Graphs and Combinatorics (2016) 32:1065–1075
DOI 10.1007/s00373-015-1636-0

ORIGINAL PAPER

Extendability of Contractible Configurations for Nowhere-Zero Flows and Modulo Orientations

Yanting Liang[1] · Hong-Jian Lai[2] · Rong Luo[2] · Rui Xu[3]

Received: 12 August 2014 / Revised: 25 September 2015 / Published online: 27 November 2015

Abstract Let H be a connected graph and G be a supergraph of H. It is trivial that for any k-flow (D, f) of G, the restriction of (D, f) on the edge subset $E(G/H)$ is a k-flow of the contracted graph G/H. However, the other direction of the question is neither trivial nor straightforward at all: for any k-flow (D', f') of the contracted graph G/H, whether or not the supergraph G admits a k-flow (D, f) that is consistent with (D', f') in the edge subset $E(G/H)$. In this paper, we will investigate contractible configurations and their extendability for integer flows, group flows, and modulo orientations. We show that no integer flow contractible graphs are extension consistent while some group flow contractible graphs are also extension consistent. We also show that every modulo $(2k+1)$-orientation contractible configuration is also extension consistent and there are no modulo $(2k)$-orientation contractible graphs.

Keywords Integer flow · Group flow · Circular flow · Modulo orientation · Group connectivity

✉ Yanting Liang
yanting.liang@uwc.edu

Hong-Jian Lai
hjlai@math.wvu.edu

Rong Luo
rluo@mail.wvu.edu

Rui Xu
xu@westga.edu

1 Department of Mathematics, University of Wisconsin-Fond du Lac, Fond Du Lac, WI 54935, USA

2 Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

3 Department of Mathematics, University of West Georgia, Carrollton, GA 30118, USA

1 Introduction

We study finite and loopless graphs and follow [1] and [18] for undefined notations and terminology. In particular, if D is a digraph, then $V(D)$ and $A(D)$ denote the vertex set and the arc set of D, respectively. An arc oriented from a vertex u to a vertex v is denoted by (u, v). Let G be a graph with a given orientation $D = D(G)$. For a vertex $v \in V(G)$, define

$$E_D^-(v) = \{(u, v) \in A(D)\}, \text{ and } E_D^+(v) = \{(v, u) \in A(D)\}.$$

The subscript D may be omitted when D is understood from the context. For graphs H and G, we denote $H \subseteq G$ when H is a subgraph of G.

Let G be a graph and let $X \subseteq E(G)$ be an edge subset. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X, and then deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$.

Throughout this paper, let A be a nontrivial (additive) Abelian group with the additive identity 0, and let $A^* = A - \{0\}$ denote the set of nonzero elements in A. We use Z_k, $\mathbb{Z}$ and $\mathbb{Q}$ to denote the cyclic group of order k, the additive group of the integers and of the rational numbers, respectively.

For a graph G with an orientation D and for subsets $X \subseteq E(G)$ and $A' \subseteq A$, define $F(X, A') = \{f | f : X \to A'\}$ to be the set of all mappings from X into A'. To emphasize orientation D, we often write a mapping $f \in F(X, A')$ as an ordered pair (D, f). When H is a subgraph of G, we define $F(H, A) = F(E(H), A)$. For $f \in F(G, A)$, define $\partial f : V(G) \to A$ by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e), \quad \text{for any vertex } v \in V(G),$$

where "$\sum$" refers to the addition in A. If $\partial f = 0$, then (D, f) is called an *A-flow* (or simple a flow, when A is understood from the context).

Definition 1.1 Let (D, f) be an A-flow of a graph G and k, d be two integers with $0 < d \leq \frac{k}{2}$. We have the following definitions.

1. (D, f) is called an *integer flow* if $A = \mathbb{Z}$; and is a *k-flow* if, in addition, we have $|f(e)| < k$ for every edge $e \in E(G)$.
2. (D, f) is called a *nowhere-zero circular* $\frac{k}{d}$*-flow* if $A = \mathbb{Q}$ and $f : E(G) \to \{\pm d, \pm(d + 1), \pm(d + 2), \ldots, \pm(k - d)\}$.
3. A flow (D, f) is *nowhere-zero* if $f(e) \neq 0$ for each edge $e \in E(G)$.

The study of nowhere-zero flows was initiated by Tutte [16,17] as a generalization of the map coloring problems and the concept of circular flow, introduced in [18], is a generalization of integer flows, and a dual version of the circular coloring problem. Tutte proved the following.

Theorem 1.2 (Tutte [17]) *Let $k > 0$ be an integer and A be an Abelian group with $|A| = k$. A graph admits a nowhere-zero k-flow if and only if it admits a nowhere-zero A-flow.*

Let $\mathcal{F}$ be a given graph family. Researchers have been investigating the problem of determining or characterizing the graphs H such that

$$\text{for any graph } G \text{ with } H \subseteq G, G/H \in \mathcal{F} \text{ implies } G \in \mathcal{F}. \quad (1)$$

In [2–4] and [10], for a given graph family $\mathcal{F}$, the set of all graphs H satisfying (1) is denoted by $\mathcal{F}^o$, called the kernel of $\mathcal{F}$. If $\mathcal{P}$ denotes a graphical property, then we also use $\mathcal{P}$ to denote the family of graphs possessing $\mathcal{P}$. In some former studies such as [5], members in $\mathcal{P}^o$ are also called $\mathcal{P}$-contractible configurations. We will also refer members in $\mathcal{P}^o$ as contractible configurations with respect to the property $\mathcal{P}$.

Definition 1.3 Let G be a graph and $k > 1$ be an integer. An orientation D of G is called a *modulo k-orientation* if for each vertex v in G, $|E_D^+(v)| \equiv |E_D^-(v)| \pmod{k}$.

Modulo orientation is closely related to circular flows and Z_k-flows. For example, a graph admits a nowhere-zero circular $\frac{2k+1}{k}$-flow if and only if it admits a modulo $(2k+1)$-orientation (See Theorem 3.3 in Sect. 3). Let M_k denote the family of all graphs admitting a modulo k-orientation.

The purpose of this paper is to investigate $\mathcal{P}^o$ for different graph families or graphical properties $\mathcal{P}$. Following Jaeger [6], for a fractional number $r > 1$, let F_r denote the family of all graphs admitting a nowhere-zero r-flow. By Theorem 1.2, if $k > 1$ is an integer, then F_k is also the family of all graphs admitting a nowhere-zero A-flow, where A is an Abelian group with $|A| = k$. By definitions, $F_3 = M_3$. Catlin [2] has proved that F_4^o contains the 4-cycle as well as graphs with 2 edge disjoint spanning trees. While F_4^o has been investigated in [2] and [9], among others, it is left as an open problem to determine F_4^o. We in this paper will present a complete characterization of the family M_{2k+1}^o, for each integer $k \geq 1$.

Flow extension is another problem that has been receiving the attention of many researchers. Thomassen [15] and Lovász, Thomassen, Wu and Zhang [12] utilized flow extension to attack Jaeger's weak 3-flow conjecture, Jaeger's modulo $(2k+1)$-orientation conjecture and other conjectures related to M_{2k+1}^o. The extension of a partial nowhere-zero 4-flow was studied [9]. We in this paper also consider the following special case of flow extensions. Let H be a connected graph. If $D(H) \subseteq D(G)$ and if G/H has a nowhere-zero A-flow $(D(G/H), f')$, is it possible for G to have a nowhere-zero A-flow (D, f) such that $f'(e) = f(e)$ for every edge $e \in E(G/H)$?

The above question proposes a new problem stronger than contractible configurations, such a property is called *extension consistent* (see Definition 1.4).

Definition 1.4 Let $\mathcal{P}$ be a labeling and/or an orientation property of graphs and H be a connected graph. A $\mathcal{P}$-contractible configuration H is *extension consistent* if for every supergraph G of H, G/H has the property $\mathcal{P}$ with a labeling f' and/or an orientation D', then G has the property $\mathcal{P}$ with a labeling f and/or an orientation D such that f and f', and/or D and D' are consistent to each other for every edge in $E(G/H)$.

It was discovered by Tutte (Theorem 1.2) that a graph G admits a nowhere-zero k-flow if and only if it admits a nowhere-zero A-flow for every Abelian group A of order k (independent of the structure of the group A). Hence, it is obvious that a connected graph H is k-flow contractible if and only if it is A-flow contractible for every Abelian group A of order k (independent of the structure of the group A). However, it is somehow surprising as we show in this paper: An A-flow extension consistent configuration may not be k-flow extension consistent. The discovery indicates that in the study of extendability, structures of groups do make significant difference.

2 Integer Flows and Group Flow

In this section, we investigate the contractible configuration and extension consistent problems for integer flows and circular flows.

2.1 Integer Flows

In this subsection we will show that there are no integer flow extension consistent configurations. Specifically we prove the following theorem.

Theorem 2.1 *Let k and d be two integers with $0 < d \leq \frac{k}{2}$. Then there is no nontrivial $\frac{k}{d}$-flow extension consistent graph.*

Proof Suppose by contradiction that there is a nontrivial $\frac{k}{d}$-flow extension consistent graph H. We will construct a supergraph G of H such that G/H admits a nowhere-zero $\frac{k}{d}$-flow which cannot be extended to a nowhere-zero $\frac{k}{d}$-flow of G.

Let Δ be the maximum degree of H and let t be an integer so that $td \geq \Delta(k-d)+1$.

Since H is nontrivial, H has at least one edge xy. Let G be a graph obtained from H by adding t vertices $u_1, \ldots, u_t$ of degree two such that each u_i is adjacent to both x and y. Let D be an orientation of G so that the path xu_iy is a directed path from x to y. Then G/H is the graph consisting of t 2-cycles sharing a common vertex, say u, and the restriction of D on G/H consists of t directed 2-cycles. For each edge $e \in E(G/H)$, let $f(e) = d$. Then (D, f) is a nowhere-zero $\frac{k}{d}$-flow of G/H. Note that, in the supergraph G, the total possible out-flow at the vertex x is at least $td \geq \Delta(k-d)+1$, while the total in-flow at x is at most $\Delta(k-d)$. Therefore, (D, f) cannot be extended to G. This proves that there are no nontrivial $\frac{k}{d}$-flow extension consistent graphs. □

Corollary 2.2 *There is no nontrivial k-flow extension consistent graph for each $k \geq 2$.*

2.2 Group Connectivity

From the previous subsection, we prove that there is no integer flow extension consistent graph. However, group flow extension consistent graphs do exist.

Definition 2.3 Let A be an Abelian group and G be a graph.

(i) A mapping $\beta : V(G) \to A$ is called an *A-boundary* of G if $\sum_{v \in V(G)} \beta(v) = 0$.

(ii) The graph G is *A-connected* if, for every A-boundary β, there is an orientation D_β and a weight $f_\beta : E(G) \to A\backslash\{0\}$ such that for every vertex $v \in V(G)$,

$$\sum_{e \in E^+_{D_\beta}(v)} f_\beta(e) - \sum_{e \in E^-_{D_\beta}(v)} f_\beta(e) = \beta(v).$$

We have the following.

Theorem 2.4 *Let A be an Abelian group with $|A| \geq 3$. Then a connected graph H is A-flow extension consistent if and only if it is A-connected.*

Proof Suppose H is A-connected. Let G be a supergraph of H such that G/H admits a nowhere-zero A-flow (D', f). For any edge e in $E(G) - E(H)$ with both ends in H, we give it an arbitrary orientation and define $f(e) \in A - \{0\}$. We obtain an orientation D'' of edges in $E(G) - E(H)$. Now D'' and f naturally generates an A-boundary β of H where $\beta(v) = \sum_{e \in E^-_{D''}(v)} f(e) - \sum_{e \in E^+_{D''(v)}} f(e)$. Since H is A-connected, H has an orientation D''' and a weight f_β such that $\partial f_\beta(v) = \beta(v)$ for each $v \in V(H)$. Combine (D'', f) and (D''', f_β), we obtain a nowhere-zero A-flow of G.

Now we prove the converse. Suppose that H is A-flow extension consistent. We are to show that H is A-connected.

Let β be an A-boundary of H. Let $V_0 = \{v : \beta(v) = 0\}$ and $V_1 = \{v : \beta(v) \neq 0\}$. Construct a supergraph G of H as follows: add a new vertex w and the following edges: for each $u_i \in V_0$, add a pair of parallel edges $\{e_i, e'_i\}$ between w and u_i; for each $v_j \in V_1$, add one edge e_j between w and v_j. For the contracted graph G/H, let (D', f') be a nowhere-zero A-flow: parallel edges $\{e_i, e'_i\}$ between w and $u_i \in V_0$ are oriented opposite to each other and $f'(e_i) = f'(e'_i) \in A - \{0\}$; each edges e_j between w and $v_j \in V_1$ is oriented towards v_j and $f'(e_j) = \beta(v_j)$. Notice that

$$\sum_{e \in E^+_{D'}(w)} f'(e) - \sum_{e \in E^-_{D'}(w)} f'(e) = \sum_{e_j = wv_j, v_j \in V_1} f'(e_j) = \sum_{v_j \in V_1} \beta(v_j) = 0.$$

Therefore, G/H admits a nowhere-zero A-flow. Since H is A-flow extension consistent, let (D, f) be a nowhere-zero A-flow of G. It is obvious that the restriction of (D, f) on H satisfies Definition 2.3. That is, H is A-connected. □

2.3 Contractibility and Extendability

For $k = 1$, we have the following which is an immediate corollary of Corollary 3.4 and Theorem 3.5.

Theorem 2.5 *A connected graph is Z_3-flow contractible if and only if it is Z_3-flow extension consistent.*

While for each $k \geq 4$ and $k \neq 5$ and for each Abelian group A of order k, there are A-flow contractible graphs which are not A-flow extension consistent. It is known [8] that for each Abelian group A with $|A| = k \geq 4$, the k-cycle C_k is not A-connected and thus it is not A-flow extension consistent. However, Catlin [2] proved that a C_4 is Z_4-flow contractible. Seymour's 6-flow theorem implies that every cycle is A-flow contractible for $|A| \geq 6$. For 5-flow problem, no graph has been found yet that it is 5-flow contractible but not 5-flow extension consistent. However, the famous 5-flow conjecture implies all bridgeless graphs that are non-Z_5-connected are such examples (which include the circuits of length at least 5). It is still an open problem to prove $C_5 \in F_5^0$.

3 Modulo Orientations

Definition 3.1 Let $k \geq 2$ be an integer.

1. A connected graph H is *modulo k-orientation contractible* if for every supergraph G of H, G admits a modulo k-orientation if and only if G/H admits a modulo k-orientation.
2. A connected graph H is *modulo k-orientation extension consistent* provided that for every supergraph G of H, if G/H admits a modulo k-orientation D, then G admits a modulo k-orientation D' so that D' is consistent with D for all edges in $E(G/H)$.

3.1 Modulo 2k-Orientation

It is easy to see that a graph G admits a modulo $2k$-orientation if and only if G is Eulerian. Since in Eulerian graphs, contraction preserves the parity of degree of each vertex. Therefore if G is Eulerian, then so is G/H for any subgraph of H. However the converse of this is not always true. For a noneulerian graph G, if G has a connected subgraph H containing all vertices of odd degrees, then G/H is Eulerian and thus admits a modulo $2k$-orientation but G does not because it is not Eulerian. In summary we have the following observation.

Observation 3.2 *Let $k \geq 1$ be an integer. There are no nontrivial modulo $2k$-orientation contractible configurations and thus no nontrivial modulo $2k$-orientation extension consistent configurations.*

3.2 Modulo (2k + 1)-Orientation

Theorem 3.3 (Jaeger [7]) *Let $k \geq 1$ be an integer. A graph G admits a circular $\frac{2k+1}{k}$-flow if and only if it admits a modulo $(2k+1)$-orientation.*

As a corollary, we have the following

Corollary 3.4 *Let $k \geq 1$ be an integer. A graph G is $\frac{2k+1}{k}$-flow contractible if and only if it is a modulo $(2k+1)$-orientation contractible.*

By Theorem 2.1, we know that there are no $\frac{2k+1}{k}$-flow extension consistent configurations. However we will show that modulo $(2k+1)$-orientation extension consistent configurations do exist as shown in the following theorem.

Theorem 3.5 *A connected graph H is a modulo $(2k+1)$-orientation contractible configuration if and only if H is modulo $(2k+1)$-orientation extension consistent.*

Proof One direction is trivial. The other direction follows immediately from Lemmas 3.8 and 3.9. □

To prove Theorem 3.5, we will make use of a special type of bipartite graphs which only have vertices of degree 1 and $2k+1$.

For a graph G, denote by $L(G)$ the set of vertices of degree one. For nonnegative integers m, n, k, we denote $\mathcal{T}_{m,n,k}$ the set of connected bipartite graphs T with bipartition (A, B) and with only vertices of degree 1 and $2k+1$ satisfying $\{|L(T) \cap A|, |L(T) \cap B|\} = \{m, n\}$.

Lemma 3.6 *Let k, m be two positive integers. Then $\mathcal{T}_{m,m,k} \neq \emptyset$.*

Proof Let $T = (A', B')$ be a $(2k+1)$-regular hamiltonian bipartite simple graph with at least $2m$ vertices. Let C be a Hamilton cycle of T. Then $T \backslash E(C)$ is a $(2k-1)$-regular bipartite simple graph. By Hall's Theorem, $T \backslash E(C)$ has a perfect matching M with $|M| \geq m$. In T, pick m edges from M and cut each of these m edges in the middle to get m more edges (m edges change to $2m$ edges). Let T' be the resulting graph. Clearly, T' contains the cycle C, and hence T' is connected. Therefore T' is a bipartite graph in $\mathcal{T}_{m,m,k}$. □

Lemma 3.7 *Let m, n, k be nonnegative integers with $k \geq 1, m \equiv n \pmod{2k+1}$ and $m+n > 0$. Then $\mathcal{T}_{m,n,k} \neq \emptyset$ and for each $T \in \mathcal{T}_{m,n,k}$, the graph obtained from T by identifying the $m+n$ vertices of degree 1 in T as a new vertex admits a modulo $(2k+1)$-orientation.*

Proof We first prove $\mathcal{T}_{m,n,k} \neq \emptyset$ by induction on $m+n = t > 0$. If $t = 2$, then $m = n = 1$. By Lemma 3.6, $\mathcal{T}_{1,1,k} \neq \emptyset$. Assume that the lemma is true for every pair $\{m, n\}$ with $m+n < t$ for some integer $t \geq 2$. Without loss of generality, we may assume $m \geq n$. By Lemma 3.6, we may further assume $m \neq n$. Since $m \equiv n \pmod{2k+1}$, we have $m \geq 2k+1$.

By the induction hypothesis, $\mathcal{T}_{m-2k,n+1,k} \neq \emptyset$. Let $T \in \mathcal{T}_{m-2k,n+1,k}$ with bipartition (A, B) and with exactly $n+1 \geq 1$ vertices of degree 1 in B. Let v be a vertex of degree 1 in B. We construct T' by adding $2k$ vertices to T each of which is adjacent to v. It's easy to see that $T' \in \mathcal{T}_{m,n,k}$. Therefore $\mathcal{T}_{m,n,k} \neq \emptyset$.

Let $T \in \mathcal{T}_{m,n,k} \neq \emptyset$ and let G be the graph obtained from T by identifying the $m+n$ vertices of degree 1 in T as a new vertex u. Orient each of the edges of T with tail in A and head in B. Let D be the corresponding orientation of G. Notice that vertex u does not have degree $2k+1$ since $deg(u) = m+n$ and $m \equiv n \pmod{2k+1}$ with $m, n > 0$. For each vertex v of degree $2k+1$, all the edges incident with v are either all oriented out at v (if $v \in A$) or all oriented in at v (if $v \in B$). Hence we have $|E_D^+(v)| \equiv |E_D^-(v)| \equiv 0 \pmod{2k+1}$ for each $v \neq u$. Since $m \equiv n \pmod{2k+1}$, we have $|E_D^+(u)| \equiv |E_D^-(u)| \pmod{2k+1}$. Therefore, D is a modulo $(2k+1)$-orientation. □

For any $i \in Z_k^*$, let $-i$ be the inverse of i. Then $Z_{2k+1} = \{0, \pm 1, \pm 2, \ldots, \pm k\}$. For convenience, the values of each Z_{2k+1}-boundary function are in $\{0, \pm 1, \pm 2, \ldots, \pm k\}$ in the rest of the paper.

Lemma 3.8 *Let H be a connected graph and a modulo $(2k+1)$-orientation contractible configuration. Then for each Z_{2k+1}-boundary b of H, H has an orientation D such that $\partial_D(v) \equiv b(v) \pmod{2k+1}$ for each $v \in V(H)$ where $\partial_D(v) = |E_D^+(v)| - |E_D^-(v)|$.*

Proof Let $M = \{v | v \in V(H), b(v) > 0\}$ and $N = \{v | v \in V(H), b(v) < 0\}$. Define $m = \sum_{v \in M} b(v)$ and $n = -\sum_{v \in N} b(v)$. If $M \cup N = \emptyset$, any modulo $(2k+1)$-orientation D of H satisfies the requirement in the theorem. Thus we may assume $m + n > 0$. Since $\sum_{v \in V(H)} b(v) \equiv 0 \pmod{2k+1}$, we have $\sum_{v \in M} b(v) + \sum_{v \in N} b(v) \equiv 0 \pmod{2k+1}$. Therefore $\sum_{v \in M} b(v) \equiv -\sum_{v \in N} b(v) \pmod{2k+1}$. This implies $m \equiv n \pmod{2k+1}$. Note that $m + n > 0$. By Lemma 3.7, let $T \in \mathcal{T}_{m,n,k}$ with bipartition $\{A, B\}$ where $|A \cap L(T)| = m$ and $|B \cap L(T)| = n$.

We partition the m vertices of degree 1 in A into $\{A_v : v \in M\}$ so that $|A_v| = b(v)$ and partition the n vertices of degree 1 in B into $\{B_u : u \in N\}$ so that $|B_u| = -b(u)$. For each vertex $v \in M$, identify all vertices in A_v with v and for each $u \in N$ identify all vertices in B_u with u. Let G be the resulting graph.

Note that G/H is the graph obtained from T by identifying all its vertices of degree 1 into one new vertex. By Lemma 3.7, G/H admits a modulo $(2k+1)$-orientation. Since H is a modulo $(2k+1)$-orientation contractible configuration, G admits a modulo $(2k+1)$-orientation D as well. Let D' be the orientation obtained from D by reverse the direction of each edge of D. Since T has only vertices of degree $2k+1$ and 1 and it is connected, all its vertices of degree $2k+1$ induced a connected subgraph. Thus under D either all the edges of T are oriented in at A and oriented out at B or are oriented out at A and oriented in at B. Therefore in the former case, the restriction of D on H is the desired orientation and in the latter case, the restriction of D' on H is the desired orientation. □

Lemma 3.9 *Let H be a connected graph. If for each Z_{2k+1}-boundary b of H, there is an orientation D of H such that $\partial_D(v) \equiv b(v) \pmod{2k+1}$ for each $v \in V(H)$, then H is modulo $(2k+1)$-extension consistent.*

Proof Let G be a supergraph of H such that G/H admits a modulo $(2k+1)$-orientation D'. We need to show that D' can be extended to a modulo $(2k+1)$-orientation D of G such that D is consistent with D' for all the edges in $E(G/H)$.

By giving an arbitrary orientation in edges $E(G) - E(H)$ with both ends in H, we obtain an orientation D'' of edges in $E(G) - E(H)$. Now D'' naturally generates a Z_{2k+1}-boundary b of H where $b(v) = |E_{D''}^-(v)| - |E_{D''}^+(v)| \pmod{2k+1}$.

By the assumption of the lemma, H has an orientation D''' of H such that $\partial_{D'''}(v) \equiv b(v) \pmod{2k+1}$ for each $v \in V(H)$. Clearly combine the orientation D'' (which is on the edges $E(G) - E(H)$) and the orientation D''' (which is on the edges of $E(H)$), we obtain a desired orientation D of G. □

We conclude this subsection by introducing a method to construct bigger modulo $(2k+1)$-orientation contractible configurations from a modulo $(2k+1)$-orientation contractible graphs.

A *t-addition* to a graph G is an operation which adds a new vertex v to G so that the degree of v is t (with multiple edges allowed).

Lemma 3.10 ([11]) *Let $k \geq 1$ be an integer and let H be a graph with two vertices u, v. Let μ denote the multiplicity of uv. Then H is modulo $(2k+1)$-orientation contractible if and only if $\mu \geq 2k$.*

Theorem 3.11 *Let H be a connected graph and H_{2k} be a graph obtained from H by a $2k$-addition. Then H is modulo $(2k+1)$-orientation contractible if and only if H_{2k} is modulo $(2k+1)$-orientation contractible.*

Proof For convenience, let v_{2k} be the new vertex added by the $2k$-addition. We first assume that H is modulo $(2k+1)$-orientation contractible. Let G be a supergraph of H_{2k} (and of H as well) such that G/H_{2k} admits a modulo $(2k+1)$-orientation and let u be the new vertex in G/H. Let $E_{2k}(u, v_{2k})$ denote the subgraph of G/H with the two vertices u, v_{2k} and with $2k$ edges. Then $G/H_{2k} = [G/H]/E_{2k}(u, v_{2k})$. Since both $E_{2k}(u, v_{2k})$ and H are modulo $(2k+1)$-orientation contractible by Lemma 3.10 and the assumption, G/H and thus G admit a modulo $(2k+1)$-orientation. Therefore H_{2k} is modulo $(2k+1)$-orientation contractible.

Now assume that H_{2k} is modulo $(2k+1)$-orientation contractible. We are to show that H is modulo $(2k+1)$-orientation contractible. By Lemma 3.9, we only need to show that for every Z_{2k+1}-boundary b of H, H has an orientation D such that $\partial_D(v) \equiv b(v) \pmod{2k+1}$ for each $v \in V(H)$.

We construct a new Z_{2k+1}-boundary of H_{2k} as follows: $b'(v_{2k}) = 1$ and for any v with $vv_{2k} \in E(H_{2k})$, define $b'(v) = b(v) + \mu(vv_{2k})$. For each other vertex v of H_{2k}, let $b'(v) = b(v)$.

Since the degree of v_{2k} is $2k$ and $\sum_{v\in V(H)} b(v) \equiv 0 \pmod{2k+1}$, we have $\sum_{v\in V(H_{2k})} b'(v) \equiv 0 \pmod{2k+1}$. Since H_{2k} is modulo $(2k+1)$-orientation contractible, by Lemma 3.8, H_{2k} has an orientation D' such that $\partial_{D'}(v) \equiv b'(v) \pmod{2k+1}$ for each $v \in V(H_{2k})$. Note that $b'(v_{2k}) = 1$ and the degree of v_{2k} is $2k$. Thus all the edges incident with v_{2k} are oriented in edges at v_{2k}. Therefore the restriction of D' on H induces a desired orientation D of H. □

Corollary 3.12 *Let G be a connected graph and H be a graph obtained from G by a sequence of $2k$-additions. Then G is modulo $(2k+1)$-orientation contractible if and only if H is modulo $(2k+1)$-orientation contractible.*

Remark Neither necessary condition nor sufficient condition of Theorem 3.11 is true for t-addition when $t < 2k$. For $t > 2k$, from Lemma 3.10 and the proof of Theorem 3.11, we still have that if H is modulo $(2k+1)$-orientation contractible, then so is H_t. However the other direction is not true as one can see from the following example:

Let G be a graph obtained from a triangle uvw by adding some parallel edges such that $\mu(uv) = t \geq 2k$ and $\mu(vw) = 2k-1$.

Clearly G can be obtained from u by a t-addition ($t \geq 2k$) adding vertex v, then a $2k$-addition adding vertex w such that $\mu(wu) = 1$ and $\mu(wv) = 2k-1$. Therefore G is modulo $(2k+1)$-orientation contractible. On the other hand, G can be obtained from multiple edge vw (with $\mu(vw) = 2k-1$) by a $(t+1)$-addition (with $t+1 \geq 2k+1$) adding vertex u. Even though G is modulo $(2k+1)$-orientation contractible, the multiple edges vw with $\mu(vw) = 2k-1$ is not modulo $(2k+1)$-orientation contractible by Lemma 3.10.

4 Concluding Remarks

Let A be an Abelian group. Let k and d be two integers with $0 < d \leq \frac{k}{2}$. By Theorems 2.1, 2.4, and 3.5, we have the following:

{k-flow extension consistent graphs}$= \emptyset$ and {k-flow contractible graphs} $\neq \emptyset$,
{$\frac{k}{d}$-flow extension consistent graphs}$= \emptyset$ and {$\frac{k}{d}$-flow contractible graphs} $\neq \emptyset$,
$\emptyset \neq$ {A-flow extension consistent graphs} $\subset$ {A-flow contractible graphs} if $|A| \geq 4$,
{modulo k-orientation extension consistent graphs} $=$ {modulo k-orientation contractible graphs} $\neq \emptyset$ (if k is odd),
{modulo k-orientation extension consistent graphs} $=$ {modulo k-orientation contractible graphs} $= \emptyset$ (if k is even).

By Corollary 3.4, Theorems 3.5 and 2.5, we have the following theorem for Z_3-flow contractible configurations.

Theorem 4.1 *Let G be a connected graph. Then the following statements are equivalent:*

1. *G is* 3*-flow contractible.*
2. *G is* Z_3*-flow contractible.*
3. *G is* Z_3*-flow extension consistent.*
4. *G is modulo* 3*-orientation contractible.*
5. *G is modulo* 3*-orientation extension consistent.*

Acknowledgments We would like to thank the referee for the comments which improved the presentation of the paper and corrected a couple of errors in the proofs of Theorem 2.4, Lemmas 3.7 and 3.9.

References

1. Bondy, J.A., Murty, U.S.R.: Graph Theory. Springer, New York (2008)
2. Catlin, P.A.: Double cycle covers and the Petersen graph. J. Graph Theory **13**, 465–483 (1989)
3. Catlin, P.A.: The reduction of graph families under contraction. Discrete Math. **160**, 67–80 (1996)
4. Catlin, P.A., Hobbs, A.M., Lai, H.-J.: Graph families operations. Discrete Math. **230**, 71–97 (2001)
5. Fan, G., Lai, H.-J., Xu, R., Zhang, C.-Q., Zhou, C.: Nowhere-zero 3-flows in triangularly connected graphs. J. Comb. Theory Ser. B **98**(6), 1325–1336 (2008)
6. Jaeger, F.: Flows and generalized coloring theorems in graphs. J. Comb. Theory Ser. B **26**, 205–216 (1979)
7. Jaeger, F.: Nowhere-zero flow problems. In: Beineke, L., Wilson, R. (eds.) Selected Topics in Graph Theory 3, pp. 71–95. Wiley, New York (1988)
8. Jaeger, F., Linial, N., Payan, C., Tarsi, M.: Group connectivity of graphs—a nonhomogeneous analogue of nowhere-zero flow properties. J. Comb. Theory Ser. B **56**, 165–182 (1992)
9. Lai, H.-J.: Extending a partial nowhere zero 4-flow. J. Graph Theory **30**, 277–288 (1999)
10. Lai, H.-J., Lai, H.Y.: Duality of graph families. Discrete Math. **110**, 165–177 (1992)
11. Lai, H.-J., Liang, Y., Liu, J., Meng, J., Miao, Z., Shao, Y., Zhang, Z.: On strongly $\mathbb{Z}_{2s+1}$-connected graphs. Discrete Appl. Math. **174**, 73–80 (2014)
12. Lovász, L.M., Thomassen, C., Wu, Y.-Z., Zhang, C.-Q.: Nowhere-zero 3-flows and modulo k-orientations. J. Comb. Theory Ser. B **103**, 587–598 (2013)
13. Seymour, P.D.: Nowhere-zero 6-flows. J. Comb. Theory Ser. B **30**, 130–136 (1981)
14. Steinberg, R., Younger, D.H.: Grötzsch's theorem for the projective plane. Ars Comb. **28**, 15–31 (1989)
15. Thomassen, C.: The weak 3-flow conjecture and the weak circular flow conjecture. J. Comb. Theory Ser. B **102**, 521–529 (2012)

16. Tutte, W.T.: On the embedding of linear graphs in surfaces. Proc. Lond. Math. Soc. Ser. **2**(51), 474–483 (1949)
17. Tutte, W.T.: A contribution to the theory of chromatical polynomials. Can. J. Math. **6**, 80–91 (1954)
18. Zhang, C.-Q.: Integer Flows and Cycle Covers of Graphs. Marcel Dekker Inc., New York (1997). ISBN 0-8247-9790-6