-neighbor in X. For simplicity, for a subdigraph H, we write $N_D^+(H) = N_D^+(V(H))$ and $N_D^-(H) = N_D^-(V(H))$. For a pair of disjoint sets $X, Y \subset V(D)$, $(X, Y)_D$ stands the set of all the arcs with tail in X and head in Y. When $Y = V - X$, we use $\partial_D^+(X) = (X, V - X)_D$, and $\partial_D^-(X) = (V - X, X)_D$. When $X = \{v\}$, we also use $\partial_D^+(v) = \partial_D^+(\{v\})$ and $N_D^+(v) = N_D^+(\{v\})$.

For any disjoint vertex sets X, Y, an (X, Y)-ditrail (or dipath) is a ditrail (or a dipath) from a vertex in X to a vertex in Y and none of whose internal vertex lies in $X \cup Y$. An (X, Y)-segment of a ditrail (or a dipath) P is an (X, Y)-ditrail (or an (X, Y)-dipath) which is a subdigraph of P. When $X = \{x\}$ and $Y = \{y\}$, we may use (x, y)-ditrail (or dipath) instead of $(\{x\}, \{y\})$-ditrail (or dipath).

In Section 2, we apply a necessary condition for a digraph to be supereulerian in [7] to find some candidates of the exceptional graphs for the main result. The proof of the main result is presented in Section 3.

2. Some classes of digraphs

Let D be a strong digraph and $U \subset V(D)$. Then in $D[U]$, the digraph induced by U, we can find some ditrails $P_1, \ldots, P_t$ such that $\bigcup_{i=1}^t V(P_i) = U$ and $A(P_i) \cap A(P_j) = \emptyset$ for any $i \neq j$. Let $\tau(U)$ be the minimum value of such t. Then $c(G(D[U])) \leq \tau(U) \leq |U|$, where $c(G(D[U]))$ is the number of components of the underlying graph of $D[U]$. For any $X \subseteq V(D) - U$, denote $Y := V(D) - U - X$ and let

$$h(U, X) := \min\{|\partial_D^+(X)|, |\partial_D^-(X)|\} + \min\{|(U, Y)_D|, |(Y, U)_D|\} - \tau(U), \text{ and}$$
$$h(U) := \min\{h(U, X) : X \cap U = \emptyset\}.$$

In [7], Hong et al. give the following proposition, and use it to find some classes of digraphs which are not supereulerian.

Proposition 2.1 ([7]). *If D has a spanning eulerian subdigraph, then for any $U \subset V(D)$, $h(U) \geq 0$.*

Hong et al. [7] used this proposition to find the following example digraphs, each of which has a large minimum degree sum but contains no spanning eulerian subdigraphs.

Example 2.2. Let $k_1, k_2 \geq 0$, $\ell \geq 2$ be integers with $(k_1 + 1)(k_2 + 1) \geq \ell - 1$, and D_1 and D_2 be two disjoint complete digraphs of order $k_1 + 1$ and $k_2 + 1$, respectively. Let U be an independent set disjoint from $V(D_1) \cup V(D_2)$ with $|U| = \ell$. Let $\mathcal{D}(k_1, k_2, \ell)$ denote the family of digraphs such that $D \in \mathcal{D}(k_1, k_2, \ell)$ if and only if D is the digraph obtained from $D_1 \cup D_2 \cup U$ by adding all arcs directed from every vertex in D_2 to every vertex in $U \cup D_1$, and all arcs directed from every vertex in U to every vertex in D_1, and then by adding a set of $\ell - 1$ arcs from some vertices in D_1 to some vertices in D_2.

Let $\mathcal{D}_1$ denote the family $\mathcal{D}(k_1, k_2, 2)$, Hong et al. [7] proved if a simple digraph D satisfying $\min\{\delta^+(D), \delta^-(D)\} \geq 4$ and $\delta^+(D) + \delta^-(D) \geq n - 4$, then D is supereulerian if and only if D is not a member in $\mathcal{D}_1$. Moreover, if the condition $\min\{\delta^+(D), \delta^-(D)\} \geq 4$ is removed, more new exceptional non-supereulerian digraphs will appear. Let $\mathcal{D}_2 \subseteq \bigcup_{i=1}^2 \mathcal{D}(i, k_2, 3) \cup \mathcal{D}(k_1, i, 3)$ be the family of digraphs with minimum out-degree or minimum in-degree 2. By using Proposition 2.1, Hong et al. [7] proved no digraph in $\mathcal{D}(k_1, k_2, \ell)$ is supereulerian, and so every one in $\mathcal{D}_1 \cup \mathcal{D}_2$ is nonsupereulerian.

Next, let $\mathcal{D}_3$ be the set of digraphs obtained from digraphs in $\mathcal{D}(0, k_2, 2) \cup \mathcal{D}(k_1, 0, 2)$ by replacing a vertex in U by a dicycle $w_1 w_2 w_1$ of length 2 and adding all the arcs from $\{w_1, w_2\}$ to $V(D_1)$ and all the arcs from $V(D_2)$ to $\{w_1, w_2\}$. By Proposition 2.1, none of the digraphs in $\mathcal{D}_3$ is supereulerian. In fact, let $D \in \mathcal{D}_3$, by the construction, $\tau(U) = 2$. Let $X = V(D_1)$ and $Y = V(D_2)$. Then $h(U, X) = \min\{|\partial_D^+(X)|, |\partial_D^-(X)|\} + \min\{|(U, Y)_D|, |(Y, U)_D|\} - \tau(U) = 1 + 0 - 2 < 0$, and so D is not supereulerian by Proposition 2.1.

Therefore, for $i = 1, 2, 3$, none of the spanning subdigraphs of digraphs in $\mathcal{D}_i$ has a spanning eulerian subdigraph. For $i = 1, 2, 3$, let $\mathcal{F}_i$ be the family of digraphs such that $D \in \mathcal{F}_i$ if and only if for some member $D' \in \mathcal{D}_i$, D is a strong spanning subdigraph of D' satisfying (1.1). Then, each $\mathcal{F}_i$ is also a family of non-supereulerian digraphs. In next section, we will show that if a digraph D satisfies this Ore-type degree condition (1.1), then D is supereulerian if and only if D is not a member of $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

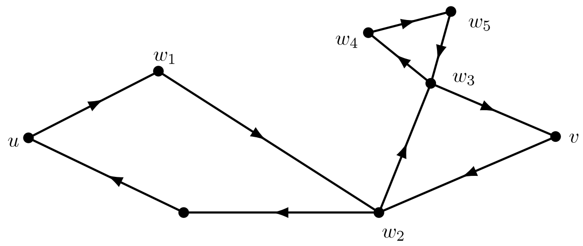

Fig. 1. An example of increment, where the (u, v)-ditrail $Q = uw_1w_2w_3v$, $\bar{Q} = uw_1w_2w_3w_4w_5w_3v$, $I_Q = \{w_1, w_3, w_4, w_5\}$.

3. An ore-type degree condition for a digraph to be supereulerian

In this section, we characterize the non-supereulerian digraphs D which satisfy (1.1). The main tool used in this paper, called *increment*, is the same to that in [7]. The formal definition is stated below.

Definition 3.1 ([7]). Let H be a eulerian subdigraph of a digraph D. Suppose for some distinct vertices $u, v \in V(H)$, Q is a (u, v)-ditrail of H. Let H' be the connected component of the underlying graph $H - A(Q)$ containing both u and v. Define $I_Q = V(H) - V(H')$, which is called the *increment of* Q *with respect to* H. If the eulerian subdigraph H is clear from context, we also say I_Q is the increment of Q.

Since H can also be viewed as a closed ditrail, H has a minimum (u, v)-ditrail that contains all arcs in $A(H[I_Q]) \cup A(Q)$. This ditrail is denoted by $\bar{Q}$. Note that it is possible that $\bar{Q} = Q$. Also, the underlying graph of $H[I_Q]$ might not be connected (see Fig. 1 for an example).

Using these definitions and notations, we have the following observation stated as the next lemma.

Lemma 3.2 ([7]). *Let D be a digraph, H be a eulerian subdigraph of D, and $X, Y \subseteq V(H)$ be two disjoint vertex sets. Then for any (X, Y)-ditrail Q, $(V(H - I_Q), I_Q)_H \cup (I_Q, V(H - I_Q))_H \subseteq A(Q)$, and for any two arc-disjoint (X, Y)-ditrails Q_1, Q_2, $I_{Q_1} \cap I_{Q_2} = \emptyset$.*

In order to make the proof be easier to read, we present a lemma first.

Lemma 3.3. *Let D be a strong digraph with order $n \geq 5$. If $\delta^-(D) \geq n - 3$ or $\delta^+(D) \geq n - 3$, then for any two vertices u, v there is a (u, v)-ditrail P of D such that $|V(P)| \geq n - 1$.*

Proof. By symmetry, we only prove the case when $\delta^-(D) \geq n - 3$. Let u, v be two arbitrary vertices of D and P be a (u, v)-ditrail such that $p := |V(P)|$ is maximized. Denote $P = v_1v_2 \dots v_t$, where $v_1 = u$, $v_t = v$. Note that t may be greater than p. We first show that $t \geq 3$. In fact, as $\delta^-(D) \geq n - 3 \geq 2$, there is a vertex $w \in N_D^-(v)$ different from u. Since D is strong, u has a dipath to w in $D - wv$. By adding the arc wv to the dipath, we obtain a (u, v)-ditrail with at least 2 arcs, which implies $t \geq 3$.

Let $R = V(D) - V(P)$. If $|R| \leq 1$, then we are done. So we may assume $|R| \geq 2$. Since D is strong, we may assume there exists a vertex $w \in R$ such that $wv_i \in A(D)$ for some $1 \leq i \leq t$. Choose such w and v_i such that i is maximized. Let $X = \{v_1, \dots, v_i\}$ and $Y = V(P) - X$.

If $|X| \geq 3$, then neither $v_iw \in A(D)$ nor $v_{i-1}w \in A(D)$. For, otherwise, either $v_1 \dots v_iwv_i \dots v_t$ or $v_1 \dots v_{i-1}wv_i \dots v_t$ is a (u, v)-ditrail with $p + 1$ vertices, contradicts the maximality of p. This, together with the fact $\delta^-(D) \geq n - 3$, forces $N_D^-(w) = V(D) - \{v_{i-1}, v_i, w\}$. Pick $w' \in R - \{w\}$. Then $w'w \in A(D)$. Similar to the above, we may show that $N_D^-(w') \cap \{v_{i-1}, v_i\} = \emptyset$. If $v_{i-2} \neq v_i$ then $v_{i-2}w' \notin A(D)$, since otherwise, $v_1 \dots v_{i-2}w'wv_i \dots v_t$ is a (u, v)-ditrail with $p + 1$ vertices, a contradiction. If $v_{i-2} = v_i$, then $v_{i-3}w' \notin A(D)$, since otherwise, $v_1 \dots v_{i-3}w'wv_{i-2} \dots v_t$ is a (u, v)-ditrail with $p + 2$ vertices (here v_{i-3} exists according to the assumption $|X| \geq 3$), a contradiction. In either cases, $d_D^-(w') \leq n - 4$, a contradiction to the fact $\delta^-(D) \geq n - 3$.

If $|X| \leq 2$, then either $i \leq 2$ or $i = 3$ and $v_1 = v_3$. Similar to the previous paragraph, it is easy to see that $N_D^-(w) \cap X = \emptyset$. Thus

$$|N_D^-(w) \cap Y| \geq n - 3 - (|R| - 1) = n - |R| - 2. \tag{3.1}$$

Also, by the assumption that $t \geq 3$ and $u \neq v$, $Y \neq \emptyset$. For any $v_j \in Y$, by the choice of v_i and X, $j > i$ and thus $N_D^-(v_j) \cap R = \emptyset$. This, together with the fact $\delta^-(D) \geq n - 3$, forces $|R| = 2$ and $N_D^-(v_j) = V(P) - \{v_j\}$. By the arbitrary of v_j, $D[Y]$ is a complete digraph. We relabel the vertices of Y as $u_1, \dots, u_m = v_t$. By (3.1), $|N_D^-(w) \cap Y| \geq 1$. Assume $u_jw \in A(D)$. Then $v_1 \dots v_iu_jwv_iu_1 \dots u_{j-1}u_{j+1} \dots u_m$ is a (u, v)-ditrail with $p + 1$ vertices, a contradiction. This completes the proof. ■

Theorem 3.4. *Let D be a strong digraph of order $n \geq 11$ satisfying (1.1). Then D has a spanning eulerian subdigraph if and only if $D \notin \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.*

Proof. By Example 2.2, no digraph in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ has a spanning eulerian subdigraph, and so the necessity is clear. To prove the sufficiency, we assume that D satisfies (1.1) and that

$$D \text{ is not supereulerian} \tag{3.2}$$

to show that $D \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Choose a eulerian subdigraph H of D such that

$$|V(H)| \text{ is maximized.} \tag{3.3}$$

For any $u \in N_D^+(H)$, since D is strong, there is a dipath from u to $V(H)$, which must visit a vertex of $N_D^-(H)$, say v. Let P be the (u, v)-segment of this dipath. Then there exist $x, y \in V(H)$ such that $xu, vy \in A(D) - (A(H) \cup A(P))$. Furthermore, we may choose H as a eulerian subdigraph satisfying (3.3) and choose $u \in N_D^+(H)$, $v \in N_D^-(H)$ and P a (u, v)-ditrail in $D - A(H)$ such that

$$\begin{array}{ll} (1) & \text{there exist } x, y \in V(H) \text{ such that } xu, vy \in A(D) - (A(H) \cup A(P)); \\ (2) & \text{subject to (1), } |V(P) - V(H)| \text{ is maximized;} \\ (3) & \text{subject to (1)(2), } |A(P)| \text{ is minimized;} \\ (4) & \text{subject to (1)(2)(3), } d_D^-(u) + d_D^+(v) \text{ is maximized.} \end{array} \tag{3.4}$$

Note that $V(P) \cap V(H) \neq \emptyset$ is possible. Let $p = |V(P) - V(H)|$. As $u \in V(P) - V(H)$, we have $p \geq 1$. Let $H_0 = D[V(H) \cup \{u, v\}] - A(D[\{u, v\}]) - \big(A(H) \cup A(P)\big)$ and define

$$X = \{x \in V(H) \mid H_0 \text{ has a dipath from } x \text{ to } u\}, \textit{ and}$$
$$Y = \{y \in V(H) \mid H_0 \text{ has a dipath from } v \text{ to } y\}.$$

With these definition, we have the following claim.

Claim 1. Each of the following holds.

(i) Each dicycle of P is vertex-disjoint with H.
(ii) $X \neq \emptyset, Y \neq \emptyset, X \cap Y = \emptyset$.
(iii) $(V(H) - X, X)_D \subseteq A(H)$, $(Y, V(H) - Y)_D \subseteq A(H)$.

(i) Suppose, to the contrary, that P contains a dicycle C such that $V(H) \cap V(C) \neq \emptyset$. Then $V(C) \subseteq V(H)$. For otherwise, $A(H) \cup A(C)$ induces a eulerian subdigraph of D with at least $|V(H)| + 1$ vertices, contradicts (3.3). Thus $P - A(C)$ is still a ditrail of D satisfying (3.4)(1) and (3.4)(2), and containing less arcs, contradicts (3.4)(3). This proves (i).

(ii) By the choice of P as described in (3.4), $X, Y \neq \emptyset$. Suppose that there exists $w \in X \cap Y$. Then by the definition of X and Y, H_0 has a dipath P_1 from w to u and a dipath P_2 from v to w. Thus each P_i is arc-disjoint with P and H. By the definition of X, Y, $V(P_1) \subseteq X \cup \{u\}$ and $V(P_2) \subseteq Y \cup \{v\}$. Thus, we may choose $w \in X \cap Y$ such that $A(P_1) \cap A(P_2) = \emptyset$. It follows that $H + wP_1uPvP_2w$ is a eulerian subdigraph with at least $|V(H)| + 1$ vertices, contradicts (3.3). This proves (ii).

(iii) It follows from the definitions of X and Y, $(V(H) - X, X)_D \cup (Y, V(H) - Y)_D \subseteq A(H) \cup A(P)$. Furthermore, if there is an arc $x'x \in A(P)$ such that $x \in X$ and $x' \notin X$, then by letting P_3 be the dipath from x to u in H_0, $H + xP_3uPx$ is a eulerian subdigraph with at least $|V(H)| + 1$ vertices, contradicts (3.3). Thus, $(V(H) - X, X)_D \subseteq A(H)$. Similarly, $(Y, V(H) - Y)_D \subseteq A(H)$. Claim 1 is proved. □

By the definition of X and Y, for any $x \in X$ and $y \in Y$, there exist an (x, u)-dipath in H_0 and a (v, y)-dipath in H_0. By Claim 1, $X \cap Y = \emptyset$. So, in the rest of the proof we may use P_x and P_y to represent the (x, u)-dipath and the (v, y)-dipath, respectively.

Claim 2. $N_D^-(u) \subseteq X \cup V(P)$, $N_D^+(v) \subseteq Y \cup V(P)$.

By symmetry, it suffices to show $N_D^-(u) \subseteq X \cup V(P)$. In fact, by the definition of X, it suffices to show that $N_D^-(u) \subseteq V(H) \cup V(P)$. Suppose, to the contrary, that there exists $w \in N_D^-(u) - (V(H) \cup V(P))$. If $w \in N_D^+(H)$, then the dipath $P' = wuPv$ is also a candidate of P with $|V(P') - V(H)| = |V(P) - V(H)| + 1$, contradicts (3.4). If there exists $w_1 \in N_D^+(H) \cap N_D^-(w)$, then let $x_1 \in V(H)$ such that $x_1w_1 \in A(D)$. If $x_1w_1 \notin A(H) \cup A(P)$, then $P' = w_1wuPv$ is also a candidate of P with $|V(P') - V(H)| \geq |V(P) - V(H)| + 1$, contradicts (3.4). So, $x_1w_1 \in A(H) \cup A(P)$. Since $w_1 \notin V(H)$, we must have $x_1w_1 \in A(P)$. Thus $x_1 \in V(H) \cap V(P)$ and $H + x_1w_1wuPx_1$ is a eulerian subdigraph with order at least $|V(H)| + 1$, contradicts (3.3). Hence, $(N_D^+(H) \cup V(H)) \cap (N_D^-(w) \cup \{w\}) = \emptyset$. It follows that $n \geq |V(H)| + |N_D^+(H)| + |N_D^-(w)| + 1$.

Let $\bar{H} = D[V(H)] - A(P)$. Then $A(\bar{H}) \cap A(P) = \emptyset$. We will use Lemma 3.3 to find a long trail in $\bar{H}$, which will result in a eulerian subdigraph violating (3.3). First, we need to verify the conditions of Lemma 3.3.

(2A) $d_{\bar{H}}^+(x) \geq |V(\bar{H})| - 3$ for all $x \in V(\bar{H})$.

For any $x \in V(\bar{H}) = V(H)$, as $xw \notin A(D)$, by (1.1), $d_D^+(x) + d_D^-(w) \geq n - 4$, it follows that $d_D^+(x) \geq n - 4 - |N_D^-(w)| \geq |V(H)| + |N_D^+(H)| - 3$. Thus $d_{\bar{H}}^+(x) \geq d_D^+(x) - |N_D^+(x) \cap N_D^+(H)| - d_P^+(x) \geq |V(H)| + |N_D^+(H) - N_D^+(x)| - d_P^+(x) - 3$.

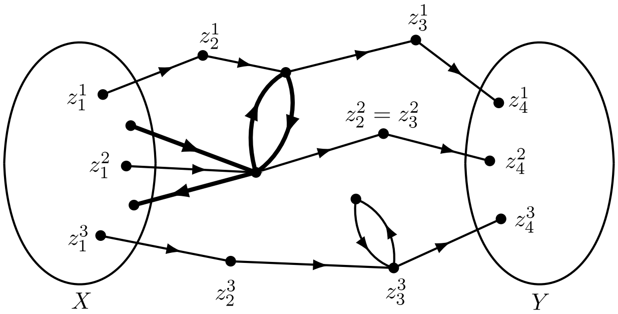

Fig. 2. An example for z_i^j, where the bold arcs are in H but not in Q_i.

If $x \notin V(P)$, then $d_P^+(x) = 0$ and thus $d_{\bar{H}}^+(x) \geq d_D^+(x) - |N_D^+(H)| \geq |V(H)| - 3 = |V(\bar{H})| - 3$. So, we may assume $x \in V(P)$. By Claim 1(i), $d_P^+(x) = 1$. Also, if $xu \in A(D)$, then $H + xuPx$ is a eulerian subdigraph with at least $|V(H)| + 1$ vertices, contradicts (3.3). So, $u \in N_D^+(H) - N_D^+(x)$. Thus $d_{\bar{H}}^+(x) \geq |V(H)| + |N_D^+(H) - N_D^+(x)| - 4 \geq |V(\bar{H})| - 3$.

(2B) $|V(\bar{H})| \geq 5$.

Suppose, to the contrary, that $|V(H)| = |V(\bar{H})| \leq 4$. Then any dicycle of D has length at most 4. Let

$$T = v_1 v_2 \ldots v_t \text{ be a longest dipath of } D. \tag{3.5}$$

Then $d_D^+(v_t) = |N_D^+(v_t) \cap V(T)| \leq 3$ and $d_D^-(v_1) = |N_D^-(v_1) \cap V(T)| \leq 3$. Since $d_D^+(v_t) + d_D^-(v_1) \leq 6 < n - 4$, we have $v_t v_1 \in A(D)$, and so $v_1 T v_t v_1$ is a dicycle of D. It follows that $t \leq 4$. Since D is strong, there is a vertex $z \in V(D) - V(T)$ such that $zv_i \in A(D)$ for some $1 \leq i \leq t$. Thus $zv_i \ldots v_t v_1 \ldots v_{i-1}$ is a dipath with $|V(T)| + 1$ vertices, contradicts (3.5). Hence, $|V(\bar{H})| = |V(H)| \geq 5$.

So, by Lemma 3.3, for any $x \in N_D^-(u) \cap V(H)$ and any $y \in N_D^+(v) \cap V(H)$, there is a (y, x)-ditrail Q in $\bar{H}$ such that $|V(Q)| \geq |V(H)| - 1$. As $A(Q) \cup A(P) \cup \{xu, vy\}$ induces a eulerian subdigraph of D with at least $|V(H)| + p - 1$ vertices, by (3.3), $p = 1$, which implies $u = v$. Since D is strong, w has a dipath to $V(H)$, which must visit a vertex w_1 of $N_D^-(H)$. Let P' be the (w, w_1)-segment of this dipath. Then $P'' := uwP'w_1$ is also a candidate of P such that $|V(P'') - V(H)| \geq 2 > p$, a contradiction to (3.4). This proves Claim 2. □

Let $\lambda_H(X, Y)$ denote the maximum number of arc-disjoint (X, Y)-dipaths in H. By Menger's Theorem (Page 170, Theorem 7.16 of [4]), $\lambda_H(X, Y) = \min\{\partial_H^+(U) \mid X \subset U \text{ and } Y \cap U = \emptyset\}$ and $\lambda_H(Y, X) = \min\{\partial_H^-(U) \mid X \subset U \text{ and } Y \cap U = \emptyset\}$. However, since H is a eulerian subdigraph of D, $|\partial_H^+(U)| = |\partial_H^-(U)|$ holds for each $U \subset V(H)$. Therefore, we have $\lambda_H(X, Y) = \lambda_H(Y, X)$. Assume $\lambda_H(X, Y) = k$ and $Q_1, \ldots, Q_k$ are k arc-disjoint (X, Y)-dipaths.

For $i = 1, \ldots, k$, let I_{Q_i} be the increment of Q_i. If $I_{Q_i} \cap (X \cup Y) \neq \emptyset$ for some i, then $\bar{Q}_i$ has some internal vertex in $X \cup Y$, where $\bar{Q}_i$ is the minimal ditrail containing $A(Q_i) \cup A(H[I_{Q_i}])$. Thus we may choose an (X, Y)-segment of $\bar{Q}_i$ as Q_i. Then all Q_i's are still pairwise arc-disjoint and the new $\bar{Q}_i$ contains less arcs. So, we may assume $Q_1, \ldots, Q_k$ arc such arc-disjoint dipaths such that $\sum_{i=1}^k |A(\bar{Q}_i)|$ is minimized. Then $I_{Q_i} \cap (X \cup Y) = \emptyset$ for $i = 1, \ldots, k$.

Notation 3.5. *Suppose that Q_i is a dipath from $z_1^i \in X$ to $z_4^i \in Y$ and that z_2^i be the first vertex of $\bar{Q}_i$ in I_{Q_i}, and z_3^i be the last vertex of $\bar{Q}_i$ in I_{Q_i}.*

Note that it is possible that $z_1^i \bar{Q}_i z_2^i$ and $z_3^i \bar{Q}_i z_4^i$ contain more than one arcs (see Fig. 2 for example). By Lemma 3.2, $I_{Q_i} \cap I_{Q_j} = \emptyset$ for any $i \neq j$. Let $q_i = |I_{Q_i}|$. We may furthermore assume $q_1 \leq q_2 \leq \cdots \leq q_k$.

Note that $z_1^1 \in X$ and $z_4^1 \in Y$. Let $H' := H - A(\bar{Q}_1) - I_{Q_1} + z_1^1 P_{z_1^1} uPv P_{z_4^1} z_4^1$. Then H' is a eulerian subdigraph of D with at least $|V(H)| - q_1 + p$ vertices. By (3.3), we must have $q_1 \geq p$ and so $q_k \geq \cdots \geq q_1 \geq p$. Let $H_1 = H - \bigcup_{i=1}^k A(\bar{Q}_i) - \bigcup_{i=1}^k I_{Q_i}$. Note that H_1 may not be connected when $k \geq 2$. As H can be viewed as a eulerian ditrail, $\lambda_H(Y, X) = \lambda_H(X, Y) = k$ and there are k arc-disjoint (Y, X)-dipaths in H which are also arc-disjoint with $Q_1, \ldots, Q_k$. Then by the definition of H_1, $\lambda_{H_1}(Y, X) = \lambda_H(Y, X) = k$. By Menger's Theorem, there is a partition (X', Y') of $V(H_1)$ such that

$$X \subseteq X', Y \subseteq Y' \quad \text{and} \quad |(Y', X')_{H_1}| = k. \tag{3.6}$$

Furthermore, subject to (3.6), we may also assume the partition (X', Y') satisfies

$$\mu(X', Y') \triangleq |(Y', X')_{D-A(H)}| \left(1 + \sum_{i=1}^k q_i\right) + \left|\left(Y', \bigcup_{i=1}^k I_{Q_i}\right)_D\right| + \left|\left(\bigcup_{i=1}^k I_{Q_i}\right), X'\right)_D\right| \text{ is minimized.} \tag{3.7}$$

As H is eulerian, $|(X', Y')_H| = |(Y', X')_H| = k$. Then by the definition of H_1, it is easy to see that $(X', Y')_{H_1} = \emptyset$. Define

$$R = V(D) - V(H) - V(P), \quad \text{and} \quad r = |R|.$$

Then $n = |X'| + |Y'| + \sum_{i=1}^{k} q_i + p + r$.

Claim 3. For each i, $\big(N_D^-(X) \cup N_D^+(Y)\big) \cap \big(R \cup I_{Q_i} \cup (V(P) - V(H))\big) = \emptyset$.

By the symmetry between X and Y, we only show the case when $N_D^-(X) \cap (R \cup I_{Q_i} \cup (V(P) - V(H))) = \emptyset$. Suppose, to the contrary, that there exist $x_1 \in X$ and $w \in R \cup I_{Q_i} \cup (V(P) - V(H))$ such that $wx_1 \in A(D)$. Then $wx_1 \notin A(Q_i)$. By Lemma 3.2, we have $wx_1 \notin A(H)$, Then, by Claim 1(iii), $w \notin V(H) - X$ and so $w \in R \cup \big(V(P) - V(H)\big)$.

If $w \in V(P) - V(H)$, then $H + x_1P_{x_1}uPwx_1$ is a eulerian subdigraph of D with at least $|V(H)| + 1$ vertices, contradicts (3.3). Hence, $w \in R$. Then as D is strong, there is a dipath P_1 from a vertex $x \in V(H)$ to w, none of whose inner vertex lies in $V(H)$. If $V(P_1) \cap V(P) \neq \emptyset$, then let $w_1 \in V(P)$ be the first vertex of P_1 and thus $H + x_1P_{x_1}uPw_1P_1wx_1$ is a eulerian subdigraph of D with at least $|V(H)| + 1$ vertices, contradicts (3.3). So, $V(P_1) \cap V(P) = \emptyset$. Let w_2 be the successor of x on P_1. Then $w_2 \notin V(H)$, and so $P' = w_2P_1wx_1P_{x_1}uPv$ is a candidate of P with $|V(P') - V(H)| > |V(P) - V(H)|$, contradicts (3.4). This proves Claim 3. □

For any vertex $x \in X$, by Claim 3, $d_D^-(x) \le |X'| - 1 + |N_D^-(x) \cap Y'|$. By Claim 2, $d_D^+(v) \le |Y| + p - 1 + |N_D^+(v) \cap V(P) \cap V(H)|$. In fact, if there exists $w \in N_D^+(v) \cap V(P) \cap V(H)$, then $H + wPvw$ is a eulerian subdigraph with at least $|V(H)| + 1$ vertices, contradicts (3.3). Thus $d_D^+(v) \le |Y| + p - 1$. Moreover, by Claim 3, $vx \notin A(D)$. So, $n - 4 \le d_D^+(v) + d_D^-(x) \le |X'| + |Y| + p + |N_D^-(x) \cap Y'| - 2 = n - \sum_{i=1}^{k} q_i - |Y' - Y| - r + |N_D^-(x) \cap Y'| - 2$. It follows that

$$\sum_{i=1}^{k} q_i + |Y' - Y| + r \le 2 + |N_D^-(x) \cap Y'|. \tag{3.8}$$

Similarly, for any $y \in Y$, by considering the pair (y, u), we also have

$$\sum_{i=1}^{k} q_i + |X' - X| + r \le 2 + |N_D^+(y) \cap X'|. \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$2\sum_{i=1}^{k} q_i + |X' - X| + |Y' - Y| + 2r \le 4 + |N_D^-(x) \cap Y'| + |N_D^+(y) \cap X'|. \tag{3.10}$$

Note that every arc in $(Y, X')_D \cup (Y', X)_D - \{yx\}$ contributes at most 1 to $|N_D^-(x) \cap Y'| + |N_D^+(y) \cap X'|$ and the arc yx (if exists) contributes 2 to $|N_D^-(x) \cap Y'| + |N_D^+(x) \cap Y'|$. Thus $|N_D^-(x) \cap X'| + |N_D^+(y) \cap Y'| \le |(Y, X')_D \cup (Y', X)_D| + 1$. By Claim 1(iii), $|(Y, X')_D \cup (Y', X)_D| = |(Y, X')_H \cup (Y', X)_H| \le |(Y', X')_{H_1}| = k$. Thus, by (3.10), $2\sum_{i=1}^{k} q_i + |X' - X| + |Y' - Y| + 2r \le 4 + |N_D^-(x) \cap X'| + |N_D^+(y) \cap Y'| \le k + 5$. It follows that

$$\begin{aligned}
|X| + |Y| &= n - |X' - X| - |Y' - Y| - \sum_{i=1}^{k} q_i - p - r \\
&\ge n - k - 5 + \sum_{i=1}^{k} q_i - p + r \\
&\ge n - 5 + (k-1)p - k + r \\
&\ge n + r - 6 \\
&\ge 5.
\end{aligned} \tag{3.11}$$

Claim 4. $p = 1$ and $|X| + |Y| \ge 6$.

Firstly, we show that $|(Y, X)_D| < |Y| \cdot |X|$. Suppose this is not true. Then $k = |(Y', X')_{H_1}| \ge |(Y, X)_{H_1}| = |(Y, X)_D| = |X| \cdot |Y|$. On the other hand, by (3.9), $k \le \sum_{i=1}^{k} q_i \le 2 + |N_D^+(y) \cap X'| - |X' - X| \le 2 + |X|$. It follows that $|X| \ge k - 2$. Similarly, $|Y| \ge k - 2$. Hence, if $k \ge 2$, then $k \ge |X| \cdot |Y| \ge (k-2)^2$ and thus $k \le 4$. So, $k \le 4$ anyway. By (3.11), we have $|X| + |Y| \ge 5$. This, together with $|X| \cdot |Y| \le k \le 4$, forces that $k = 4$ and either $|X| = 1, |Y| = 4$ or $|X| = 4, |Y| = 1$. However, we have deduced that $|X| \ge k - 2 = 2$ and $|Y| \ge k - 2 = 2$, a contradiction. So, $|(Y, X)_D| < |Y| \cdot |X|$, which implies there is a vertex $x_1 \in X$ and a vertex $y_1 \in Y$ such that $y_1x_1 \notin A(D)$. So, $d_D^-(x_1) + d_D^+(y_1) \ge n - 4$. On the other hand, by Claim 3, $d_D^-(x_1) \le |X'| - 1 + |N_D^-(x_1) \cap Y'|$ and $d_D^+(y_1) \le |Y'| - 1 + |N_D^+(y_1) \cap X'|$. Hence,

$$\begin{aligned} n - 4 &\le |X'| + |Y'| - 2 + |N_D^-(x_1) \cap X'| + |N_D^-(y_1) \cap Y'| \\ &= n - \sum_{i=1}^{k} q_i - p - r - 2 + |N_D^-(x_1) \cap X'| + |N_D^+(y_1) \cap Y'|. \end{aligned}$$

It follows that $\sum_{i=1}^{k} q_i + p + r \le 2 + |N_D^-(x_1) \cap Y'| + |N_D^+(y_1) \cap X'|$. As every arc in $(Y', X')_D$ contributes at most 1 to $|N_D^-(x_1) \cap Y'| + |N_D^+(y_1) \cap X'|$ (note that $y_1x_1 \notin A(D)$), we have $|N_D^-(x_1) \cap Y'| + |N_D^+(y_1) \cap X'| \le |(Y', X)_D \cup (Y, X')_D| = |(Y', X)_H \cup (Y, X')_H| \le k$. Thus $\sum_{i=1}^{k} q_i + p + r \le k + 2$, and so $(k+1)p \le k+2$, which implies $p = 1$. Also, by using the pair (y_1, x_1) instead (y, x) in (3.10), similar to (3.11), we also deduce that $|X| + |Y| \ge 6$, which completes the proof of the claim. □

By Claim 4, $u = v$. If $V(P) \cap V(H) \ne \emptyset$, then $H + uPu$ is a eulerian subdigraph with $|V(H)| + 1$ vertices, contradicts (3.3). So, $V(P) \cap V(H) = \emptyset$ and thus P is in fact a trivial dipath.

Claim 5. $k \le 2$ and $|X| + |Y| \ge n - 4$.

By Claim 4, we may assume, without loss of generality, that $|X| \ge 3$. Then there is a vertex $x_1 \in X$ such that $|N_D^-(x_1) \cap Y'| \le k/|X| \le k/3$. By (3.8), $|Y' - Y| + k + r \le |Y' - Y| + \sum_{i=1}^{k} q_i + r \le 2 + k/3$. It follows that $k \le 3 - 3(r + |Y' - Y|)/2$. If $k \ge 3$, then $|X| = k = 3$, $r = |Y' - Y| = 0$ and $q_3 = q_2 = q_1 = p = 1$. By Claim 4, $|Y| \ge 6 - |X| = 3$. Similarly, we also have $|Y| = k = 3$ and $|X' - X| = 0$, which implies $n = |X| + |Y| + q_1 + q_2 + q_3 + p = 10$, contradicts the assumption that $n \ge 11$. Hence, $k \le 2$. Thus $N_D^-(x_1) \cap Y' = \emptyset$.

For the second part of the claim, if $q_1 = 1$, then $z_2^1 = z_3^1$. Let $H' = H - z_1^1\bar{Q}_1z_1^4 + z_1^1P_{z_1^1}uPvP_{z_4^1}z_4^1$ and $P' = z_2^1$. It is easy to verify that H' is a eulerian digraph with maximum number of vertices and P' is a dipath satisfying (3.4)(1), (3.4)(2) and (3.4)(3). Thus $d_D^+(z_2^1) + d_D^-(z_2^1) \le d_D^+(v) + d_D^-(u) \le |X| + |Y|$. As $z_2^1u, vz_2^1 \notin A(D)$, we have $2(|X| + |Y|) \ge d_D^+(z_2^1) + d_D^-(u) + d_D^+(v) + d_D^-(z_2^1) \ge 2(n-4)$ and the result follows. So, we may assume that $q_1 \ge 2$. Then $\sum_{i=1}^{k} q_i \ge 2k$. By (3.9), $|X' - X| \le 2 + |N_D^+(y) \cap X'| - \sum_{i=1}^{k} q_i \le 2 + k - 2k \le 1$. Then, as $vx_1 \notin A(D)$, $n - 4 \le d_D^+(v) + d_D^-(x_1) \le |Y| + |X'| - 1$. It follows that $|X| + |Y| \ge n - 4$. □

Claim 6. $(Y', X')_{D-A(H)} = \emptyset$.

Suppose, to the contrary, that there exist $x' \in X'$ and $y' \in Y'$ such that $y'x' \in A(D) - A(H)$. Then by Claim 1, $x' \in X' - X$ and $y' \in Y' - Y$. By Claim 5, $|X' - X| = |Y' - Y| = q_1 = p = k = 1$. Assume $y_0 \in Y'$ such that $N_{H_1}^+(y_0) \cap X' \ne \emptyset$.

If $N_{H_1}^+(y') \cap Y = \emptyset$, then, by $y' \notin I_{Q_1}$, $d_{H_1}^+(y') \ge 1$, which forces $y' = y_0$ and $|N_{H_1}^-(y') \cap Y| = d_{H_1}^-(y') = d_{H_1}^+(y') = |N_{H_1}^+(y') \cap X| = 1$. Thus $|(Y, X' \cup \{y'\})_{H_1}| = |(Y', X')_{H_1}|$ and $(Y, X' \cup \{y'\})_{D-A(H)} = \emptyset$ by Claim 1(iii). However, noting that $k = 1$,

$$\begin{aligned} \mu(X', Y') - \mu(X' \cup \{y'\}, Y) &= |(Y', X')_{D-A(H)}|(1 + q_1) + |(Y', I_{Q_1})_D| + |(I_{Q_1}, X')_D| - |(Y, I_{Q_1})_D| - |(I_{Q_1}, X' \cup \{y'\})_D| \\ &\ge (1 + q_1) - |(I_{Q_1}, \{y'\})_D| > 0, \end{aligned}$$

contradicts (3.7). Hence, $N_{H_1}^+(y') \cap Y \ne \emptyset$. Similarly, $N_{H_1}^-(x') \cap X \ne \emptyset$. Furthermore, if there exists $y_1 \in N_{H_1}^+(y') \cap N_{H_1}^-(y') \cap Y$, then picking $x_1 \in N_{H_1}^-(x') \cap X$, we observe that $H - y'y_1y' - x_1x' + x_1P_{x_1}uPvP_{y_1}y_1y'x'$ is a spanning eulerian subdigraph of D, a contradiction. So, $N_{H_1}^+(y') \cap N_{H_1}^-(y') \cap Y = \emptyset$. Moreover, for any $y \in Y - \{y_0\}$, as $yu \notin A(D)$, $d_D^+(y) \ge n - 4 - d_D^-(u) \ge n - |Y' - Y| - |X' - X| - q_1 - p - |X| \ge |Y|$. This forces $yy' \in A(D)$ and, furthermore, $yy' \in A(H_1)$ by Claim 1(iii), and so $y'y \notin A(H_1)$. Thus $d_{H_1}^-(y') \ge |Y - \{y_0\}|$ and $d_{H_1}^+(y') \le 1$. So, $|Y| \le 2$. Similarly, $|X| \le 2$, a contradiction to Claim 4. □

By Claims 1 and 6, we see that $|(Y', X')_D| = k$. Next, we show that X' (and Y') together with some vertices of $\bigcup_{i=1}^{k} I_{Q_i}$ will take places of D_2 (and D_1) in Example 2.2 and the rest of the vertices will take places of U in Example 2.2, as shown in the cases below based on the values of k and q_1.

Case 1. $k = 2$.

In this case, we will show $D \in \mathcal{F}_2$. By Claim 4, without loss of generality, we may assume that $|Y| \ge 3 > k$. Then there exists a vertex $y_0 \in Y$ such that $N_D^+(y_0) \cap X' = \emptyset$. Applying (3.9) to the vertex y_0, $|X' - X| + \sum_{i=1}^{k} q_i + r \le 2$. It follows that $X' = X$, $r = 0$ and $q_1 = q_2 = 1$. Then $z_2^1 = z_3^1$ and $z_2^2 = z_3^2$. By Claim 5, $|Y' - Y| \le 1$.

Recall that the z_i^j's are defined in Notation 3.5. It suffices to show that $(Y' - Y, \{z_2^1, z_2^2\})_D = \emptyset$ and $z_2^1z_2^2, z_2^2z_2^1 \notin A(D)$ and $|X| \le 2$. In fact, if $z_2^1z_2^2 \in A(D)$, then by $z_2^1z_2^2 \in A(D) - A(H)$. Thus $D - z_1^1\bar{Q}_1z_4^1 - z_1^2\bar{Q}_2z_4^2 + z_1^1\bar{Q}_1z_2^1z_2^2\bar{Q}_2z_4^2 + z_1^2P_{z_1^2}uPvP_{z_4^1}z_4^1$ is a spanning eulerian subdigraph, contradicts (3.2). Similarly, $z_2^2z_2^1 \notin A(D)$. Furthermore, suppose that there exists a vertex $y' \in Y' - Y$ such that $y'z_2^1 \in A(D)$. If $z_2^1y' \in A(D)$, then by Lemma 3.2, $z_2^1y', y'z_2^1 \notin A(H_1)$. Thus $H - z_1^1\bar{Q}_1z_4^1 + z_1^1P_{z_1^1}uPvP_{z_4^1}z_4^1 + y'z_2^1y'$ is a spanning eulerian subdigraph, contradicts (3.2). Hence, $N_D^+(z_2^1) \subseteq Y' - \{y'\}$ by Claim 3. Note that

$d^+_D(z^1_2) \geq n-4-d^-_D(u) \geq |Y'|-1$. This forces that $N^+_D(z^1_2) = Y'-\{y'\}$. If there exists $y_1 \in N^+_{H_1}(y') \cap Y$, then $z^1_2 y_1 \in A(D)$. Thus $H' = (H - z^1_1\bar{Q}_1 z^1_4 + z^1_1 P_{z^1_1} uPvP_{z^1_4} z^1_4) - y'y_1 + y'z^1_2 y_1$ is a spanning eulerian subdigraph, contradicts (3.2). So, $N^+_{H_1}(y') \cap Y = \emptyset$. It follows that $|N^-_{H_1}(y') \cap Y| = d^-_{H_1}(y') = d^+_{H_1}(y') = |N^+_{H_1}(y') \cap X|$, and so we have both $|(Y, X' \cup \{y'\})_{H_1}| = |(Y', X')_{H_1}|$ and $|(Y, X' \cup \{y'\})_{D-A(H)}| = 0$. By (3.7),

$$\begin{aligned}0 \leq& \mu(X \cup \{y'\}, Y) - \mu(X', Y')\\ =&|(Y, \{z^1_2, z^2_2\})_D| + |(\{z^1_2, z^2_2\}, X \cup \{y'\})_D| - |(Y', \{z^1_2, z^2_2\})_D| - |(\{z^1_2, z^2_2\}, X')_D|\\ =&|(\{z^1_2, z^2_2\}, \{y'\})_D| - |(\{y'\}, \{z^1_2, z^2_2\})_D|.\end{aligned}$$

Thus $z^2_2 y' \in A(D)$ since $y'z^1_2 \in A(D)$ and $z^1_2 y' \notin A(D)$. Then $H - z^1_1\bar{Q}_1 z^1_4 - z^2_1\bar{Q}_2 z^2_4 + z^2_1\bar{Q}_2 z^2_2 y' z^1_2\bar{Q}_1 z^1_4 + z^1_1 P_{z^1_1} uPvP_{z^2_4} z^2_4$ is a spanning eulerian subdigraph, contradicts (3.2).

Finally, if $|X| \geq 3$, then as $k = 2$, there exists a vertex $x_0 \in X$ such that $N^-_D(x_0) \cap Y' = \emptyset$. Then $n-4 \leq d^+_D(y_0) + d^-_D(x_0) \leq |Y'| - 1 + |X'| - 1 = n - 5$, a contradiction. So, $D \in \mathcal{F}_2$.

Case 2. $k = 1$ and $q_1 = 2$.

In this case, we will show that either $D \in \mathcal{F}_1$ or $D \in \mathcal{F}_3$. By Claim 4, without loss of generality, we may assume that $|Y| \geq 3 > k$. Thus there exists a vertex $y_0 \in Y$ such that $N^+_D(y_0) \cap X' = \emptyset$. Applying (3.9) to the vertex y_0, $|X'-X| + q_1 + r \leq 2$. It follows that $X' = X$, $r = 0$ and $q_1 = 2$. Let $I_{Q_1} = \{z_2, z_3\}$, where $z_2 = z^1_2$. Note that $z^1_3 = z_2$ is possible. By Claim 5, $|Y' - Y| \leq 1$. Moreover, we claim that $|X| = 1$ and thus $X = \{z^1_1\}$. For otherwise, there exists a vertex $x_0 \in X$ such that $N^-_D(x_0) \cap Y' = \emptyset$, then as $y_0 x_0 \notin A(D)$, $n - 4 \leq d^-_D(x_0) + d^+_D(y_0) \leq |X| - 1 + |Y'| - 1 = n - 5$, a contradiction. If $(Y' - Y, I_{Q_1})_D = \emptyset$, then by Claims 3 and 6, $D \in \mathcal{F}_3$, where $\{z_2, z_3\}$ take the place of the set of vertices of the 2-cycle in Example 2.2. So, we may assume that $Y' - Y = \{y'\}$ and either $y'z_2 \in A(D)$ or $y'z_3 \in A(D)$.

As $k = 1$, let $y_1 \in Y'$ such that $N^+_D(y_1) \cap X' \neq \emptyset$. For any $y \in Y - \{y_1\}$, as $yu \notin A(D)$, $d^+_D(y) \geq n - 4 - d^-_D(u) \geq |Y'| - 1$. This implies $D[Y - \{y_1\}]$ is a complete digraph and $Y - \{y_1\} \subseteq N^-_D(y') \cap N^-_D(y_1)$. So, it is easy to find a (z^1_4, y_1)-dipath P_1 and a (z^1_4, y')-dipath P_2 such that $A(P_1) \cap A(P_2) = \emptyset$ and $V(P_1) \cup V(P_2) = Y'$. Thus, if $z_3 z_2 \in A(D)$ then, as either $y'z_2 \in A(D)$ or $y'z_3 \in A(D)$, $y_1 z^1_1 uPvP_{z^1_4} z^1_4 P_2 y'(z_3) z_2 \bar{Q}_1 z^1_4 P_1 y_1$ is a spanning eulerian subdigraph, contradicts (3.2). So, $z_3 z_2 \notin A(D)$ and thus $D \in \mathcal{F}_1$ where $Y' \cup \{z_3\}$ takes the place of D_1 and X takes the place of D_2 in Example 2.2.

Case 3. $k = 1$ and $q_1 = 1$.

In this case, we show that $D \in \mathcal{F}_1$. Assume $x_0 \in X'$, $y_0 \in Y'$ such that $y_0 x_0 \in A(H_1)$. Then by Claim 6 $y_0 x_0$ is the only arc from Y' to X' in D and it suffices to show $(Y' - Y, I_{Q_1})_D = (I_{Q_1}, X' - X)_D = \emptyset$. Suppose, to the contrary, that $y'z^1_2 \in A(D)$, where $y' \in Y' - Y$. If $z^1_2 y' \in A(D)$, $H - z^1_1\bar{Q}_1 z^1_4 + z^1_1 P_{z^1_1} uPvP_{z^1_4} z^1_4 + y'z^1_2 y'$ is a spanning eulerian subdigraph, contradicts (3.2). Hence $z^1_2 y' \notin A(D)$. Moreover, for any $y \in Y - \{y_0\}$, as $yu \notin A(D)$, $d^+_D(y) \geq n - 4 - d^-_D(u) \geq |X' - X| + |Y'| - 2$. Next, we consider two subcases.

Subcase 3.1 $X' - X \neq \emptyset$.

Then $d^+_D(y) \geq |Y'| - 1$ for all $y \in Y - \{y_0\}$, which implies $D[Y - \{y_0\}]$ is a complete digraph and $Y - \{y_0\} = N^-_D(y') \cap Y = N^-_D(y_0) \cap Y$. It is easy to see that there is a (z^1_4, y')-dipath P_1 and a (z^1_4, y_0)-dipath P_2 such that $A(P_1) \cap A(P_2) = \emptyset$ and $V(P_1) \cup V(P_2) = Y'$. Let T be the (x_0, z^1_1)-ditrail in H. Then T spans all the vertices in X'. Thus $y_0 x_0 T z^1_1 P_{z^1_1} uPvP_{z^1_4} z^1_4 P_1 y' z^1_2 \bar{Q}_1 z^1_4 P_2 y_0$ is a spanning eulerian subdigraph, contradicts (3.2). This proves that this subcase cannot occur.

Subcase 3.2 $X' - X = \emptyset$.

In this case, $d^+_D(y) \geq |Y'| - 2$ for all $y \in Y - \{y_0\}$. Also, as $z^1_2 u \notin A(D)$, $d^+_D(z^1_2) \geq n - 4 - d^-_D(u) \geq |Y'| - 2$. If $N^+_{H_1}(y') \cap Y' = \emptyset$, then as $d^+_{H_1}(y') \geq 1$, $y' = y_0$ and thus $|N^-_{H_1}(y') \cap Y'| = |N^-_{H_1}(y')| = |N^+_{H_1}(y')| = |N^+_{H_1}(y') \cap X'| = 1$. So $(X' \cup \{y'\}, Y' - \{y'\})$ is a partition of $V(H_1)$ so that $|(Y' - \{y'\}, X' \cup \{y'\})_D| = 1$. Moreover, if there is a vertex $y_1 \in Y'$ such that $y_1 y' \in A(D) - A(H)$, then by Claim 1(iii) $y_1 \notin Y$. Thus, by the fact $d^+_{H_1}(y_1) > 0$ and $d^-_{H_1}(y') = 1$, there is a vertex $y_2 \in Y$ such that $y_1 y_2 \in A(H_1)$. Thus, $H - z^1_1\bar{Q}_1 z^1_4 + z^1_1 P_{z^1_1} uPvP_{z^1_4} z^1_4 - y_1 y_2 + y_1 y' z^1_2 y_2$ is a spanning eulerian subdigraph, a contradiction. Hence, $(Y' - \{y'\}, X' \cup \{y'\})_{D-A(H)} = \emptyset$. Then, by the facts $y'z^1_2 \in A(D)$ and $z^1_2 y' \notin A(D)$,

$$\mu(X' \cup \{y'\}, Y' - \{y'\}) - \mu(X', Y') = |(I_{Q_1}, \{y'\})_D| - |(\{y'\}, I_{Q_1})_D| < 0,$$

a contradiction to (3.7). Hence, $|N^+_{H_1}(y') \cap Y'| \geq 1$. Also, $N^+_{H_1}(y') \cap N^+_D(z^1_2) \cap Y' = \emptyset$, since otherwise, let $y_1 \in N^+_{H_1}(y') \cap N^+_D(z^1_2) \cap Y'$ and then $H - z^1_1\bar{Q}_1 z^1_4 + z^1_1 P_{z^1_1} uPvP_{z^1_4} z^1_4 - y'y_1 + y'z^1_2 y_1$ is a spanning eulerian subdigraph, contradicts (3.2). This, together with $|N^+_D(z^1_2) \cap Y'| = d^+_D(z^1_2) \geq |Y'| - 2$, forces $|N^+_{H_1}(y') \cap Y'| = 1$, say $N^+_{H_1}(y') \cap Y' = \{y'_1\}$, and $N^+_D(z^1_2) = Y' - \{y', y'_1\}$. Denote $\{y'_2\} = N^-_{H_1}(y') \cap Y'$ as $d^-_{H_1}(y') = d^+_{H_1}(y') = 1$.

Moreover, we claim that $|X| = 1$. For otherwise, there exists a vertex $x_1 \in X - \{x_0\}$ and thus as $z^1_2 x_1 \notin A(D)$, $n - 4 \leq d^-_D(x_1) + d^+_D(z^1_2) \leq |X| - 1 + |Y'| - 2 = n - 5$, a contradiction. So, $|X| = 1$. Thus, by Claim 5 $|Y'| \geq |Y| + 1 \geq n - 4 - |X| + 1 \geq 7$.

For any $y \in Y' - \{y', y_1', y_2', y_0\}$, as $d_D^+(y) \geq |Y'| - 2$, we see that either $yy' \in A(D)$ or $yy_1' \in A(D)$. If $yy' \in A(D)$, then $yy' \notin A(H_1)$ and thus $H - z_1^1\bar{Q}_1 z_4^1 + z_1^1 P_{z_1^1} uPvP_{z_4^1} z_4^1 + y'z_2^1 yy'$ is a spanning eulerian subdigraph, contradicts (3.2). So, $yy_1' \in A(D)$. Furthermore, if $yy_1' \notin A(H_1)$, then $H - z_1^1\bar{Q}_1 z_4^1 + z_1^1 P_{z_1^1} uPvP_{z_4^1} z_4^1 - yy_1' + y'z_2^1 yy_1'$ is a spanning eulerian subdigraph, contradicts (3.2) again. Hence, $yy_1' \in A(H_1)$. Then, as y is arbitrary, we have $d_{H_1}^+(y_1') = d_{H_1}^-(y_1') \geq |N_{H_1}^-(y_1') \cap Y'| \geq |Y' - \{y_1', y_2', y_0\}| \geq |Y'| - 3 \geq 4$. This forces there exists a vertex $y_3' \in N_{H_1}^-(y_1') \cap N_{H_1}^+(y_1') \cap Y$. Thus $H - z_1^1\bar{Q}_1 z_4^1 + z_1^1 P_{z_1^1} uPvP_{z_4^1} z_4^1 - y'y_1'y_3' + y'z_2^1 y_3'$ is a spanning eulerian subdigraph, contradicts (3.2) once more.

Similarly, $(I_{Q_1}, X' - X)_D = \emptyset$ and $D \in \mathcal{F}_1$. The proof is completed. ■

Let $\mathcal{D}_4 = \mathcal{D}(k_1, 0, 2) \cup \mathcal{D}(0, k_2, 2)$. Then $\mathcal{D}_4 \subseteq \mathcal{D}_1$. It is easy to verify that every digraph D in $(\mathcal{D}_1 - \mathcal{D}_4) \cup \mathcal{D}_2 \cup \mathcal{D}_3$ has a pair of vertices (u, v) such that $uv \notin A(D)$ and $d_D^+(u) + d_D^-(v) = n - 4$. So only the digraph in $\mathcal{D}_4$ may satisfies the condition $d_D^+(u) + d_D^-(v) \geq n - 3$ for every two vertices u, v with $uv \notin A(D)$. Let $\mathcal{F}_4$ be the family of spanning subdigraphs of digraphs in $\mathcal{D}_4$ such that $D \in \mathcal{F}_4$ if and only if D is strong and $d_D^+(u) + d_D^-(v) \geq n - 3$ for every pair of vertices (u, v) with $uv \notin A(D)$. Then we have the following corollary.

Corollary 3.6. *Let D be a digraph of order $n \geq 11$. If $d_D^+(u) + d_D^-(v) \geq n - 3$ for every two vertices u, v with $uv \notin A(D)$, then either $D \in \mathcal{F}_4$ or D has a spanning eulerian subdigraph.*

Also, note that digraphs in $\mathcal{F}_4$ have minimum degree 1. So, we get the following corollary.

Corollary 3.7. *Let D be a digraph with order $n \geq 11$ and $\min\{\delta^+, \delta^-\} \geq 2$. If $d_D^+(u) + d_D^-(v) \geq n - 3$ for every two vertices u, v with $uv \notin A(D)$, then D is supereulerian.*

Another consequence of the main results is the following degree sum condition, which is a stronger version of the main result of [7].

Corollary 3.8 ([7]). *Let D be a digraph of order $n \geq 11$. If $\delta^+(D) + \delta^-(D) \geq n - 4$ then either $D \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ or D has a spanning eulerian subdigraph.*

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