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Supereulerian graphs with small circumference and 3-connected hamiltonian claw-free graphs[☆]

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ABSTRACT

A graph G is supereulerian if it has a spanning eulerian subgraph. We prove that every 3-edge-connected graph with the circumference at most 11 has a spanning eulerian subgraph if and only if it is not contractible to the Petersen graph. As applications, we determine collections $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ of graphs to prove each of the following
(i) Every 3-connected $\{K_{1,3}, Z_9\}$-free graph is hamiltonian if and only if its closure is not a line graph $L(G)$ for some $G \in \mathcal{F}_1$.
(ii) Every 3-connected $\{K_{1,3}, P_{12}\}$-free graph is hamiltonian if and only if its closure is not a line graph $L(G)$ for some $G \in \mathcal{F}_2$.
(iii) Every 3-connected $\{K_{1,3}, P_{13}\}$-free graph is hamiltonian if and only if its closure is not a line graph $L(G)$ for some $G \in \mathcal{F}_3$.

1. Introduction

We consider finite loopless graphs and follow [3] for undefined terminology and notations. In particular, $\kappa(G)$ and $\kappa'(G)$ denote connectivity and edge connectivity of G, respectively. We define $\kappa'(K_1) = \infty$. For a graph G which contains at least one cycle, the **circumference** of G, denoted by $c(G)$, is the length of a longest cycle contained in G; and the **girth** of G, denoted by $g(G)$, is the length of a shortest cycle contained in G. For an integer $i \geq 0$ and $v \in V(G)$, define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}, \quad \text{and} \quad E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

For a vertex $v \in V(G)$, define $N_G(v) = \{u \in V(G) : vu \in E(G)\}$, and for $X \subseteq V(G)$, $N_G(X) = \cup_{x\in X} N_G(x)$. If H is a subgraph of G, the set of **vertices of attachment of** H **in** G is

$$A_G(H) = \{v \in V(H) : N_G(v) - V(H) \neq \emptyset\}.$$

The subscript G in the notations above might be omitted if G is understood from the context.

For a graph G, let $O(G)$ denote the set of all odd degree vertices in G. A graph G is eulerian if G is connected with $O(G) = \emptyset$, and G is supereulerian if G has a spanning eulerian subgraph. In 1977, Boesch et al. [2] raised a problem to determine when a graph is supereulerian. They commented in [2] that such a problem would be a difficult one. In 1979, Pulleyblank [25] confirmed this remark by showing that the problem to determine if a graph is supereulerian, even within planar graphs,

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is NP-complete. For more literature on supereulerian graphs, see Catlin's excellent survey [6] and its supplements [11] and [18]. Catlin [7] and Jaeger [16] independently showed that every 4-edge-connected graph is supereulerian. Therefore, the problem is to determine which 3-edge-connected or 2-edge-connected graph is supereulerian. Characterizations of 2 or 3-edge-connected supereulerian graphs for certain classes of graphs have been widely investigated. See [4,17,20–23], and [30], among others. A main result of this paper is the following.

Theorem 1.1. *Let G be a graph with $\kappa'(G) \geq 3$. If the circumference of G is at most 11, then G is supereulerian if and only if G is not contractible to the Petersen graph $P(10)$.*

Since $P(10)$ has circumference 9, Theorem 1.1 immediately implies Theorem 4 of [19] that if a 3-edge-connected graph has circumference at most 8, then G is supereulerian. Theorem 1.1 also has a number of applications in 3-connected hamiltonian claw-free graphs. For an integer $k > 0$, P_k denotes a path of k vertices and Z_k denotes the graph obtained from the disjoint union of a P_{k+1} and a 3-cycle K_3 by identifying one end vertex of P_{k+1} with a vertex of K_3. For graphs $H_1, H_2, \ldots, H_s$, a graph G is $\{H_1, H_2, \ldots, H_s\}$-free if it contains no induced subgraph isomorphic to a copy of H_i for any i. A graph G is called **claw-free** if it is $K_{1,3}$-free.

The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a common vertex. Beineke [1] and Robertson [14] showed that line graphs are $K_{1,3}$-free graphs.

Two fascinating conjectures on hamiltonian line graphs and hamiltonian claw-free graphs have attracted the attention of many researchers.

Conjecture 1.2 (*Thomassen, [28]*)**.** *Every 4-connected line graph is hamiltonian.*

Conjecture 1.3 (*Matthews and Sumner, [24]*)**.** *Every 4-connected $K_{1,3}$-free graph is hamiltonian.*

Ryjáček [26] introduced the line graph **closure** $cl(G)$ of a claw-free graph G and used it to show that Conjectures 1.2 and 1.3 are equivalent. Motivated by Conjectures 1.2 and 1.3, many researchers have investigated forbidden induced subgraph conditions for hamiltonicity. In 1999, Brousek, Ryjáček and Favaron proved the following theorem.

Theorem 1.4 (*Brousek, Ryjáček and Favaron, [5]*)**.** *Every 3-connected $\{K_{1,3}, Z_4\}$-free graph is hamiltonian.*

Theorem 1.4 is extended to Theorem 1.5, and further to Theorem 1.6.

Theorem 1.5 (*Lai, Xiong, Yan, Yan, [19]*)**.** *Every 3-connected $\{K_{1,3}, Z_8\}$-free graph is hamiltonian.*

Theorem 1.6 (*Fujisawa [13]*)**.** *Let Q^* be the graph obtained from the Petersen graph by adding one pendant edge to each vertex. Let G be a 3-connected $\{K_{1,3}, Z_9\}$-free graph. Then G is hamiltonian unless G is the line graph of Q^*.*

In 2004, Łuczak and Pfender proved another type of forbidden subgraph condition for 3-connected hamiltonian claw-free graphs.

Theorem 1.7 (*Łuczak and Pfender, [29]*)**.** *Every 3-connected $\{K_{1,3}, P_{11}\}$-free graph is hamiltonian.*

As there exist 3-connected nonhamiltonian $\{K_{1,3}, Z_9\}$-free graphs and 3-connected nonhamiltonian $\{K_{1,3}, P_{12}\}$-free graphs, some natural problems arise: can we characterize 3-connected nonhamiltonian $\{K_{1,3}, Z_9\}$-free graphs and 3-connected nonhamiltonian $\{K_{1,3}, P_{12}\}$-free graphs? In a later section of this paper, we shall apply Theorem 1.1 to determine collections of graph $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ to prove the following. Note that Theorem 1.8(i) provides an independent proof of Theorem 1.6.

Theorem 1.8. *Each of the following holds.*

(i) *Every 3-connected $\{K_{1,3}, Z_9\}$-free graph is hamiltonian if and only if its closure is not a line graph $L(G)$ for some $G \in \mathcal{F}_1$.*

(i) *Every 3-connected $\{K_{1,3}, P_{12}\}$-free graph is hamiltonian if and only if its closure is not a line graph $L(G)$ for some $G \in \mathcal{F}_2$.*

(i) *Every 3-connected $\{K_{1,3}, P_{13}\}$-free graph is hamiltonian if and only if its closure is not a line graph $L(G)$ for some $G \in \mathcal{F}_3$.*

Part of our approach is a modification of that in [19]. However, we have noticed that the proof of a key theorem in [19] has a gap: when analyzing Case 1.2 in the proof of Theorem 4 in [19], an important subcase when G_1 or G_2 has only three vertices is missing. This subcase turns out to be the most complicated one. In this paper, we will fix this gap by proving Lemma 3.1 and Theorem 1.1. In Section 2, we display the basics of Catlin's reduction method, and utilize this reduction method to prove Theorem 1.1 in the next section. The applications of Theorem 1.1 to Hamiltonian claw-free graphs will be given in the last section.

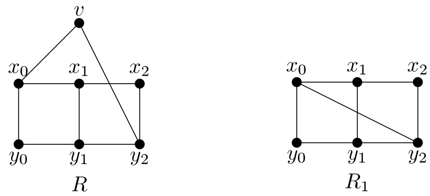

Fig. 1. The graphs R and R_1.

2. Preliminaries

Let G be a graph and $X \subseteq E(G)$ be an edge subset. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If K is a subgraph of G, then we write G/K for $G/E(K)$. If K is a connected subgraph of G, and if v_K is the vertex in G/K onto which K is contracted, then the subgraph $G[V(K)]$ is the **preimage** of v_K in G, and is denoted by $PI_G(v_K)$. The subscript G is often omitted when G is understood from the context. A vertex v in a contraction of G is *nontrivial* if $PI(v)$ has at least one edge. If L' is a path (or a cycle, respectively) of G/X, then by the definition of contraction, and by the connectedness of each component of $G[X]$, G has a path L (or a cycle, respectively) such that $L/(E(L) \cap X) = L'$. We then say that L' is lifted to L in G. Note that the same L' in G/X may have more than one lifts in G.

In [7], Catlin discovered collapsible graphs. A graph G is **collapsible** if for any $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph T_R with $O(T_R) = R$. Catlin showed in [7] that every vertex of G lies in a unique maximal collapsible subgraph of G. The **reduction** of G, denoted by G', is obtained from G by contracting all maximal collapsible subgraphs of G. A graph is **reduced** if it is the reduction of some graph.

Theorem 2.1 (*Catlin, [7]*). *Let G be a connected graph, let G' be the reduction graph of G, H be a collapsible subgraph of G and v_H be the vertex in G/H onto which H is contracted. Each of the following holds:*

(i) *(Theorem 8 of [7]) G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if the reduction G' is K_1.*

(ii) *(Theorem 5 of [7]) G is reduced if and only if G has no nontrivial collapsible subgraphs.*

(iii) *(Theorem 8 of [7]) $g(G') \ge 4$ and $\delta(G') \le 3$.*

(iv) *(Theorem 8 of [7]) If L' is an open (or closed, respectively) trail of G/H such that $v_H \in V(L')$, then G has an open (or closed, respectively) trail L with $E(L') \subseteq E(L)$ and $V(H) \subseteq V(L)$.*

Theorem 2.2. *Let G be a connected graph and let G' be the reduction graph of G. Let $K_{3,3}^-$ denote the graph obtained from $K_{3,3}$ by removing an edge, let R denote the graph (see Fig. 1) with $V(R) = \{x_0, x_1, x_2, y_0, y_1, y_2, v\}$ and $E(R) = \{x_0y_0, x_1y_1, x_2y_2, x_0x_1, x_1x_2, y_0y_1, y_1y_2, x_0v, vy_2\}$, and let $R_1 = R/\{vy_2\}$ (see Fig. 1). Then each of the following holds:*

(i) *(Catlin, Theorem 11 of [8]) The graphs K_3, $K_{3,3}^-$, and R are collapsible.*

(ii) *(Catlin, Theorem 7 of [7]) The reduction G' does not have a nontrivial collapsible subgraph.*

(iii) *R_1 is collapsible.*

Proof. (iii) Since R is collapsible, any contraction of R is also collapsible. □.

Theorem 2.3 (*Chen, [10]*). *If G is a 3-edge-connected simple graph with at most 13 vertices, then either G is supereulerian or G is contractible to the Petersen graph $P(10)$.*

Definition 2.4. Let $C = x_1x_2y_1y_2x_1$ be a 4-cycle in G with a partition $\pi(C) = \langle\{x_1, y_1\}, \{x_2, y_2\}\rangle$. Following [8], we define $G/\pi(C)$ to be the graph obtained from $G - E(C)$ by identifying x_1 and y_1 to form a vertex v_1, by identifying x_2 and y_2 to form a vertex v_2, and by adding an edge $e_{\pi(C)} = v_1v_2$.

Theorem 2.5 (*Catlin, [8]*). *Let G be a graph containing a 4-cycle C and let $G/\pi(C)$ be defined as above. Each of the following holds.*

(a) *If $G/\pi(C)$ is collapsible, then G is collapsible.*

(b) *If $G/\pi(C)$ has a spanning eulerian subgraph, then G has a spanning eulerian subgraph.*

Definition 2.6. Let s_1, s_2, s_3, m, l, t be integers with $t \ge 2$ and $m, l \ge 1$, $M \cong K_{1,3}$ with $D_3(M) = \{a\}$ and $D_1(M) = \{a_1, a_2, a_3\}$. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from M by adding s_i vertices with neighbors $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3 \pmod 3$. Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t. Let $K'_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to u' only. Let $K''_{2,t}(u, u', u'')$ be the graph obtained

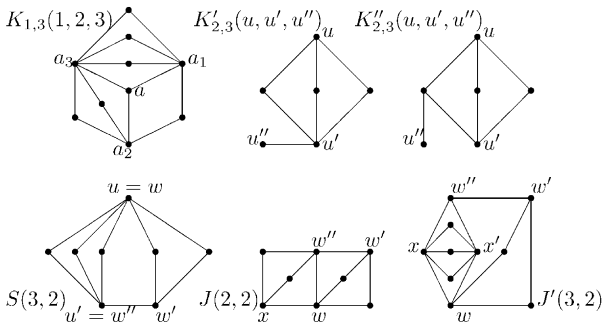

Fig. 2. Some graphs in $\mathcal{F}$ with small parameters.

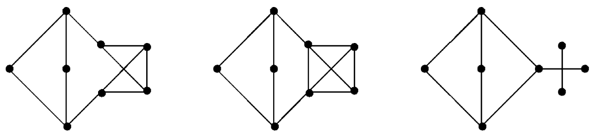

Fig. 3. These graphs are for Lemma 2.8.

from a $K_{2,t}(u,u')$ by adding a new vertex u'' that joins to a vertex of degree 2 of $K_{2,t}$. Hence u'' has degree 1 and both u and u' have degree t in $K''_{2,t}(u,u'')$. We shall use $K'_{2,t}$ and $K''_{2,t}$ for a $K'_{2,t}(u,u',u'')$ and a $K''_{2,t}(u,u',u'')$, respectively. Let $S(m,l)$ be the graph obtained from a $K_{2,m}(u,u')$ and a $K'_{2,l}(w,w',w'')$ by identifying u with w, and w'' with u'; let $J(m,l)$ denote the graph obtained from a $K_{2,m+1}$ and a $K'_{2,l}(w,w',w'')$ by identifying w, w'' with the two ends of an edge in $K_{2,m+1}$, respectively; let $J'(m,l)$ denote the graph obtained from a $K_{2,m+2}$ and a $K'_{2,l}(w,w',w'')$ by identifying w, w'' with two vertices of degree 2 in $K_{2,m+2}$, respectively. See Fig. 2 for examples of these graphs. Let

$$\mathcal{F}=\{K_1,\ K_2,\ K_{2,t},\ K'_{2,t},\ K''_{2,t},\ K_{1,3}(s,s',s''),\ S(m,l),\ J(m,l),\ J'(m,l),\ P(10)\},$$

where $t,s,s',s'',m,l\geq 0$ are integers.

For a graph G, let $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees.

Theorem 2.7 (*Catlin, [7]*). (i) *If G is reduced, then $F(G)=2|V(G)|-|E(G)|-2$.*

(ii) *(Catlin et al., Theorem 1.3 of [9]) If G is 2-edge-connected, and if $F(G)\leq 2$, then the reduction of G is either K_1 or a $K_{2,t}$ for some integer $t\geq 2$.*

(iii) *(Chen and Lai, Theorem 2.4 of [12]) If G is connected reduced graph with $|V(G)|\leq 11$ and $F(G)\leq 3$, then $G\in\mathcal{F}$, where $\mathcal{F}$ is defined as above.*

Lemma 2.8 (*Lemma 2.1 of [22]*). *Let G be a connected simple graph with $n\leq 8$ vertices and with $D_1(G)=\emptyset$, $|D_2(G)|\leq 2$. Then either G is one of three graphs depicted in Fig. 3, or the reduction of G is K_1 or K_2.*

Theorem 2.9 (*Theorem 4 of [19]*). *Let G be a graph. If $\kappa'(G)\geq 3$ and $c(G)\leq 8$, then G is supereulerian.*

It should be noted that the proof of Theorem 4 in [19] misses a case. This gap will be filled by the validity of Lemma 3.1. Therefore, Theorem 2.9 remains a valid statement.

3. Proof of Theorem 1.1

Let $P=v_0v_1v_2\cdots v_n$ denote a path in a graph G. For any $0\leq i<j\leq n$, we use the following notations of subpaths in our proof:

$$P[v_i,v_j]=v_iv_{i+1}v_{i+2}\cdots v_j,\qquad P(v_i,v_j]=v_{i+1}v_{i+2}\cdots v_j,$$
$$P[v_i,v_j)=v_iv_{i+1}v_{i+2}\cdots v_{j-1}\quad\text{and}\quad P(v_i,v_j)=v_{i+1}v_{i+2}\cdots v_{j-1}.$$

Thus P is also denoted by $P[v_0,v_n]$, usually referred as a (v_0,v_n)-path. For discussion convenience, cycles are often given with an orientation. For a cycle $C=u_1u_2\cdots u_lu_1$, we use the following notations in our proof:

$$C[u_i,u_j]=u_iu_{i+1}u_{i+2}\cdots u_j,\qquad C(u_i,u_j]=C[u_i,u_j]-\{u_i\},$$
$$C[u_i,u_j)=C[u_i,u_j]-\{u_j\}\quad\text{and}\quad C(u_i,u_j)=C[u_i,u_j]-\{u_i,u_j\}.$$

We also view $P = v_0v_1v_2\cdots v_n$ as a path with an orientation. The path with the same vertices but in the reverse order is denoted by $\overleftarrow{P}$. If $X, Y \subseteq V(G)$, then for any $x \in X$ and $y \in Y$, an (x, y)-path is called an (X, Y)-path; when $X = \{x\}$, then an (x, y)-path is also called an (x, Y)-path.

Lemma 3.1. *Let G be a graph with $c(G) \le 11$ such that*

$$\kappa(G) \ge 2,\ \kappa'(G) \ge 3, \text{ and } G \text{ is reduced.} \tag{1}$$

Let $C = x_1x_2y_1y_2x_1$ be a 4-cycle in G with a partition $\pi(C) = \langle\{x_1, y_1\}, \{x_2, y_2\}\rangle$. Each of the following holds.
(i) *$\kappa'(G/\pi(C)) \ge 2$.*
(ii) *If*

$$\text{the choice of } C \text{ maximizes } \kappa'(G/\pi(C)), \tag{2}$$

then $\kappa'(G/\pi(C)) \ge 3$.

Proof. (i) Assume that by contradiction, $e_{\pi(C)}$ is a cut edge of $G/\pi(C)$. Thus $G - E(C)$ has two components G_1 and G_2. We may assume that $x_i, y_i \in V(G_i)$, $(i \in \{1, 2\})$. For each $i \in \{1, 2\}$, let $P_i[x_i, y_i]$ be a longest (x_i, y_i)-path in G_i with length $p_i \ge 2$. By (1), G is simple, and so as $\kappa'(G) \ge 3$, we have $|V(G_i)| \ge 4$, $(1 \le i \le 2)$.

Claim 1. *For each $i \in \{1, 2\}$, $p_i \ge 5$.*
By symmetry, it suffices to show that $p_1 \ge 5$. If $p_1 = 2$, then every $w \in V(G_1) - \{x_1, y_1\}$ must be adjacent to both x_1 and y_1, and so by $\kappa'(G) \ge 3$, G_1 must have a cycle of length at most 3, contrary to (1).

Claim 1 Case A. $p_1 = 3$.
Denote $P_1 = x_1w_1w_2y_1$. Since $\kappa'(G) \ge 3$, and by (1), there must be a vertex $w' \in N_{G_1}(w_2) - \{x_1, y_1, w_1\}$. By $\kappa(G) \ge 2$, $G - w_2$ has a (w', w'')-path Q_1 with such that $V(Q_1) \cap \{x_1, y_1, w_1\} = \{w''\}$. If $w'' \in \{y_1, w_1\}$, then by (1), G contains no 3-cycles and so $|E(Q_1)| \ge 2$. Let

$$Q_1' = \begin{cases} x_1w_1w_2Q_1[w', y_1] & \text{if } w'' = y_1 \\ x_1\overleftarrow{Q_1}[w_1, w']w_2y_1 & \text{if } w'' = w_1. \end{cases}$$

Then as $|E(Q_1)| \ge 2$, $|E(Q_1')| \ge 5$, contrary to the assumption that $p_1 = 3$. Hence we must have $w'' = x_1$. Since $p_1 = 3$, we must have $n_1 = 1$ and so $Q_1 = w'x_1$.
By $\kappa'(G) \ge 3$ and by (1), there must be a vertex $w_1' \in N_G(w_1) - \{x_1, y_1, z_1, z_1'\}$. By $\kappa(G) \ge 2$, $G - w_1$ has a p(w_1', w_1'')-path Q_2 with $V(Q_2) \cap \{x_1, y_1, w_2, w'\} = \{w_1''\}$. Since G is reduced, G contains no 3-cycles and so either $w_1'' = y_1$, whence G_1 has a path $x_1w'w_2w_1Q_2[w_1', y_1]$; or $w_1'' = w'$, whence G_1 has a path $x_1w_1Q_2[w_1', w']w_2y_1$. In either case, a contradiction to the assumption $p_1 = 3$ is obtained. This proves Claim 1 Case A.

Claim 1 Case B. $p_1 = 4$.
Denote $P_1 = x_1w_1w_2w_3y_1$. Since $\kappa'(G) \ge 3$ and by (1), there must be a vertex $w_2' \in N_G(w_2) - \{w_1, x_1, y_1, z_1\}$. By $\kappa(G) \ge 2$, $G - w_2$ has a (w_2', w_2'')-path Q_3 with $V(Q_3) \cap \{w_1, x_1, y_1, z_1\} = \{w_2''\}$. Since G is reduced, G contains no 3-cycles, and so if $w_2'' \in \{w_1, w_2\}$, $|E(Q_3)| \ge 2$, implying that G_1 has an (x_1, y_1)-path of length at least 6, contrary to $p_1 = 4$. Therefore, by symmetry and by $p_1 = 4$, we may assume that $w_2'' = x_1$ and $Q_3 = w_2'x_1$. Again by $\kappa'(G) \ge 3$ and since G is reduced, there must be a vertex $w \in N_G(w_2') - \{w_1, w_2, x_1, w_2\}$. By $\kappa(G) \ge 2$, $G - w_2'$ has a (w, w')-path Q_4 with $V(Q_4) \cap \{w_1, w_2, x_1, y_1, z_1\} = \{w'\}$. Since $p_1 = 4$, a similar argument to the above leads to $w_2'y_1 \in E(G)$.
Again by $\kappa'(G) \ge 3$ and by (1), there must be a vertex $w_1' \in N_G(w_1) - \{w_1, w_2, w_2', x_1, w_3\}$. If $w_1' = y_1$, then $G[\{w_1, w_2, w_2', x_1, y_1, w_3\}] \cong K_{3,3}^-$, contrary to (1), (see Theorem 2.2). Thus $w_1' \ne y_1$ as well. By $\kappa(G) \ge 2$, $G - w_1$ has a (w_1', w_1'')-path Q_5 with $V(Q_5) \cap \{w_2, w_2', x_1, y_1, w_3\} = \{w_1''\}$. Since G is reduced, if $w_1'' \in \{w_2, x_1\}$, $|E(Q_5)| \ge 2$. By Theorem 2.2 and by (1), G cannot have R as a subgraph, and so if $w_1'' = y_1$, then $|E(Q_5)| \ge 2$ also. Define

$$Q_5' = \begin{cases} x_1w_1Q_5[w_1', w_2]w_3y_1 & \text{if } w_1'' = w_2 \\ x_1w_1Q_5[w_1', w_2']w_2w_3y_1 & \text{if } w_1'' = w_2' \\ \overleftarrow{Q_5}[x_1, w_1']w_1w_2w_3y_1 & \text{if } w_1'' = x_1 \\ x_1w_2'w_2w_1Q_5[w_1', y_1] & \text{if } w_1'' = y_1 \\ x_1w_2'w_2w_1Q_5[w_1', w_3]y_1 & \text{if } w_1'' = w_3. \end{cases}$$

In any case, $|E(Q_5')| \ge 5$, contrary to the assumption of $p_1 = 4$. This proves Claim 1.
By Claim 1, $\min\{p_1, p_2\} \ge 5$, and so G has a cycle of length at least 12, contrary to the assumption $c(G) \le 11$. This proves (i).
(ii). We argue by contradiction and assume that $\kappa'(G/\pi(C)) \le 2$. By (i), we must have $\kappa'(G/\pi(C)) = 2$. Then $G - E(C)$ has a cut edge $e = z_1z_2$ and $G - (E(C) \cup \{e\})$ has two components G_1 and G_2 such that $x_i, y_i, z_i \in G_i$, $(1 \le i \le 2)$.

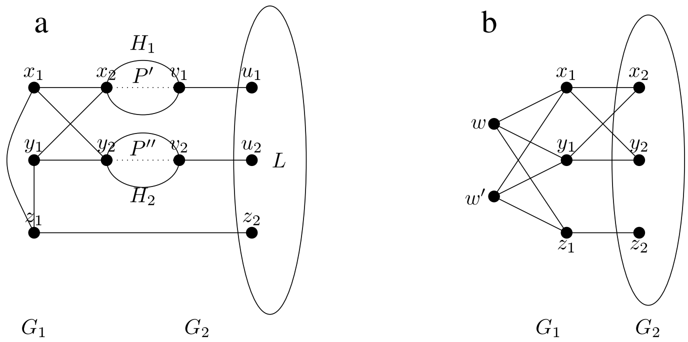

Fig. 4. Graphs in Case 1 and Case 2 of the proof for Lemma 3.1, respectively.

Case 1. $\min\{|V(G_1)|, |V(G_2)|\} = 3$.

We may assume that $|V(G_2)| \geq |V(G_1)| = 3$. Since $\kappa'(G) \geq 3$, $x_1z_1, y_1z_1 \in E(G)$. Thus $C_1 = x_1x_2y_1z_1x_1$, $C_2 = x_1y_2y_1z_1x_1$ are two 4-cycles (see Fig. 4(a)). Let $\pi(C_1) = \langle\{x_1, y_1\}, \{x_2, z_1\}\rangle$ and $\pi(C_2) = \langle\{x_1, y_1\}, \{y_2, z_1\}\rangle$. By Lemma 3.1(i), both $\kappa'(G/\pi(C_1)) = 2$ and $\kappa'(G/\pi(C_2)) = 2$. Therefore, $(G/\pi(C_1)) - e_{\pi(C_1)}$ has a cut edge v_2u_2 separating the two ends of $e_{\pi(C_1)}$. If $v_2u_2 = z_1z_2$, then x_2 would be a cut-vertex of G, contrary to (1). Hence $v_2u_2 \neq z_1z_2$. Similarly, $(G/\pi(C_2)) - e_{\pi(C_2)}$ has a cut edge $v_1u_1 \neq z_1z_2$ separating the two ends of $e_{\pi(C_2)}$, as depicted in Fig. 4(a), where the subgraphs H_1 and H_2 are possibly trivial. Hence

$$D = \{z_1z_2, u_1v_1, u_2v_2\} \text{ is an edge cut of } G.$$

Let L be the component of $G-D$ with $u_1, u_2, z_2 \in V(L)$, and let $P[u_1, u_2]$ be a longest (u_1, u_2)-path in L with length p. Choose an (x_2, v_1)-path P' in H_1 and a (v_2, y_2)-path P'' in H_2.

Claim 2. $p \geq 5$.

If $p = 1$, then $u_1u_2 \in E(G)$. By $\kappa'(G) \geq 3$ and by (1), there exist $u_i' \in N_G(u_i) - \{u_{3-i}, v_{3-i}, v_i\}$, for $1 \leq i \leq 2$. As $p = 1$, every (u_1', u_2')-path of G must use either u_1u_1' or u_2u_2' (but not both), and so either u_1 or u_2 would be a cut vertex of G, contrary to (1). Hence $p \geq 2$.

Suppose that $p = 2$ and $P = u_1u_1'u_2$. Since $\kappa'(G) \geq 3$, there must be a $z \in N_G(u_1') - V(P)$. By $\kappa(G) \geq 2$, G has a cycle C_z containing both u_1u_1', $u_1'z$, and so G has a (u_1, z)-path $Q_1 = C_z - u_1'$. If $u_2 \notin V(Q_1)$, then $Q_1[u_1, z]u_1'u_2$ has length at least 3, contrary to $p = 2$. Therefore, we assume $u_2 \in V(Q_1)$, and so $u_1u_1'\overleftarrow{Q_1}[z, u_2]$ is a (u_1, u_2)-path in L of length at least 3, contrary to $p = 2$.

Assume that $p = 3$ and $P = u_1u_1'u_2'u_2$. If we have both $w_1 \in (N_L(u_1) - V(P)) \cap N_L(u_2')$ and $w_2 \in (N_L(u_2) - V(P)) \cap N_L(u_1')$, then $u_1w_1u_2'u_1'w_2u_2$ is a path of length 5. Hence we assume that $(N_L(u_2) - V(P)) \cap N_L(u_1') = \emptyset$.

By $\kappa'(G) \geq 3$, $N_L(u_1') - V(P)$ contains a vertex w_1 and $N_L(u_2) - V(P)$ contains a vertex w_2 such that $w_1 \neq w_2$. Since $\kappa(G) \geq 2$, G has a cycle C containing w_1 and w_2. Since $\{u_1v_1, u_2v_2, z_1z_2\}$ is an edge cut, $|E(C) \cap \{u_1v_1, u_2v_2, z_1z_2\}| \in \{0, 2\}$, and so either $C[w_1, w_2]$ or $C[w_2, w_1]$ is a path in L. Assume that $C[w_1, w_2]$ is a path in L. Then $L - u_1'$ has a shortest path $Q_2[w_1, w']$ for some $w' \in V(P)$. If $w_2 \in V(Q_2)$, then as $w_1 \neq w_2$, $u_1u_1'Q_2[w_1, w_2]u_2$ has length at least 4. Hence we assume that $w_2 \notin Q_2$. Let

$$Q_2' = \begin{cases} \overleftarrow{Q_2}[u_1, w_1]u_1'u_2'u_2 & \text{if } w' = u_1 \\ u_1u_1'Q_2[w_1, u_2']u_2 & \text{if } w' = u_2' \\ u_1u_1'Q_2[w_1, u_2] & \text{if } w' = u_2, \ (\text{as } (N_L(u_2) - V(P)) \cap N_L(u_1') \neq \emptyset, |E(Q_2[w_1, u_2])| \geq 2). \end{cases}$$

Note that $|E(Q_2')| \geq 4$, and so a contradiction to $p = 3$ occurs.

Assume that $p = 4$ and $P = u_1u_1'u_2'u_3'u_2$. For $1 \leq i \leq 3$, since $\kappa'(G) \geq 3$, there exists a $w_i \in N_L(u_i') - V(P)$, (possibly $w_1 = w_3$). Suppose first that $N_L(u_1') \cap N_L(u_3') - V(P) \neq \emptyset$, which contains a vertex w. By $\kappa'(G) \geq 3$, there exists a $w' \in N_L(w) - \{u_1', u_3'\}$. By (1), G is reduced, and so $w' \notin V(P)$. As $\kappa(G) \geq 2$, $G - w$ has a (w', w'')-path Q_3 with $V(Q_3) \cap V(P) = \{w''\}$. If $z_2 \notin V(Q_3)$, then Q_3 is a path in L. Let

$$Q_3' = \begin{cases} \overleftarrow{Q_3}[u_1, w']wu_1'u_2'u_3'u_2 & \text{if } w'' = u_1 \\ u_1\overleftarrow{Q_3}[u_1', w']wu_3'u_2 & \text{if } w'' = u_1' \\ u_1u_1'wQ_3[w', u_2']u_3'u_2 & \text{if } w'' = u_2' \\ u_1u_1'wQ_3[w', u_3']u_2 & \text{if } w'' = u_3' \\ u_1u_1'u_2'u_3'wQ_3[w', u_2] & \text{if } w'' = u_2. \end{cases}$$

In any case, Q_3' has length at least 5, contrary to $p = 4$. Hence $z_2 \in V(Q_3)$. Then $x_1x_2y_1 \overleftarrow{P''}[y_2, v_2]\, u_2u_3'u_2'u_1'wQ_3[w', z_2]\, z_1x_1$ is a cycle of length at least 12, contrary to the assumption that $c(G) \leq 11$.

Thus we must have $N_L(u_1') \cap N_L(u_3') - V(P) = \emptyset$. Since $\kappa(G) \geq 2$, G has a cycle C with $w_1, w_3 \in V(C)$. Since $\{u_1v_1, u_2v_2, z_1z_2\}$ is an edge cut, either $C[w_1, w_3]$ or $C[w_3, w_1]$ is a path in L. Let T be a (w_1, w_3)-path in L. If $V(T) \cap V(P) = \emptyset$, then since $w_1 \neq w_3$, the path $u_1u_1'T[w_1, w_3]u_3'u_2$ is of length at least 5, contrary to $p = 4$. Hence $V(T) \cap V(P) \neq \emptyset$, and so C contains a (w_1, w_1')-path T_1 and a (w_3, w_3')-path T_3 such that for $i \in \{1, 3\}$, $V(T_i) \cap V(P) = \{w_i'\}$. If $w_1' \in \{u_1, u_2'\}$, then either $\overleftarrow{T_1}[u_1, w_1]\, u_1''u_2'u_3'u_2$ or $u_1u_1'T_1[w_1, u_2']u_3'u_2$ is a (u_1, u_2)-path of length at least 5 in L. Hence we have $w_1' \in \{u_3', u_2\}$. Similarly, $w_3' \in \{u_1', u_1\}$. If $(w_1', w_3') = (u_2, u_1)$, then $\overleftarrow{T_2}[u_1, w_3']\, u_3'u_2'u_1'T_1[w_1, u_2]$ has length at least 6. Thus we only need to discuss the following cases.

Claim 2 Case A. $(w_1', w_3') = (u_3', u_1')$.

As $\{w_1, u_2', w_3\} \subseteq N_L(u_1') \cap N_L(u_3')$ and as G is reduced, we may assume that $u_2' \neq z_2$.

By $\kappa(G) \geq 2$, $G - u_2'$ has a (w_2, w_2')-path T_2 for some $w_2' \in V(P)$ such that $V(T_2) \cap V(P) = \{w_2'\}$. If $z_2 \in V(T_2)$, then $x_1x_2y_1\overleftarrow{P''}[y_2, v_2]u_2u_3'T_3[w_3, u_1']u_2'T_2[w_2, z_2]\, z_1x_1$ is a cycle of length $10 + |E(\overleftarrow{P''}[y_2, v_2])| + |E(T_3[w_3, u_1'])| + |E(T_2[w_2, z_2])|$. Since $w_3 \neq u_1'$ and since $c(G) \leq 11$, we must have $u_2'z_2 \in E(G)$. By symmetric arguments, we also have $w_1z_2 \in E(G)$. It follows that $u_1u_1'w_1z_2u_2'u_3'u_2$ is of length 6, contrary to $p = 4$. Thus $z_2 \notin V(T_2)$ and T_2 is a path of L. If $w_2' \in \{u_1, u_1'\}$, then $\overleftarrow{T_2}[u_1, w_2]\, u_2'u_1'w_1u_3'u_2$ (if $w_2' = u_1$) or $u_1\overleftarrow{T_2}[u_1', w_2]\, u_1'w_3u_3'u_2$ (if $w_2' = u_1'$) are (u_1, u_2)-paths of length at least 5 in L. Hence $w_2' \notin \{u_1, u_1'\}$. By symmetry, $w_2' \notin \{u_2, u_3'\}$, and so $w_2' \notin V(P)$, contrary to the assumption that $w_2' \in V(P)$. Hence Case A does not occur.

Claim 2 Case B. $(w_1', w_3') \in \{(u_3', u_1), (u_2, u_1')\}$.

By symmetry, we assume that $(w_1', w_3') = (u_3', u_1)$. By $\kappa(G) \geq 2$, $G - u_2'$ has a (w_2, w_2')-path T_2 for some $w_2' \in V(P)$ such that $V(T_2) \cap V(P) = \{w_2'\}$. As shown in the proof of Case A, $w_2' \notin \{u_1, u_1'\}$. If $w_2' = u_3'$, then $u_1u_1'u_2'T_2[w_2, u_3']u_2$ has length at least 5. If $w_2' = u_2$, then $\overleftarrow{T_2}[u_1, w_3]\,\overleftarrow{T_1}[u_3', w_1]u_1'u_2'T_2[w_2, u_2]$ has length at least 5. Therefore, in any case, a contradiction to $p = 4$ is obtained. This shows that Case B does not occur either, which completes the proof of Claim 2.

We continue our proof for Case 1. By Claim 2, $p \geq 5$. If $p \geq 6$, then G has a cycle $z_1x_1P'[x_2, v_1]P[u_1, u_2]\, P''[v_2, y_2]y_1z_1$ with length at least 12, contrary to the assumption that $c(G) \leq 11$. Hence $p = 5$. Let $P = u_1u_1'u_2'u_3'u_4'u_2$. By $\kappa'(G) \geq 3$, for $1 \leq i \leq 4$, there exists $u_i'' \in N_G(u_i') - N_P(u_i')$, (possibly $u_1'' = u_3''$, $u_2'' = u_4''$, or $u_1'' = u_4''$).

Case 1.1 $[(N_G(u_1') \cap N_G(u_3')) \cup (N_G(u_2') \cap N_G(u_4'))] - V(P) \neq \emptyset$.

By symmetry, we assume that there exists a $w \in N_G(u_1') \cap (N_G(u_3') - V(P))$. By $\kappa'(G) \geq 3$, there exists a $w' \in N_G(w) - \{u_1', u_3'\}$. (We can view $u_1'' = u_3'' = w$.) By (1), G is reduced, and so $w' \notin V(P) - \{u_2\}$. By $\kappa(G) \geq 2$, $G - w$ has a (w', w'')-path Q_4 such that $V(Q_7) \cap V(P) = \{w''\}$. If $z_2 \in V(Q_4)$, then $y_1y_2x_1P'[x_2, v_1]\, u_1u_1'u_2'u_3'wQ_4[w, z_2]z_1y_1$ is a cycle of length $10 + |E(P'[x_2, v_1])| + |E(Q_4[w, z_2])|$. By $c \leq 11$, we must have $w' = z_2$ and so $wz_2 \in E(G)$. By symmetry, $u_2'z_2 \in E(G)$. This implies that $u_1u_1'u_2'z_2wu_3'u_4'u_2$ is a path of length 7, contrary to $p = 5$. Hence $z_2 \notin V(Q_4)$ (and so Q_4 is a path in L). If $w' \notin \{w'', u_2\}$ and $w'' \neq u_2$, then define

$$Q_4' = \begin{cases} \overleftarrow{Q_4}[u_1, w']wu_1'u_2'u_3'u_4'u_2 & \text{if } w'' = u_1 \\ u_1\overleftarrow{Q_4}[u_1', w']wu_3'u_4'u_2 & \text{if } w'' = u_1' \\ u_1u_1'wQ_4[w', u_2']u_3'u_4'u_2 & \text{if } w'' = u_2' \\ u_1u_1'wQ_4[w', u_3']u_4'u_2 & \text{if } w'' = u_3' \\ u_1u_1'u_2'u_3'wQ_4[w', u_4']u_2 & \text{if } w'' = u_4' \\ u_1u_1'u_2'u_3'wQ_4[w', u_2] & \text{if } w'' = u_2 \text{ and } w' \neq w''. \end{cases}$$

In each of these cases, a (u_1, u_2)-path of length at least 6 in L is found, contrary to $p = 5$. Therefore, we may assume that $w' = w'' = u_2$, and so $wu_2 \in E(G)$. By symmetry, $u_2'u_2 \in E(G)$. But then $G[(V(P) - \{u_1\}) \cup \{w\}]$ contains a $K_{3,3}^-$, and so G is not reduced, contrary to (1).

Case 1.2 $(N_G(u_1') \cap N_G(u_3')) \cup (N_G(u_2') \cap N_G(u_4')) \subseteq V(P)$.

Suppose first that $[(N_G(u_1) \cap N_G(u_2')) \cup (N_G(u_2) \cap N_G(u_3'))] - V(P) \neq \emptyset$. By symmetry, we assume that there exists a vertex $w_1 \in N_G(u_1) \cap N_G(u_2') - V(P)$. We first show that

$$u_1'u_4' \notin E(G). \tag{3}$$

By contradiction, we assume that $u_1'u_4' \in E(G)$. Since G is reduced and has no $K_{3,3}^-$, $u_3'' \notin V(P)$. By $\kappa(G) \geq 2$, $G - \{u_3'\}$ has a (u_3'', u_3''')-path T_3 such that $V(T_3) \cap V(P) = \{u_3'''\}$. Define

$$T_3' = \begin{cases} \overleftarrow{T_3}[u_1, u_3'']u_3'u_2'u_1'u_4'u_2 & \text{if } u_3''' = u_1 \\ u_1w_1u_2'\overleftarrow{T_3}[u_1', u_3'']u_3'u_4'u_2 & \text{if } u_3''' = u_1' \\ u_1u_1'u_2'\overleftarrow{T_3}[u_2'', u_3'']u_3'u_4'u_2 & \text{if } u_3''' = u_2' \\ P[u_1, u_3']T_3[u_3'', u_4']u_2 & \text{if } u_3''' = u_4' \\ u_1w_1u_2'u_1'u_4'u_3'T_3[u_3'', u_2] & \text{if } u_3''' = u_2. \end{cases}$$

Thus in any case, a (u_1, u_2)-path of length longer than 5 is found. This justifies (3).

By $\kappa(G) \geq 2$, $G - \{u'_1\}$ has a (u''_1, u'''_1)-path T_1 such that $V(T_1) \cap V(P) = \{u'''_1\}$. Since G is reduced, and by (3), $u''_1 \notin V(P) - \{u_2\}$. Thus if $u'''_1 \in V(P) - \{u_2\}$, then $u''_1 \neq u'''_1$. It follows by $p = 5$ that we must have $u''_1 = u'''_1 = u_2$. Thus $u'_1 u_2 \in E(G)$. By symmetry, we also have $u'_4 u_1 \in E(G)$.

Let W be the subgraph of G with $V(W) = V(P) \cup \{w_1\}$ and $E(W) = E(P) \cup \{w_1 u_1, w_1 u'_2, u_1 u'_4, u'_1 u_2\}$. Hence $F(W) = 3$. As $|V(W)| = 7$, by Theorem 2.7(iii), G is not reduced, contrary to (1). Thus we may assume that

$$(N_G(u_1) \cap N_G(u'_2)) \cup (N_G(u_2) \cap N_G(u'_3)) \subseteq V(P). \tag{4}$$

Subcase 1.2A $u'_2 \in N_G(u_2)$ and $u'_3 \in N_G(u_1)$.

Since G is reduced, $u''_1, u''_4 \notin V(P)$. Since $\kappa(G) \geq 2$, both u''_1, u''_4 are contained in a cycle C_{14}. Since $\{u_1v_1, u_2v_2, z_1z_2\}$ is an edge cut of G, either $C_{14}[u''_1, u''_4]$ or $C_{14}[u''_4, u''_1]$ is a path in L, and so L contains a (u''_4, u''_1)-path Q_5. If $z_2 \notin V(Q_5)$, then $u_1u'_3u'_4Q_5[u''_4, u''_1]u'_1u'_2u_2$ has length at least 6, contrary to $p = 5$. Hence $z_2 \in V(Q_5)$, and so $y_1y_2x_1P'[x_2, v_1]P[u_1, u'_4]Q_5[u''_4, z_2]z_1y_1$ is a cycle of length $11 + |E(P'[x_2, v_1])| + |E(Q_5[u''_4, z_2])|$. By $c(G) \leq 11$, we must have $u''_4 = z_2$. By symmetry, we also have $u''_1 = z_2$. It follows that $u_1u'_3u'_4z_2u'_1u'_2u_2$ has length 6, contrary to $p = 5$. This proves Case 1.2A.

Subcase 1.2B $u'_2 \in N_G(u_2)$ and $u'_3 \notin N_G(u_1)$ (or $u'_2 \notin N_G(u_2)$ and $u'_3 \in N_G(u_1)$). By symmetry, we assume that $u'_2 \in N_G(u_2)$ and $u'_3 \notin N_G(u_1)$. Since G is reduced, $u''_3 \notin V(P)$.

Suppose first that $u''_1 \in V(P)$. Since G is reduced, we must have $u''_1 = u'_4$. As $\kappa(G) \geq 2$, $G - u'_3$ has a (u''_3, u'''_3)-path Q_6 with $V(Q_6) \cap V(P) = \{u'''_3\}$. Since $p = 5$ and by (4), $u'''_3 \notin V(P) - \{u_1, u'_1\}$. If $u'''_3 = u'_1$, then by the assumption of Case 1.2, $N_G(u'_1) \cap N_G(u'_3) - V(P) = \emptyset$, and so $|E(Q_6)| \geq 2$. This implies that $u_1\overleftarrow{Q_6}[u'_1, u'''_3]u'_3u'_4u_2$ has length at least 6, contrary to $p = 5$. Thus $u'''_3 = u_1$ and $|E(Q_6)| \geq 2$, and so $\overleftarrow{Q_6}[u_1, u''_3]u'_3u'_2u'_1u'_4u_1$ has length at least 6, contrary to $p = 5$.

Therefore, $u''_1 \notin V(P)$. By $\kappa(G) \geq 2$, G has a cycle C_{13} with $u''_1, u''_3 \in V(C_{13})$. Since $\{u_1v_1, u_2v_2, z_1z_2\}$ is an edge-cut of G, we may assume that $C_{13}[u''_1, u''_3]$ is a path in L.

If $V(C_{13}[u''_1, u''_3]) \cap V(P) = \emptyset$, then by the assumption of Case 1.2, $u''_1 \neq u''_3$ and so $u_1u'_1C_{13}[u''_1, u''_3]u'_3u'_4u_2$ has length at least 6, contrary to $p = 5$. Thus $V(C_{13}[u''_1, u''_3]) \cap V(P) \neq \emptyset$. Hence $C_{13}[u''_1, u''_3]$ contains a (u'_1, x')-path Q'_6 with $V(Q'_6) \cap V(P) = \{x'\}$ and a (u''_3, x'')-path Q''_6 with $V(Q''_6) \cap V(P) = \{x''\}$. By $p = 5$ and since G is reduced, $x' \notin V(P) - \{u_2, u'_4\}$ and $x'' \notin V(P) - \{u_1\}$. If $x' = u'_4$, then $u_1u'_1Q'_6[u''_1, u'_4]u'_3u'_2u_2$ has length at least 6. Hence we must have $x' = u_2$ and $x'' = u_1$. Since $u'_3 \notin N_G(u_1)$, $|E(Q''_6)| \geq 2$, and so $\overleftarrow{Q''_6}[u_1, u''_3]u'_3u'_2u'_1Q'_6[u''_2, u_2]$ is of length at least 6, contrary to $p = 5$. This proves Case 1.2B.

Subcase 1.2C $u'_2 \notin N_G(u_2)$ and $u'_3 \notin N_G(u_1)$.

Then $u''_2, u''_3 \notin V(P)$. Since $\kappa(G) \geq 2$, G contains a cycle C_{23} with $u''_2, u''_3 \in V(C_{23})$. Since $\{u_1v_1, u_2v_2, z_1z_2\}$ is an edge-cut of G, we may assume that $C_{23}[u''_2, u''_3]$ is a path in L. If $V(C_{23}[u''_2, u''_3]) \cap V(P) = \emptyset$, then as G is reduced, $u''_2 \neq u''_3$, and so $u_1u'_1u'_2C_{23}[u''_2, u''_3]u'_3u'_4u_2$ is of length at least 6, contrary to $p = 5$.

Hence $V(C_{23}[u''_2, u''_3]) \cap V(P) \neq \emptyset$, and so $C_{23}[u''_2, u''_3]$ contains a (u''_2, x')-path T'_9 with $V(T'_9) \cap V(P) = \{x'\}$ and a (u''_3, x'')-path T''_9 with $V(T''_9) \cap V(P) = \{x''\}$. By $p = 5$, (4), and by the assumption of Subcase 1.2C, we must have $x' = u_2$ and $x'' = u_1$ with $|E(T'_9)| = |E(T''_9)| = 2$.

If $u''_1 = u_2$, then by Theorem 2.7(i), $F(G[V(P) \cup V(T'_9) \cup V(T''_9)]) \leq 3$, and so by Theorem 2.7(iii), G is not reduced, contrary to (1). Hence $u''_1 \neq u_2$. By (3) and (1), $u''_1 \notin V(P)$. Since $\kappa(G) \geq 2$, $G - u'_1$ has a (u''_1, u'''_1)-path $V(Q_7) \cap (V(P) \cup V(T'_9) \cup V(T''_9)) = \{u'''_1\}$. By $p = 5$ and by the assumption of Case 1.2, $u'''_1 \notin V(P) - \{u_2, u'_4\}$. If $u'''_1 = u'_4$, then $\overleftarrow{T''_9}[u_1, u''_3]u'_3u'_2u'_1Q_7[u''_1, u'_4]u_2$ has length at least 7. Similarly, if $u'''_1 \in V(T'_9) \cup V(T''_9) - V(P)$ or $u'''_1 = u'_4$, then a (u_1, u_2)-path of length at least 7 in L can be found. Hence $u'''_1 = u_2$. But then, $\overleftarrow{T''_9}[u_1, u''_3]u'_3u'_2u'_1Q_7[u''_1, u_2]$ has length at least 6, contrary to $p = 5$. This proves Subcase 1.2C, and completes the proof of Case 1.

Case 2. $4 \leq \min\{|V(G_1)|, |V(G_2)|\} \leq 6$.

We again assume that $|V(G_2)| \geq |V(G_1)|$, and so $4 \leq |V(G_1)| \leq 6$. If $|V(G_1)| = 4$, then $V(G_1) - \{x_1, y_1, z_1\} = \{w\}$. As $\kappa'(G) \geq 3$, $N(w) = \{x_1, y_1, z_1\}$ and $N(z_1) \cap \{x_1, y_1\} \neq \emptyset$. It follows that G has a 3-cycle, contrary to (1).

Assume that $|V(G_1)| = 5$ and denote $V(G_1) - \{x_1, y_1, z_1\} = \{w, w'\}$. If $ww' \in E(G)$, then as $\kappa'(G) \geq 3$ and as $N(w) \cup N(w') \subseteq \{x_1, y_1, z_1\}$, $N(w) \cap N(w') \neq \emptyset$, forcing G to have a 3-cycle, contrary to (1). Hence $N(w) = N(w') = \{x_1, y_1, z_1\}$, and so $G[\{w, w', x_2; x_1, y_1, z_1\}] \cong K^-_{3,3}$, contrary to (1). (See Fig. 4(b)).

Hence $|V(G_1)| = 6$. Let $V(G_1) - \{x_1, y_1, z_1\} = \{w_1, w_2, w_3\}$, and let $G^1 = G[\{w_1, w_2, w_3, x_1, x_2, y_1, y_2, z_1, z_2\}]$. By Theorem 2.7(i), $F(G^1) = 2|V(G^1)| - |E(G^1)| - 2$. If $d_G(z_1) \geq 4$, then $F(G^1) \leq 2$, and if $d_G(z_1) = 3$, then $F(G^1) \leq 3$. It follows from Theorem 2.7 that G^1 is not reduced, contrary to (1).

Case 3. $\min\{|V(G_1)|, |V(G_2)|\} \geq 7$.

Let $W_i = \{x_i, y_i\}$. For each $i \in \{1, 2\}$, let P_i be a longest (z_i, W_i)-path in G_i, where that both $x_i, y_i \in V(P_i)$ are possible. Without loss of generality, assume that $|E(P_2)| \geq |E(P_1)|$. If $|E(P_1)| \geq 5$, then by combining P_1 and P_2 with one suitable edge in $x_1x_2y_1y_2x_1$ and the edge z_1z_2, G would have a cycle of length at least 12, contrary to the assumption that $c(G) \leq 11$. Hence we must have

$$|E(P_1)| \leq 4. \tag{5}$$

Claim 3. $\kappa(G_1) \geq 2$.

Table 1
Contradictions to (5) in the proof of Claim 4B.

w_1	w_2	a (z_1, W_1)-path longer than 4	Symmetric cases and explanations
x_1	x_1	$Q_1[z_1, w_1]\overleftarrow{Q_2}[w_2, y_1'']y_1'$	The cases when $w_1, w_2 \in \{x_1, y_1\}$ can be excluded similarly.
x_1	x_1'	$Q_1[z_1, w_1]y_1'\overleftarrow{Q_2}[y_1'', w_2]y_1$	The case when $w_1 = y_1$ can be excluded similarly.
x_1'	x_1	$Q_1[z_1, w_1]y_1y_1'Q_2[y_1'', w_2]$	The case when $w_2 = y_1$ can be excluded similarly.
x_1'	x_1'	$Q_1[z_1, w_1]\overleftarrow{Q_2}[w_2, y_1'']y_1'y_1$	

Let $T(G_1)$ be the block-cut-vertex tree of G_1. By contradiction, we assume that $T(G_1)$ is a nontrivial tree. By (1), $\kappa(G) \geq 2$. Thus every block of G_1 corresponding to a vertex in $D_1(T(G_1))$ (referred as an end block of G_1) must contain x_1, y_1 or z_1. We may assume that x_1 and z_1 are in two different end blocks B_{x_1}, B_{z_1}, respectively. Let $B_1, b_1, B_2, b_2, \ldots, b_{k-1}, B_k$ be the unique (B_1, B_k)-path of $T(G_1)$ with $B_1 = B_{x_1}$ and $B_k = B_{z_1}$. Since $\kappa'(G) \geq 3$ and since B_{z_1} is an end block, $N_G(z_1) \cap V(B_k)$ contains a vertex z' which is not a cut vertex of G_1, and so $B_k \neq K_2$. If $|V(B_k)| \leq 8$, then by Lemma 2.8, B_k contains a nontrivial collapsible subgraph, contrary to (1). Hence $|V(B_k)| \geq 9$.

Let P denote a longest (z_1, b_{k-1})-path in B_k. If $|E(P)| \geq 4$, then P can be extended to a (z_1, x_1)-path of length at least 5, contrary to (5). Hence $|E(P)| \leq 3$. Since B_k is reduced and 2-connected, $|E(P)| \geq 2$. If $|E(P)| = 2$, then since P is longest and since $\kappa(B_k) \geq 2$, B_k is spanned by a $K_{2,t}$ for some $t \geq 7$. By $\kappa'(G) \geq 3$, every vertex in $V(B_k) - \{z_1, b_{k-1}\}$ has degree 3 in B_k, and so B_k must have a 3-cycle, contrary to (1). Thus we assume that $P = z_1v_1v_2b_{k-1}$ is a path of length 3. By $\kappa'(G) \geq 3$ and (1), there exists a $v_i' \in N_G(v_i) - V(P)$, for each $i \in \{1, 2\}$, and $v_1' \neq v_2'$. By $\kappa(B_k) \geq 2$ and by (1), B_k has a cycle C of length at least 4 containing both v_1v_1' and v_2v_2'. It follows that $P \cup C$ contains a (z_1, b_{k-1})-path of length at least 4, whence G_1 has a (z_1, x_1)-path of length at least 5, contrary to (5). This proves Claim 3.

Claim 4. *$N_{G_1}(x_1) \cap N_{G_1}(y_1) = \emptyset$.*

We shall prove Claim 4 by justifying several subclaims. We first show that $|N_{G_1}(x_1) \cap N_{G_1}(y_1)| \leq 1$. In Claims 4A and 4B, we assume that $N_{G_1}(x_1) \cap N_{G_1}(y_1)$ has distinct vertices x_1', y_1' to find contradictions.

Claim 4A. *$z_1 \notin N_G(x_1) \cap N_G(y_1)$.*

By contradiction, we assume that $x_1' = z_1$. By (1), $y_1'z_1 \notin E(G)$, and so there exists a $y_1'' \in N_{G_1}(y_1') - \{x_1, y_1, z_1\}$. By Claim 3, G_1 has a cycle C^1 containing x_1z_1 and $y_1'y_1''$, with an orientation so that the edge x_1z_1 is oriented from x_1 to z_1.

If $y_1' \in V(C^1[z_1, y_1''])$, then since G contains no $K_{3,3}^-$, $|E(C^1[z_1, y_1'])| \geq 3$; and since G has no 3-cycles, $|E(C^1[y_1'', x_1])| \geq 2$. It follows that $|E(C^1[z_1, x_1])| \geq 6$, contrary to (5). Hence $y_1' \in V(C^1[y_1'', x_1])$, and so by (5), $|E(C^1[z_1, y_1''])| \leq 2$. As G contains no $K_{3,3}^-$, we must have $|E(C^1[z_1, y_1''])| = 2$. Assume that $C^1[z_1, y_1''] = z_1z_1'y_1''$.

Since $\kappa'(G) \geq 3$, $N_G(z_1') - \{y_1'', z_1\}$ contains a vertex w_1. Since G has no K_3, $w_1 \notin \{x_1, y_1, y_1'\}$. By Claim 3, G_1 contains a (w_1, w_1')-path Q_1 such that $V(Q_1) \cap \{x_1, y_1, y_1', y_1'', z_1\} = \{w_1'\}$. By (5), we must have $w_1' = y_1'$ and $Q_1 = w_1y_1'$. Arguing similarly with z_1' replaced by y_1'', we conclude that there must be a vertex $w_2 \notin \{w_1, x_1, y_1, y_1', y_1'', z_1\}$ such that $w_2y_1'', w_2z_1 \in E(G)$. Thus $z_1w_2y_1''z_1'y_1'x_1$ has length 5, contrary to (5). This proves Claim 4A.

Claim 4B. *$|N_{G_1}(x_1) \cap N_{G_1}(y_1)| \leq 1$.*

If $x_1', y_1' \in N_{G_1}(z_1)$, then $G[\{x_1, y_1, z_1, x_2, x_1', y_1'\}] \cong K_{3,3}^-$, contrary to (1). Hence we assume that $y_1' \notin N_{G_1}(z_1)$. By $\kappa'(G) \geq 3$, $N_{G_1}(y_1') - \{x_1, y_1\}$ contains a vertex y_1''. Since $y_1' \notin N_{G_1}(z_1)$, $y_1'' \neq z_1$. Since G is reduced, $y_1'' \notin \{x_1, x_1', y_1\}$. By $\kappa(G_1) \geq 2$, G_1 has a cycle C^2 containing both $y_1'y_1''$ and z_1. Without loss of generality, we may assume that $y_1' \notin V(C^2[z_1, y_1''])$. If $x_1', x_1, y_1 \notin V(C^2[z_1, y_1''])$, then $C^2[z_1, y_1'']y_1'y_1x_1'x_1$ is a violation to (5). Therefore, there must be $w_1, w_2 \in \{x_1', x_1, y_1\}$ as well as a (z_1, w_1)-path Q_1 and a (w_2, y_1'')-path Q_2 in $C^2[z_1, y_1'']$, such that for $i = 1, 2$, $V(Q_i) \cap \{x_1', x_1, y_1\} = \{w_i\}$. Since G does not have a K_3 or a $K_{3,3}^-$, $|E(Q_2)| \geq 2$. The table below indicates a contradiction to (5) can always be found, which completes the proof of Claim 4B (see Table 1).

Claim 4C. *$N_{G_1}(x_1) \cap N_{G_1}(y_1) = \emptyset$.*

By contradiction and by Claims 4A and 4B, we assume that $N_{G_1}(x_1) \cap N_{G_1}(y_1) = \{x_1'\}$ with $x_1' \neq z_1$. By $\kappa(G_1) \geq 2$, $G_1 - x_1'$ has a $(z_1, \{x_1, y_1\})$-path T_1. By symmetry, we assume that $V(T_1) \cap \{x_1, y_1\} = \{y_1\}$. As the path $T_1[z_1, y_1]x_1'x_1$ has length $|E(T_1)| + 2$, by (5), $|E(T_1)| \leq 2$. By $\kappa'(G) \geq 3$, $N_{G_1}(x_1') - \{x_1, y_1\}$ contains a x_1''. By $\kappa(G_1) \geq 2$, $G_1 - x_1'$ has a (x_1'', x_1''')-path T_2 with $V(T_2) \cap V(T_1) \cup \{x_1\} = \{x_1'''\}$.

Assume first that $x_1''' = z_1$. Since G has no K_3, $T_1 = z_1z_1'y_1$. By $\kappa'(G) \geq 3$, $N_{G_1}(z_1') - \{z_1, y_1\}$ contains a z_1''. By $\kappa(G_1) \geq 2$, $G_1 - z_1'$ has a (z_1'', z_1''')-path T_3 with $V(T_3) \cap \{x_1, x_1', y_1, z_1\} = \{z_1'''\}$. By (5), $z_1''' = x_1'$ and $T_3 = z_1''x_1'$. If $|E(T_1)| \geq 1$, then $\overleftarrow{T_1}[z_1, x_1'']x_1'z_1''z_1'y_1$ has length at least 5. Thus we must have $z_1x_1' \in E(G)$. Now by Theorem 2.7(i), $F(G[\{x_1, x_2, y_1, y_2, z_1\} \cup V(T_2)]) = 3$, and so by Theorem 2.7(iii), G is not reduced, contrary to (1).

Hence we must have $x_1''' \neq z_1$. If $x_1''' = x_1$, then since G has no K_3, $|E(T_2)| \geq 3$, and so $T_1[z_1, y_1]x_1'T_2[x_1', x_1]$ has length at least 5. Thus $x_1''' \neq x_1$. Similarly, $x_1''' \neq y_1$. Hence we must have $T_1 = z_1z_1'y_1$ and $x_1''' = z_1'$. By (5), $|E(T_2)| \leq 2$. As G has no K_3,

$T_2 = x_1''z_1'$. By $\kappa(G_1) \geq 2$, $G_1 - z_1'$ has a (z_1, z_1^4)-path T_4 with $V(T_4) \cap \{x_1, y_1, x_1', x_1'', z_1\} = \{z_1^4\}$. By (5), $z^4 \neq z_1'$. If $z_1^4 = x_1$, then $T_4[z_1, x_1]x_1'x_1''z_1'y_1$ has length at least 5. Hence $z_1^4 \neq x_1$. Similarly $z_1^4 \neq y_1$, and so $z_1^4 = x_1''$. As G has no K_3, $|E(T_4)| \geq 2$, and so $T_4[z_1, x_1'']z_1'y_1x_1'x_1$ has length at least 6, contrary to (5). This contradiction justifies Claim 4C and proves Claim 4.

By Claim 4, there exist $x_1' \in N_G(x_1) - N_{G_1}(y_1)$, and $y_1' \in N_{G_1}(y_1) - N_{G_1}(x_1)$. Thus $x_1' \neq y_1'$ and $x_1'y_1, x_1y_1' \notin E(G)$.

Claim 5. *Each of the following holds.*

(i) $N_{G_1}(x_1') \cap N_{G_1}(y_1') - \{z_1\} = \emptyset$.

(ii) $N_{G_1}(x_1') \cup N_{G_1}(y_1') - \{z_1\}$ *is an independent set.*

(iii) *For any* $x \in N_{G_1}(x_1) - \{z_1\}$ *and for any* $y \in N_{G_1}(y_1) - \{z_1\}$, $xy \notin E(G)$.

To prove Claim 5(i) and (ii), we assume that there exist $x_1'' \in N_{G_1}(x_1')$ and $y_1'' \in N_{G_1}(y_1')$. In the proof of (i), we assume that $x_1'' = y_1''$ with the notational convention that $x_1''y_1''$ denoting a single vertex x_1'', and in the proof of (ii), we assume $x_1''y_1'' \in E(G)$. We put some useful observations in Claim 5A.

Claim 5A. *Each of the following holds.*

(i) G_1 *has no* $(z_1, \{x_1, y_1\})$*-path disjoint from* $\{x_1', x_1'', y_1', y_1''\}$.

(ii) G_1 *has no* $(z_1, \{x_1', y_1'\})$*-path of length at least* 2 *disjoint from* $\{x_1'', y_1''\}$.

(iii) G_1 *has no* $(z_1, \{x_1'', y_1''\})$*-path of length at least* 3.

If G_1 has a path $Q[z_1, x_1]$ disjoint from $\{x_1', x_1'', y_1', y_1''\}$, then $Q[z_1, x_1]x_1'x_1''y_1''y_1'y_1$ violates (5). This proves Claim 5A(i). The proofs of Claim 5A(ii) and (iii) are similar and will be omitted. This justifies Claim 5A.

By Claim 3, G_1 has a cycle C'' containing z_1 and $x_1'x_1''$, and so by Claim 5A(ii), we must have $z_1x_1' \in E(C'') \subseteq E(G)$. By (1), G has no 3-cycles, C'' must contain a path $z_1z_1'x_1''$, for some $z_1' \notin \{x_1', x_1'', y_1', y_1\}$. By symmetry, $z_1y_1' \in E(G)$. By $\kappa'(G) \geq 3$, $N_G(z_1') - \{z_1, x_1''\}$ has a vertex z_1''. By Claim 3, G_1 has a cycle C^3 containing $z_1'z_1''$ and x_1x_1'. Hence C^3 contains either a path $Q_1^3[z_1', x_1]$ such that $x_1', x_1'', y_1', y_1 \notin V(Q_1^3)$, contrary to Claim 5A(i); or a path $Q_2^3[x_1'', x_1]$ such that $x_1', y_1', y_1 \notin V(Q_2^3)$, whence $z_1x_1'\overleftarrow{Q_2^3}[x_1, x_1'']y_1'y_1$ violates (5); or a path $Q_3^3[x_1, y]$ with $y \in \{y_1, y_1'\}$ such that $[\{x_1', x_1''\} \cup (\{y_1, y_1'\} - \{y\})] \cap V(Q_3^3) = \emptyset$, whence $z_1z_1'x_1''x_1'Q_3^3[x_1, y]y_1$ violates (5). This proves Claim 5(i) and (ii).

To prove Claim 5(iii), we assume that Claim 5(iii) does not hold by assuming that $x_1' \in N_{G_1}(x_1) - \{z_1\}$ and $y_1' \in N_{G_1}(y_1) - \{z_1\}$ with $x_1'y_1' \in E(G)$. By $\kappa'(G) \geq 3$ and by Claim 5(ii), there exist $x_1'' \in N_{G_1}(x_1') - \{x_1, y_1'\}$ and $y_1'' \in N_{G_1}(y_1') - \{y_1, x_1'\}$. Since G has no K_3, $x_1'' \neq y_1''$. By $\kappa(G_1) \geq 2$, G_1 has a cycle C^1 containing both $x_1'x_1''$ and $y_1'y_1''$.

Claim 5 Case A. $\{x_1', y_1'\} \cap V(C^1[x_1'', y_1'']) = \emptyset$, or $\{x_1', y_1'\} \cap V(C^1[y_1'', x_1'']) = \emptyset$.

By symmetry, we may assume that $\{x_1', y_1'\} \cap V(C^1[x_1'', y_1'']) = \emptyset$, Then $Q^1 = C^1[x_1'', y_1'']$ is an (x_1'', y_1'')-path, with $x_1', y_1' \notin V(Q^1)$. By $\kappa(G_1) \geq 2$, G_1 has paths $Q_i^2[z_1, w_i]$ with $V(Q_i^2) \cap (V(Q^1) \cup \{x_1, y_1, x_1', y_1'\}) = \{w_i\}$, for $i = 1, 2$.

Suppose first that $\{x_1, y_1\} \cap V(Q^1) = \{w'\}$, for some $w' \in \{x_1, y_1\}$. By symmetry, we may assume that $w' = y_1$. Thus $Q_1^2[z_1, w_1]Q^1[w_1, y_1'']y_1'x_1'x_1$ (if $w_1 \in V(Q^1)[x_1'', w']$), or $Q_1^2[z_1, w_1]Q^1[w_1, y_1'']y_1'x_1'Q^1[x_1'', w']$ (if $w_1 \in V(Q^1)[w', y_1''] - \{w'\}$) or $Q_1^2[z_1, w']Q^1[w', y_1'']y_1'x_1'x_1$ (if $w_1 = w'$) violates (5).

Next we assume $\{x_1, y_1\} \subset V(Q^1)$. If $y_1 \notin V(Q^1[x_1'', x_1])$ (which is equivalent to $x_1 \notin V(Q^1[y_1, y_1''])$), then $Q_1^2[z_1, w_1]\overleftarrow{Q^1}[w_1, x_1'']x_1'y_1'Q^1[y_1'', y_1]$ (if $w_1 \in V(Q^1[x_1'', x_1])$), or $Q_1^2[z_1, w_1]Q^1[w_1, y_1'']y_1'x_1'Q^1[x_1'', x_1]$ (if $w_1 \in V(Q^1[x_1, y_1''])$) violates (5). If $y_1 \in V(Q^1[x_1'', x_1])$ (which is equivalent to $x_1 \in V(Q^1[y_1, y_1''])$), then $Q_1^2[z_1, w_1]\overleftarrow{Q^1}[w_1, x_1'']x_1'y_1'\overleftarrow{Q^1}[y_1'', x_1]$ (if $w_1 \in V(Q^1[x_1'', x_1]) - \{x_1\}$), or $Q_1^2[z_1, w_1]Q^1[w_1, y_1'']y_1'x_1'Q^1[x_1'', y_1]$ (if $w_1 \in V(Q^1[x_1, y_1''])$), violates (5).

Thus we may assume that $\{x_1, y_1\} \cap V(Q^1) = \emptyset$. If some $w_i \in \{x_1, y_1, x_1', y_1'\}$, then, without loss of generality, we assume $w_1 \in \{x_1, x_1'\}$. Thus $Q_1^2[z_1, x_1]x_1'Q^1[x_1'', y_1'']y_1'y_1$ (if $w_1 = x_1$) or $Q_1^2[z_1, x_1']Q^1[x_1'', y_1'']y_1'y_1$ (if $w_1 = x_1'$) is a violation to (5). Hence we must have $w_1, w_2 \in V(Q^1)$. Without loss of generality, we assume that $|E(Q_1^2)| \geq |E(Q_2^2)|$ and $w_2 \in V(Q^1[w_1, y_1''])$. Since G has not K_3, when both $|E(Q_1^2)| = |E(Q_2^2)| = 1$, w_1 and w_2 are not adjacent. It follows that $Q_1^2[z_1, w_1]Q^1[w_1, y_1'']y_1'x_1'x_1$ is a violation to (5). This settles Case A.

Claim 5 Case B. $\{x_1', y_1'\} \cap V(C^1[x_1'', y_1'']) \neq \emptyset$ and $\{x_1', y_1'\} \cap V(C^1[y_1'', x_1'']) \neq \emptyset$.

By symmetry, assume that $x_1' \in V(C^1[x_1'', y_1''])$. Then $y_1' \in V(C^1[y_1', x_1''])$. As G has not K_3, $|E(C^1[x_1', y_1''])| \geq 2$ and $|E(C^1[y_1', x_1''])| \geq 2$. By symmetry, we assume that $z_1 \notin V(C^1[x_1', y_1''])$. By $\kappa(G_1) \geq 2$, for $i = 1, 2$, G_1 has paths $Q_i^3[z_1, w_i]$ with $V(Q_i^3) \cap (V(C^1) \cup \{x_1, y_1, x_1', y_1'\}) = \{w_i\}$ such that $w_1 \neq w_2$. Hence we may assume that $w_1 \neq x_1'$ (if $w_1 = x_1'$, then we relabel w_2 as w_1).

Suppose first that $\{x_1, y_1\} \cap V(C^1[y_1', x_1'']) = \{w'\}$, for some $w' \in \{x_1, y_1\}$. Note that if $w' = x_1$, then $x_1' \in V(C^1[x_1'', x_1])$, and if $w' = y_1$, then $y_1' \in V(C^1[y_1'', y_1])$. Table 2 shows that a contradiction to (5) can always be found.

Next we assume $\{x_1, y_1\} \subset V(C^1)$.

Table 2
Claim 5 Case B, when $\{x_1, y_1\} \cap V(C^1[y_1', x_1'']) = \{w'\}$.

w'	w_1 is in	a (z_1, W_1)-path longer than 4	Symmetric cases and explanations
x_1	$\{y_1', y_1\}$	$Q_1^3[z_1, w_1]y_1'\overleftarrow{C^1}[y_1', x_1'']x_1'x_1$	when $w' = y_1$ and $w_1 \in \{x_1', x_1\}$
x_1	$V(C^1[y_1', x_1'']) - \{y_1'\}$	$Q_1^3[z_1, w_1]C^1[w_1, x_1']y_1'\overleftarrow{C_1}[y_1'', x_1]$	when $w' = y_1$ and $w_1 \in V(C^1[x_1', y_1'']) - \{x_1'\}$
			Thus we assume that $z_1 \notin V(C^1[y_1', x_1''])$.
x_1	x_1	$Q_1^3[z_1, x_1]x_1'\overleftarrow{C_1}[x_1'', y_1']y_1$	when $w' = w_1 = y_1$
x_1	$V(C^1[x_1', y_1'']) - \{x_1'\}$	$Q_1^3[z_1, w_1]C^1[w_1, x_1']x_1$	when $w' = y_1$ and $w_1 \in V(C^1[y_1', x_1'']) - \{y_1'\}$
y_1	y_1'	$Q_1^3[z_1, y_1']y_1C^1[y_1, x_1']x_1$	

Subcase B1. $x_1, y_1 \notin V(C^1[x_1', y_1''])$. (The case when $x_1, y_1 \notin V(C^1[y_1', x_1''])$ is similar.)

Since G has not K_3, the distance between x_1 and y_1 in $C^1[x_1', y_1'']$ is at least 2. Note that $w_1, w_2, x_1, y_1 \in V(C^1[y_1', x_1''])$. If for some $i \in \{1, 2\}$, $|V(C^1[y_1', w_i]) \cap \{x_1, y_1\}| \le 1$, (say, $y_1 \notin V(C^1[y_1', w_i])$), then $Q_i^3[z_1, w_i]\overleftarrow{C^1}[w_i, x_1']\, C^1[x_1'', y_1]$ has length at least 5. Hence we may assume that $x_1, y_1 \in V(C^1[y_1', w_1])$. Then $Q_1^3[z_1, w_1]\, C^1[w_1, y_1'']y_1'y_1$ has length at least 5. This proves Subcase B1.

Subcase B2. $\{x_1, y_1\} \cap V(C^1[x_1', y_1'']) \ne \emptyset$ and $\{x_1, y_1\} \cap V(C^1[y_1', x_1'']) \ne \emptyset$.

We may assume that $x_1 \in V(C^1[x_1', y_1''])$ and $y_1 \in V(C^1[y_1', x_1''])$.

If $w_1, w_2 \cap \{x_1, y_1\} \ne \emptyset$, then by symmetry, we may assume that $w_1 = x_1$, and so $Q_1^3[z_1], [x_1]C^1[x_1', y_1'']y_1'y_1$ has length at least 5. Therefore, we may assume that $w_1, w_2 \cap \{x_1, y_1\} = \emptyset$. If $\{w_1, w_2\} \ne \{x_1', y_1'\} \ne \emptyset$, say $w_1 = x_1'$, then $Q_1^3[z_1, x_1']\overleftarrow{C^1}[x_1', y_1'']y_1'y_1$ has length at least 5. Therefore by symmetry, we assume that $w_1 \in V(C^1[y_1', x_1'']) - \{y_1, y_1'\}$. Thus $Q_1^3[z_1, w_1]C^1[w_1, x_1'']c^1[x_1', y_1']y_1$ has length at least 5, contrary to (5). This proves Subcase B2.

Finally we may assume that $\{x_1, y_1\} \cap V(C^1) = \emptyset$. If some $w_i \in \{x_1, y_1, x_1', y_1'\}$, then, without loss of generality, we assume $w_1 \in \{x_1, x_1'\}$. Thus $Q_1^3[z_1, x_1]C^1[x_1', y_1'']y_1'y_1$ (if $w_1 = x_1$) or $Q_1^3[z_1, x_1']Q^1[x_1'', y_1'']y_1'y_1$ (if $w_1 = x_1'$) is a violation to (5). Hence we must have $w_1, w_2 \in V(C^1) - \{x_1', y_1'\}$. By symmetry, we may assume that $w_1 \in V(C^1[y_1', x_1'']) - \{y_1'\}$. Hence $Q^3 - 1[z_1, w_1]C^1[w_1, x_1']y_1'y_1$ has length at least 5, contrary to (5). This justifies Claim 5(iv), and completes the proof of Claim 5.

We are now back to the proof of Case 3. By Claim 3, $G_1 - z_1$ has an (x_1, y_1)-path $T = u_0u_1u_2 \ldots u_s$ with $u_0 = x_1$ and $u_s = y_1$. By Claims 4 and 5(i)–(iii), respectively, we have

$$u_0u_{s-1}, u_1u_s \notin E(G), \quad u_2 \ne u_{s-2}, \quad \text{and} \quad u_2u_{s-2}, u_1u_{s-1} \notin E(G).$$

Hence $s \ge 6$. By Claim 3, G_1 has internally disjoint paths $Q_i[z_1, w_t]$, $(1 \le t \le 2)$, for some distinct $w_1, w_2 \in V(T)$ such that $V(Q_t) \cap V(T) = \{w_t\}$. Let $w_1 = u_i$ and $w_2 = u_j$. By symmetry, we may assume that $i < j$ and $j \ge \lceil \frac{s}{2} \rceil + 1 \ge 4$. It follows that $T[u_0, u_j]z_1$ has length at least 5, contrary to (5). This completes the proof of Lemma 3.1. □

Proof of Theorem 1.1. By contradiction, assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{6}$$

In particular,

$$G \text{ is non-supereulerian and } G \text{ is not contractible to } P(10). \tag{7}$$

Suppose that G has a nontrivial collapsible H. Since $\kappa'(G/H) \ge \kappa'(G)$ and the circumference of G/H is not bigger than that of G, it follows by (6) that G/H is either supereulerian, whence by Theorem 2.1(iv), G is supereulerian; or G/H is contractible to $P(10)$, implying that G is contractible to $P(10)$. Thus G is not a counterexample to Theorem 1.1. If G has a cut vertex, then by (6), either each block of G is supereulerian, whence G is supereulerian; or one block of G is contractible to $P(10)$, whence G is contractible to $P(10)$. Therefore, we may assume that

$$G \text{ is reduced with } \kappa(G) \ge 2 \text{ and } G \ne K_1. \tag{8}$$

Claim 6. $g(G) \ge 5$.

Suppose that G has a 4-cycle $C' = x_1x_2y_1y_2x_1$, and we shall use the same notation as in Definition 2.4. By Lemma 3.1, $\kappa'(G/\pi(C')) \ge 3$. As any cycle of $G/\pi(C')$ can be (possibly trivially) extended to a cycle of G, and so $c(G/\pi(C')) \le c(G) \le 11$. By (6), either $G/\pi(C')$ is supereulerian, whence by Theorem 2.5(b), G is also supereulerian, contrary to (6); or $G/\pi(C')$ is contractible to the $P(10)$. When $G/\pi(C')$ is contractible to the $P(10)$, if the edge $e_{\pi(C')}$ is being contracted, then by the definition of contraction, G is also contractible to $P(10)$, contrary to (6). Hence $e_{\pi(C')}$ must be an edge in $P(10)$, as depicted in Fig. 5.

We adopt the notation in Fig. 5, where for $1 \le i \le 2$, H_i is the preimage of the vertex in $P(10)$ such that $x_i, y_i \in V(H_i)$. Since H_i is connected, H_i has an (x_i, y_i)-path P_i. If a $|E(P_i)| = 1$, then G has a K_3; if $|E(P_1)| = |E(P_2)| = 2$, then $G[V(P_1) \cup V(P_2)] \cong K_{3,3}^-$. Hence by (1), we may assume that $|E(P_1)| \ge 3$ and $|E(P_2)| \ge 2$. Let e_i be an edge in $G - (E(C') \cup E(H_1) \cup E(H_2))$ incident

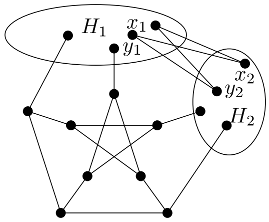

Fig. 5. The graph G when $e_{\pi(C')}$ is an edge of $P(10)$.

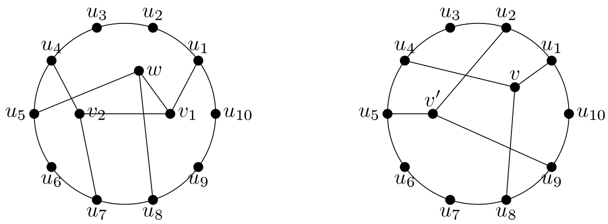

Fig. 6. Graphs in the proof of Claim 8.

with a vertex in H_i. It follows that $G[E(P_1) \cup E(P_2) \cup E(C') \cup \{e_1, e_2\}]$ contains a path Q from e_1 to e_2 with $|E(Q)| \geq 7$. As $P(10) - e_{\pi(C')}$ has a path Q' from e_1 to e_2 of length 8, it follows that $Q \cup Q'$ is a cycle of G of length at least 13, contrary to the assumption $c(G) \leq 11$. This completes the proof of Claim 6.

By Theorems 2.3 and 2.9 and Claim 6, we may assume that

$$|V(G)| \geq 14, g(G) \geq 5 \quad \text{and} \quad c(G) \geq 9. \tag{9}$$

Let $p = c(G)$ and $C = u_1u_2 \cdots u_pu_1$ be a longest cycle in G. In the discussions below, the subscripts for u_i will be taken mod p. By $\kappa(G) \geq 2$ and as G is non-supereulerian, we may assume that $G - V(C)$ has a path $P^1 = v_1v_2 \cdots v_s$ with $s > 1$ such that for some $1 \leq j_1 < j_2 \leq p$, $v_1u_{j_1}, v_su_{j_2} \in E(G)$. Choose a longest such path P^1, and assume without loss of generality that $j_1 = 1$.

If $s \geq 5$, then as $c(G) \leq 11$, both $|E(C[u_1, u_{j_2}])| \leq 5$ and $|E(C[u_{j_2}, u_1])| \leq 5$. It follows that $C' = P^1[v_1, v_s]C[u_{j_2}, u_1]v_1$ is a cycle on G, and $|E(C)| < |E(C')|$, contrary to the fact that C is a longest cycle in G. Hence we must have $1 \leq s \leq 4$.

Suppose that $s \in \{3, 4\}$. If for each v_k, $1 \leq k \leq 4$, $N_G(v_k) \subseteq V(P^1) \cup \{u_1, u_{j_2}\}$, then by $\kappa'(G) \geq 3$ and by Lemma 2.8, G is not reduced, contrary to (1). Thus there must be a path P^2 in G with $V(P^2) \cap (V(C) \cup V(P^1)) = \{v_t, u_{t'}\}$, where $1 \leq t \leq 4$ and $1 < t' < j_2$ or $j_2 < t' \leq p$. By symmetry, we assume that $1 < t' < j_2$. Since C is longest, $t' \geq t + |E(P^2)|$ and $j_2 - t' \geq |E(P^2)| + (s - t + 1)$. It follows by $|E(P^2)| \geq 1$ that $|E(C[u_1, u_j])| = j_2 + 1 \geq s + 2 + 2|E(P^2)| \geq s + 4$, and so $|E(C[u_{j_2}, u_1])| = p - |E(C[u_1, u_{j_2}])| \leq 11 - (s + 4) = 7 - s \in \{3, 4\}$. Since C is longest, and since replacing $C[u_{j_2}, u_1]$ by $u_1P^1u_{j_2}$ in C results in another cycle C', we must have $s = 3$, $|E(C[u_{j_2}, u_1])| = 4$, $|E(P^2)| = 1$, $t' \in \{p - 7, p - 6\}$ and $10 \leq p \leq 11$.

Since $s = 3$, by symmetry, we assume that $t \neq 3$. By $\kappa'(G) \geq 3$ and since G is reduced, $N_G(v_3) - \{v_2, u_j\}$ contains a vertex $w \notin V(P^1)$. Since C is longest, $w \notin V(C[u_{j_2}, u_1]) - \{u_1\}$. By Claim 6, $w \neq u_1$, and so $w = u_{t''}$ with $2 \leq t'' \leq j_2 - 1$. Since G contains no K_3, $t' \neq t''$. If $2 \leq t'' \leq t'$, then $u_1P^1[v_1, v_3]C[u_{t''}, u_1]$ is a cycle of G, and so $t'' \in \{p - 5, p - 6\}$. As $t' \neq t''$, we must have $t' = p - 7 < t''$, contrary to the fact $t' > t''$. If $t' < t'' \leq j_2 - 1$, a contradiction will be obtained with a similar argument. Thus we must have $s \leq 2$ and $N_G(v_s) \subset V(P^1) \cup V(C)$. By $\kappa'(G) \geq 3$, by (1) and by Claim 6, there exists $u_i, u_j \in N_G(v_s) \cap V(C)$ with $1 < i < j < p$.

Claim 7. $c(G) \geq 10$.

By contradiction, we assume that $c(G) = 9$. If $s = 2$, then as $c(G) = 9$, both $C[u_1, u_i]$ and $C[u_j, u_1]$ has length at least 3. By $g(G) \geq 5$, we must have $u_i = u_4$ and $u_j = u_7$. By $\kappa'(G) \geq 3$, by $g(G) \geq 5$ and with a similar argument, $N_G(v_1) - (V(C) \cup \{v_2\})$ has a vertex w' with $u_4, u_7 \in N_G(w')$, forcing the existence of a 4-cycle, contrary to $g(G) \geq 5$.

Hence $s = 1$, and so for any $w \in V(G) - V(C)$, $N(w) \subseteq V(C)$. By $g(G) \geq 5$, for each w, there exists an i, such that $N(w) = \{u_i, u_{i+3}, u_{i+6}\}$, where the subscripts are taken mod 9. By (9), there must be at least $14 - 9 = 5$ vertices in $V(G) - V(C)$. Therefore, there must exist $w_1, w_2 \in V(G) - V(C)$ and an i such that $u_i \in N(w_1) \cap N(w_2)$. Hence $N(w_1) = N(w_2)$, contrary to $g(G) \geq 5$. This proves Claim 7.

Claim 8. $c(G) = 11$.

Assume that $c(G) = 10$. Suppose $s = 2$. By $g(G) \geq 5$, by symmetry and since C is a longest cycle, we may assume that $u_4, u_j \in N(v_2)$ with $j \in \{7, 8\}$. (See Fig. 6(a).) As $\kappa'(G) \geq 3$, there exists a vertex $w \in N(v_1) - \{u_1, v_2\}$. If $j = 8$ and $w = u_6$,

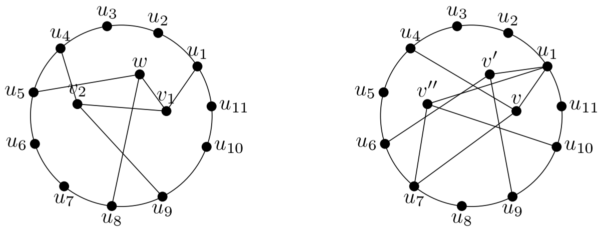

Fig. 7. Possible cases when $c(G) = 11$ in the proof of Theorem 1.1.

then $C[u_1, u_6]v_1v_2C[u_8, u_1]$ is a cycle longer than C. Hence when $j = 8$, $w \neq u_6$. It follows by (1) and by $g(G) \geq 5$ that $w \notin V(C)$ no matter whether u_7 or $u_8 \in N_G(v_2)$. By (1) and since C is longest, if $u_i \in N_G(v_2)$, then $w \notin \{u_i, u_{i\pm1}, u_{i\pm2}\}$, where the subscripts are taken mod 10. Therefore by $s = 2$, $\emptyset \neq N(w) - \{v_1\} \subseteq V(C)$. Suppose that some $u_k \in N_G(w)$. As $u_4v_2v_1wu_k$ is a path of length 4, the distance between u_k and u_4 on C is at least 4, forcing $8 \leq k \leq 10$. As $u_1v_1wu_k$ has length 3, the distance between u_1 and u_k on C is at least 3, and so $k \notin \{9, 10\}$. It follows that $k = 8$. Since $j \in \{7, 8\}$, either $j = 8$, whence G has a 4-cycle $wu_8v_2v_1w$, contrary to Claim 6; or $j = 7$, whence $C[u_1, u_7]v_2v_1wC[u_8, u_1]$ is a cycle of 13 in G, contrary to $c(G) \leq 11$. These contradictions indicate that $s < 2$.

Hence $s = 1$, and so for any $v \in V(G) - V(C)$, $N(v) \subseteq V(C)$. By $g(G) \geq 5$, by symmetry and since C is longest, for each $v \in V(G) - V(C)$, if $u_i \in N(v)$, then either $u_{i+3} \in N(v)$ and $|N(v) \cap \{u_{i+6}, u_{i+7}\}| = 1$, or $u_{i-3} \in N(v)$ and $|N(v) \cap \{u_{i+3}, u_{i+4}\}| = 1$, where the subscripts are taken mod 10 (see Fig. 6(b)). Assume that $N(v) \cap N(v') \neq \emptyset$ and (without loss of generality) $u_1 \in N(v) \cap N(v')$. Then by $g(G) \geq 5$ and by symmetry, we must have $u_4v, u_5v' \in E(G)$, and consequently, $u_7v, u_8v' \in E(G)$. It follows that G has a cycle $C[u_1, u_4]v\overleftarrow{C}[u_7, u_5]v'C[u_8, u_1]$ of length 12, contrary to $c(G) = 10$.

Hence we may assume that $N(v) \cap N(v') = \emptyset$. By (9), $|V(G) - V(C)| \geq 4$, and so we may assume that $v, v' \in V(G) - V(C)$ such that $N(v) = \{u_{i_1}, u_{i_2}, u_{i_3}\}$ and $N(v') = \{u_{i_1+1}, u_{i_2+1}, u_{i_3+1}\}$ or $N(v') = \{u_{i_1+1}, u_{i_2+1}, u_{i_3+2}\}$. (See Fig. 6(b).) Without loss of generality, we assume that $i_1 = 1$ $i_2 = 4$, and $i_3 \in \{7, 8\}$.

If $i_3 = 8$, then by $g(G) \geq 5$, we have $u_9v' \in E(G)$, and so G has a cycle $v'C[u_9, u_4]v\overleftarrow{C}[u_{i_3}, u_5]v'$ of length at least 11, contrary to $c(G) = 10$. Hence $i_3 = 7$, and $|\{u_8, u_9\} \cap N(v')| = 1$. If $v'u_8 \in E(G)$, then G has a cycle $v\overleftarrow{C}[u_4, u_8]v'C[u_5, u_7]v$ of length 12, contrary to $c(G) = 10$. Thus $u_9v' \in E(G)$, and so G has a cycle $v'C[u_9, u_4]v\overleftarrow{C}[u_7, u_5]v'$ of length 11, contrary to $c(G) = 10$. This proves Claim 8.

By Claims 7 and 8, we must have $c(G) = 11$. Suppose first that $s = 2$. (See Fig. 7(a).) By $g(G) \geq 5$, and by symmetry, we may assume that $|\{u_4, u_5\} \cap N(v_2)| = 1$ and $|\{u_7, u_8, u_9\} \cap N(v_2)| = 1$. As $\kappa'(G) \geq 3$, $N(v_1) - \{u_1, v_2\}$ has a vertex w. By $g(G) \geq 5$, we have $w \notin \{u_1, u_2, u_3, u_{10}, u_{11}\}$. As C is longest, if $u_i \in N_G(v_2)$ and $w \in V(C)$, then the distance between u_i and w on C must be at least 3. If follows that $w \notin V(C)$.

By $g(G) \geq 5$, $N_G(w) \cap \{u_1, u_2, u_{11}\} = \emptyset$. Since C is longest, if $u_i \in N_G(v_2)$, then $N_G(w) \cap \{u_i, u_{i\pm1}, u_{i\pm2}, u_{i\pm3}\} = \emptyset$. It follows by $|\{u_4, u_5\} \cap N(v_2)| = 1$ and $|\{u_7, u_8, u_9\} \cap N(v_2)| = 1$ that $N_G(w) \cap V(C) = \emptyset$, forcing $s \geq 3$, contrary to $s = 2$. Thus we must have $s = 1$, and so for any $v \in V(G) - V(C)$, $N(v) \subseteq V(C)$. By $g(G) \geq 5$, for each $v \in V(G) - V(C)$, if $u_i, u_j \in N(v)$, then the distance of u_i and u_j on C must be at least 3. These lead to the following observations.

Observation 3.2. *If $u_i \in N_G(v)$, $(1 \leq i \leq 11)$, then either $u_{i+3} \in N(v)$ and $|N(v) \cap \{u_{i+6}, u_{i+7}, u_{i+8}\}| = 1$, or $u_{i+4} \in N(v)$ and $|N(v) \cap \{u_{i+7}, u_{i+8}\}| = 1$, or $\{u_{i+5}, u_{i+8}\} \subset N(v)$, where the subscripts are taken mod* 11.

By (9), $V(G) - V(C)$ has at least 3 distinct vertices v, v', v''. Since every vertex in $V(G) - V(C)$ has at least 3 neighbors on $V(C)$, and since $|V(C)| = 11$, either for two vertices v and v' (say) $N(v) \cap N(v') \neq \emptyset$, or there exists at least one i such that $u_i \in N(v)$, $u_{i+1} \in N(v')$ and $u_{i+2} \in N(v'')$.

Claim 9. *If $u_1 \in N(v) \cap N(v')$, then $u_4, u_7 \in N(v)$ and $u_6, u_9 \in N(v')$.*

By Observation 3.2, we may assume that $u_{i_1}, u_{i_3} \in N(v)$ and $u_{i_2}, u_{i_4} \in N(v')$ with $4 \leq i_1 < i_2 \leq 6 < 7 \leq i_3 < i_4 \leq 9$. If $i_2 - i_1 = 1$ or $i_4 - i_3 = 1$, then $C[u_1, u_{i_1}]v\overleftarrow{C}[u_{i_3}, u_{i_2}]v'C[u_{i_4}, u_1]$ has length at least 12, contrary to $c(G) = 11$. Hence we must have that $u_4, u_7 \in N(v)$ and $u_6, u_9 \in N(v')$ (see Fig. 7(b)). This proves Claim 9.

Claim 10. *For any distinct $x, y \in \{v, v', v''\}$, $N(x) \cap N(y) = \emptyset$.*

Suppose not, and without loss of generality, we assume that $u_1 \in N(v) \cap N(v')$. By Claim 9, $u_4, u_7 \in N(v)$ and $u_6, u_9 \in N(v')$. If $v'' \in N(u_1)$, then by $g(G) \geq 5$, we must have $u_5, u_8 \in N(v'')$, and so $C[u_1, u_4]v\overleftarrow{C}[u_7, u_5]v''C[u_8, u_1]$ has length 13, contrary to $c(G) = 11$. If $v'' \in N_G(u_4)$, then by Claim 9 with u_4 replacing u_1 and v, v'' replacing v, v', we have $u_7, u_{10} \in N_G(v'')$ (see Fig. 7(b)), whence G has 4-cycle $u_4v''u_7vu_4$, contrary to $g(G) \geq 5$. If $v'' \in N_G(u_6)$, then by Claim 9 with u_6 replacing u_1 and v'', v' replacing v, v', we have $u_3, u_{11} \in N_G(v'')$, and so $C[u_6, u_{11}]v''\overleftarrow{C}[u_3, u_1]vC[u_4, u_6]$ is a cycle of length 13, contrary to $c(G) = 11$. Thus $N_G(v'') \cap \{u_1, u_4, u_6\} = \emptyset$. By symmetry, $N_G(v'') \cap \{u_7, u_9\} = \emptyset$, and so $N_G(v'') \subseteq \{u_2, u_3, u_5, u_8, u_{10}, u_{11}\}$. If $u_2, u_5 \in N_G(v'')$, then $C[u_6, u_1]v\overleftarrow{C}[u_4, u_2]v''u_5u_6$ is a cycle of length 13, contrary to $c(G) = 11$. This, together with $g(G) \geq 5$, implies $|N_G(v'') \cap \{u_2, u_3, u_5\}| \leq 1$. By Symmetry, $|N_G(v'') \cap \{u_8, u_{10}, u_{11}\}| \leq 1$. It follows that $3 \leq |N_G(v'')| = |N_G(v'') \cap \{u_2, u_3, u_5, u_8, u_{10}, u_{11}\}| \leq 2$. This contradiction justifies Claim 10.

By Claim 10 and by $c(G) = 11$, we may assume that $u_1 v, u_2 v' \in E(G)$. Suppose that $u_1, u_i, u_j \in N_G(v)$ $(1 < i < j < 11)$ and $u_2, u_{i'}, u_{j'} \in N_G(v')$ $(1 < i' < j' < 11)$. If $i \geq 7$, then by $g(G) \geq 5$, $j \geq 10$, and so $\overleftarrow{C}[u_j, u_1]vu_j$ is a cycle of length at most 4. By symmetry and by $g(G) \geq 5$, we may assume that

$$4 \leq i \leq 6 < j < 10, \quad \text{and} \quad 5 \leq i' \leq 7 < j' < 11. \tag{10}$$

Claim 11. *None of the following holds.*

(i) *Both $j = i' + 1$ and $i = 4$.*

(ii) *Either $i < i' \leq i + 2$, or $j < j' \leq j + 2$, or $j < i' \leq j + 2$.*

If we have both $j = i' + 1$ and $i = 4$, then $vC[u_j, u_2]v'\overleftarrow{C}[u_{i'}, u_4]v$ has length 12. This justifies Claim 11(i). We again argue by contradiction to prove Claim 11(ii). If $i < i' \leq i + 2$, then $C[u_2, u_i]v\overleftarrow{C}[u_1, u_{i'}]v'u_2$ has length at least 12. If $j < j' \leq j + 2$, then $C[u_2, u_j]v\overleftarrow{C}[u_1, u_{j'}]v'u_2$ has length at least 12. If $j < i' \leq i + 2$, then $C[u_2, u_j]v\overleftarrow{C}[u_1, u_{i'}]v'u_2$ has length at least 12. As any of these lead to a contradiction, Claim 11 must hold.

By (10), $4 \leq i \leq 6$. If $i \geq 5$, then by Claim 11(ii), $i' \geq i + 3 \geq 8$, contrary to (10). Hence $i = 4$. By Claim 11(ii), $i' = 7$, and so $j \geq 8$. By Claim 11(i), we must have $j \geq 9$. By $c(G) = 11$, we must have $j' = 11$, and so G has a 4-cycle $v'u_{11}u_1u_2v'$, contrary to $g(G) \geq 5$. This completes the proof of Theorem 1.1. □

4. Applications to 3-connected hamiltonian claw-free graphs

A subgraph H of G is dominating if $E(G - V(H)) = \emptyset$. Harary and Nash-Williams proved a useful relationship between dominating eulerian subgraphs and hamiltonian line graphs.

Theorem 4.1 (*Harary and Nash-Williams, [15]*)**.** *Let G be a connected graph with at least 3 edges. The line graph $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

Let G be a graph such that $\kappa(L(G)) \geq 3$ and such that $L(G)$ is not complete. A vertex cut X in $L(G)$ is an edge in G such that both sides of $G - X$ have at least one edge. An edge-cut X with at least two nontrivial components in $G - X$ is an **essential edge cut** of G. A graph G is essentially k-edge-connected if G does not have an essential edge cut of size less than k. For each $v \in D_2(G)$, let $E_G(v) = \{e_1^v, e_2^v\}$ and $X_2(G) = \{e_2^v : v \in D_2(G)\}$. Since $\kappa(L(G)) \geq 3$, $D_2(G)$ is an independent set of G and for any $v \in D_2(G)$, $|X_2(G) \cap E_G(v)| = 1$. Define

$$\begin{aligned} G_0 &= G/((\cup_{v \in D_1(G)} E_G(v)) \cup X_2(G)) = (G - D_1(G))/X_2(G) \\ NE(G) &= \cup_{v \in D_2(G)} E_G(v) - X_2(G). \end{aligned} \tag{11}$$

The graph G_0 is called the **core** of G, and edges in $NE(G)$ are called the **nontrivial edges** in G_0. Let $V(NE(G))$ denote the set of vertices in G incident with an edge in $NE(G)$. By the definition of G_0, vertices in G adjacent to a vertex in $D_1(G)$ can be viewed as vertices in G_0, which are the contraction images of edges in $\cup_{v \in D_1(G)} E_G(v)$. Let G_0' be the reduction of G_0. Then G_0' is a contraction of G_0 as well as G, and so we can view $E(G_0') \subseteq E(G_0) \subseteq E(G)$. Define

$$\begin{aligned} \Lambda(G_0) &= \{v \in V(G_0) : PI_G(v) \neq K_1 \text{ or } PI_{G_0}(v) \cap V(NE(G)) \neq \emptyset\} \\ \Lambda'(G_0) &= \{v \in V(G_0') : PI_G(v) \neq K_1 \text{ or } PI_G(v) \cap V(NE(G)) \neq \emptyset\}. \end{aligned} \tag{12}$$

Applying Theorem 4.1, Shao proved the following.

Theorem 4.2 (*Shao, Section 1.4 of [27]*)**.** *Let G be a connected graph with $|E(G)| \geq 3$ and let G_0 be the core of graph G, then each of the following holds:*

(i) *G_0 is nontrivial and $\delta(G_0) \geq \kappa'(G_0) \geq 3$.*

(ii) *G_0 is well defined.*

(iii) *$L(G)$ is hamiltonian if and only if G_0 has a dominating eulerian subgraph H such that $\Lambda(G_0) \subseteq V(H)$.*

Proof. The justifications of (i) and (ii), and of the sufficiency of (iii) can be found in Section 1.4 of [27]. We shall only show the necessity of (iii). Suppose that $L(G)$ is hamiltonian, then by Theorem 4.1, G must have a dominating eulerian subgraph H. By the definition of dominating eulerian subgraphs, H must contain all nontrivial vertices of G_0. Since every nontrivial edge of G_0 is the contraction image of a path of length 2 in G, both ends of any nontrivial edge must also be in H. □

Applying Theorems 1.1 and 4.2, one can derive the following on hamiltonian line graphs.

Theorem 4.3. *Let G be a graph such that $\kappa(L(G)) \geq 3$. Let G_0 be the core of G. Then one of the following must hold.*

(i) *$L(G)$ is hamiltonian.*

(ii) *G_0 is contracted to the Petersen graph $P(10)$.*

(iii) *$c(G_0) \geq 12$ and G_0 does not have a dominating eulerian subgraph.*

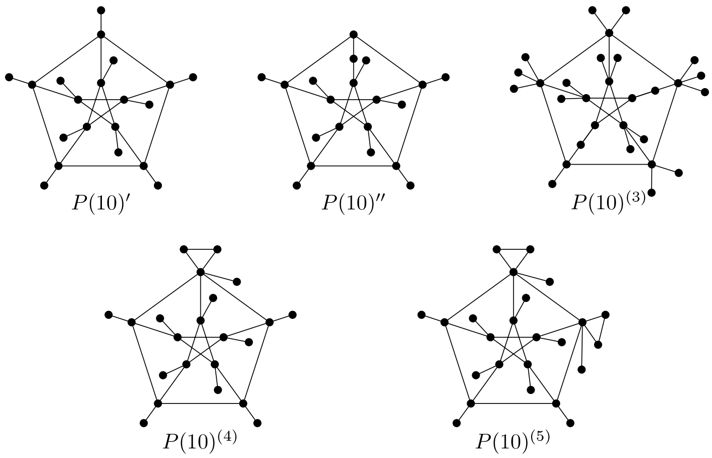

Fig. 8. Graphs in Definition 4.6.

Proof. Suppose that $L(G)$ is not hamiltonian. By Theorem 4.2, G_0 does not have any dominating eulerian subgraphs, and $\kappa'(G_0) \geq 3$. By Theorem 1.1, and since G_0 cannot be supereulerian, either G_0 is contracted to the Petersen graph $P(10)$, whence (ii) must hold; or $c(G_0) \geq 12$, whence Theorem 4.3(iii) must hold. □

Let G be a claw-free graph. Then for any $v \in V(G)$, $G[N_G(v)]$ is either connected (in this case, v is called a **locally connected vertex of** G) or is a disjoint union of two cliques. If $G[N_G(v)]$ is connected and not a clique, then the **local completion** of G at x is a graph obtained from G by adding edges to join nonadjacent vertices in $N_G(v)$. The closure of G, denoted by $\mathrm{cl}(G)$, is the graph obtained from G by repeated applications of local completions, until every locally connected vertex has its neighborhood being a clique. This construction was introduced by Ryjáček [26], and he proved the following useful result.

Theorem 4.4 (*Ryjáček, [26]*)**.** *Let G be a claw-free graph. Then*

(i) $\mathrm{cl}(G)$ *is uniquely determined.*

(ii) $\mathrm{cl}(G)$ *is the line graph of some triangle-free simple graph.*

(iii) *G is hamiltonian if and only if* $\mathrm{cl}(G)$ *is hamiltonian.*

Theorem 4.5 (*Brousek, Ryjáček and Favaron, [5]*)**.** *Let G be a claw-free graph. Then*

(i) *If G is Z_k-free, then* $\mathrm{cl}(G)$ *is also Z_k-free for any integer $k \geq 1$.*

(ii) *If G is P_i-free, then* $\mathrm{cl}(G)$ *is also P_i-free for any integer $i \geq 3$.*

Definition 4.6. Let $P(10)'$ be the graph obtained from the Petersen graph $P(10)$ by attaching exactly one pendant edge to every vertex of $P(10)$; let $P(10)''$ be the graph by replacing one edge $e = v_iv_j \in E(P(10))$ by a (v_i, v_j)-path of length 2, by attaching exactly one pendant edge to every vertex of $P(10) - \{v_i, v_j\}$ and by attaching at most one pendant edge to each of v_i and v_j. Let $P(10)^{(3)}$ denote any member in the family of graphs each of which is obtained from $P(10)$ by attaching at least one pendant edge to every vertex of $P(10)$ such that one vertex of $P(10)$ is attached with at least 2 pendent edges; and $P(10)^{(4)}$ denote any member in the family of graphs each of which is obtained from $P(10)$ by attaching to a vertex (say, v_1) of $P(10)$ by a non-tree simple graph H_{v_1} spanned by a $K_{1,t}$ for some $t \geq 2$, and by attaching exactly one pendant edge to every vertex of $P(10) - v_1$. Let $P(10)^{(5)}$ denote any member in the family of graphs each of which is obtained from $P(10)$ by replacing each of the two vertices of an edge $e = v_iv_j \in E(P(10))$ by non-tree simple graphs H_{v_i} and H_{v_j}, spanned by a K_{1,t_1}, and a K_{1,t_2}, respectively, for some $t_1, t_2 \geq 2$, and by attaching exactly one pendant edge to every vertex of $P(10) - \{v_i, v_j\}$ (see Fig. 8).

The next lemma summarizes some observations which follow from Definition 4.6 and Theorem 4.1.

Lemma 4.7. *Each of the following holds.*

(i) *If $G = P(10)'$, then $L(G)$ is $\{Z_9, P_{12}\}$-free.*

(ii) *If $G = P(10)^{(3)}$, then $L(G)$ is $\{Z_{10}, P_{12}\}$-free.*

(iii) *If $G \in \{P(10)'', P(10)^{(4)}, P(10)^{(5)}\}$, then $L(G)$ is $\{Z_{10}, P_{13}\}$-free.*

(vi) *If $G \in \{P(10)', P(10)'', P(10)^{(3)}, P(10)^{(4)}, P(10)^{(5)}\}$, then $L(G)$ is not hamiltonian.*

We are now ready to prove Theorem 1.8, restated as the following corollaries of Theorem 1.1.

Corollary 4.8. *Let Γ be a 3-connected $\{K_{1,3}, P_k\}$-free graph.*

(i) *Suppose that $k = 12$. Then Γ is hamiltonian if and only if $cl(\Gamma) \notin \{L(P(10)'), L(P(10)^{(3)})\}$.*

(ii) *Suppose that $k = 12$. Then Γ is hamiltonian if and only if $cl(\Gamma) \notin \{L(P(10)'), L(P(10)''), L(P(10)^{(3)}), L(P(10)^{(4)}), L(P(10)^{(5)})\}$.*

Corollary 4.9. *Let Γ be a 3-connected $\{K_{1,3}, Z_9\}$-free graph. Then Γ is hamiltonian if and only if $cl(\Gamma) \neq L(P(10)')$.*

Let $k > 0$ be an integer. Let $P_{k+1} = v_0v_1v_2 \ldots v_k$ denote a path on $k+1$ vertices, and let Y_k be the graph with

$$V(Y_k) = V(P_{k+1}) \cup \{v_{k+1}\} \quad \text{and} \quad E(Y_k) = E(P_{k+1}) \cup \{v_{k-1}v_{k+1}\}.$$

By definition, $Y_2 \cong K_{1,3}$. If $k = 2$, then any vertex in $D_1(Y_2)$ is a root of Y_2. If $k \geq 3$, then the unique vertex in $D_1(Y_k)$ which is not adjacent to the vertex in $D_3(Y_k)$ is the **root** of Y_k. Thus for a connected simple graph G and an integer $k > 0$, $L(G)$ is Z_k-free if and only if G does not have Y_{k+2} as a subgraph.

Before we prove Corollaries 4.8 and 4.9, we investigate some properties of graphs whose cores are contractible to $P(10)$. The observations below follow from the definition of the Petersen graph $P(10)$.

Lemma 4.10. *Each of the following holds.*

(i) *Let $u, v \in V(P(10))$ be distinct vertices. If $uv \notin E(P(10))$, then $P(10)$ contains a Hamilton (u, v)-path; if $uv \in E(P(10))$, then for any $w \in N(u) - v$, $P(10) - w$ has a Hamilton (u, v)-path.*

(ii) *For any $v \in V(P(10))$ and $e \in E(P(10))$, $P(10)$ contains a Hamilton (v, v')-path Q with $e \in E(Q)$ such that e is not incident with v'.*

(iii) *For any pair of edges $e, e' \in E(P(10))$, there exists a vertex w such that $P(10)$ has a Hamilton path Q with $e, e' \in E(Q)$ such that no end of Q is incident with e or e', and such that one end of Q is adjacent to w.*

(iv) *For any $v \in V(P(10))$ and for any $e \in E(P(10))$, $P(10)$ has a Y_8 rooted at v with $e \in E(Y_8)$ but e is not incident with any vertex in $D_1(Y_8) - \{v\}$.*

(v) *For any pair of distinct vertices $u, v \in V(P(10))$, $P(10)$ has a Hamilton path Q from u with v being the next to the last vertex in Q.*

Proof. All can be verified routinely using the definition of $P(10)$. We only sketch the justification for (iii).

(iii). For given $e, e' \in E(P(10))$, by the definition of $P(10)$, there exists a cycle C of length 9 containing e and e'. Let $w \in V(P(10)) - V(C)$. As w has degree 3, there must be a vertex $v \in N_{P(10)}(w)$ such that $\{e, e'\} \cap E_C(v) = \emptyset$ and $E_C(v)$ has an edge e'' not adjacent to either e or e'. Thus $P(10)[E(C - e'') \cup \{wv\}]$ is the desirable path. $\square$

Definition 4.11. Let G be a connected, essentially 3-edge-connected simple graph whose core G_0 is contracted to the Petersen graph $P(10)$. Let $V(P(10)) = \{v_1, v_2, \ldots, v_{10}\}$. For each i, let $L_i = PI_{G_0}(v_i)$, and $n_i = |V(L_i)|$. A star on $n \geq 2$ vertices is a graph isomorphic to $K_{1,n-1}$. For each i, let $E_{P(10)}(v_i) = \{e^i_1, e^i_2, e^i_3\}$, and let $A_G(L_i) = \{w^i_1, w^i_2, w^i_3\}$ such that w^i_j is incident with e^i_j. Since edges in $E_{P(10)}(v_i)$ may be adjacent in G, the w^i_j's may not be distinct.

(i) If L_i is not spanned by a star, then define L'_i to be the reduction of $L_i - D_1(L_1)$. If v_i is spanned by a star, then define $L'_i = K_1$.

(ii) If $L'_i \neq K_1$, then v_i is of Type 1A; if $L'_i = K_1$ and L_i is not spanned by a star, then v_i is of Type 1B.

(iii) Assume that $L'_i = K_1$ and L_i is spanned by a star. If $L_i \neq K_{1,n_i-1}$ and every cycle of L_i is a 3-cycle, then v_i is of Type 2A; if L_i is a star with $n_i \geq 2$, then v_i is of Type 2B; if $V(L_i) = \{v_i\}$, and $v_i \in V(NE(G))$, then v_i is of Type 3A; and if $V(L_i) = \{v_i\}$, and $v_i \notin V(NE(G))$, then v_i is of Type 3B.

Remark 4.12. From Definition 4.11, we have the following remarks.

(i) By the definition of collapsible graphs, if a nontrivial collapsible graph is not spanned by a star, then it must have a cycle of length at least 4. Thus if v_i is of Type 1B, then $c(L_i) \geq 4$.

(ii) A vertex can be both of Types 1A, 1B, 2A, 2B and of Type 3A.

(iii) Since G is essentially 3-edge-connected, if $L_i \in \{C_4, C_5, C_6\}$, then $V(L_i) - A_G(L_i)$ is an independent set.

(iv) If v_i is of Type 1B, then $L_i - D_1(L_i)$ is a nontrivial collapsible graph. For any $v \in V(L_i) - D_1(L_i)$, if any cycle containing v is of length 3, then $L_i - D_1(L_i)$ must be a collection of K_3's commonly sharing v. As G is essentially 3-edge-connected, L_i must be spanned by a star. Hence L_i must have a cycle C of length at least 4, and so L_i has a P_k ($k \geq 4$) from v. Since $L_i - D_1(L_i)$ is collapsible, either C has a chord or C is adjacent to a vertex of L_i not in C. Hence L_i has a $Y_{k'}$ ($k' \geq 3$) rooted at v.

Throughout the rest of this section, we always assume that G is a connected, essentially 3-edge-connected simple graph whose core G_0 is contracted to the Petersen graph $P(10)$. In the arguments, we shall use $P(10)$ to denote both the Petersen graph as well as the contraction image of G_0, for notational convenience.

Lemma 4.13. *If v_i is of Type* 1A, *then each of the following holds.*

(i) *For any $w_j^i \in A_G(L_i)$, L_i has a path from w_j^i with length at least* 3. *Furthermore, the length of any longest path in L_i from w_j^i is* 3 *if and only if L_i is a* 4*-cycle.*

(ii) *If $L_i' \notin \{C_4, C_5, C_6\}$, then for any $w_j^i \in A_G(L_i)$, L_i has a Y_k rooted at w_j^i with $k \geq 2$. Furthermore, L_i does not have a Y_k rooted at w_j^i with $k \geq 3$ for any j if and only if both $L_i \in \{K_{2,3}, S(1,2)\}$ and*

$$A_G(L_i) = \begin{cases} D_2(K_{2,3}) & \text{if } L_i' = K_{2,3} \\ N_{L_i'}(z_0) \text{ for some } z_0 \in D_3(S(1,2)) & \text{if } L_i' = S(1,2), \end{cases}$$

where $S(1,2)$ is defined in Definition 2.6.

Proof. We first claim that $\kappa'(L_i') \geq 2$. If e is a cut edge of L_i', then as L_i' is the reduction of $L_i - D_1(L_i)$, e must be an essential edge cut of L_i. Hence e an edge in $\{e_1^i, e_2^i, e_3^i\}$ will form an essential edge cut of G contrary to the assumption that G is essentially 3-edge-connected. Thus $\kappa'(L_i') \geq 2$, and so L_i' has a cycle. Let C be a longest cycle of L_i' and $c = |V(C)|$. Since G is reduced, we have $c \geq 4$.

(i) For any $w_j^i \in A_G(L_i)$, L_i has a path $P = P[w_j^i, w]$ for some $w \in V(C)$ such that P is internally disjoint from $V(C)$. It follows that $P \cup C$ contains a path from w_j^i with length at least $|E(C)| - 1 \geq 3$. If the length of any longest path from w_j^i in L_i is 3, then the path $P = P[w_j^i, w]$ has length 0, and so L_i must be a 4-cycle.

(ii) If L_i' has a cut vertex z, then L_i' has two subgraphs Z', Z'' such that $V(Z') \cap V(Z'') = \{z\}$. Since $\kappa'(L_i) \geq 2$ and since L_i is reduced, each of Z' and Z'' has a cycle of length at least 4. Therefore, no matter whether $w_j^i \in V(Z')$ or $w_j^i \in V(Z'')$, L_i will always have a Y_k with $k \geq 3$ rooted at w_j^i. Hence we may assume that $\kappa(L_i') \geq 2$.

Assume first that $c = 4$ and $L_i' \not\cong C_4$. By $\kappa(L_i') \geq 2$, L_i' must have a graph $J = K_{2,3}$ as a subgraph. If $w_j^i \notin D_2(K_{2,3})$, then L_i' has a path from w_j^i to a vertex in J, internally disjoint from J. Thus L_i' always has a Y_k ($k \geq 3$) rooted at w_j^i. Therefore, we may assume that $D_2(K_{2,3}) = \{w_1^i, w_2^i, w_3^i\}$. If $L_i' \neq K_{2,3}$, then by $\kappa(L_i') \geq 2$, J is contained in a $K_{2,4}$, and so for each w_j^i, L_i' has a Y_3 rooted at w_j^i. This completes the proof when $c = 4$.

Now assume that $c = 5$ and $L_i' \not\cong C_5$. By $c = 5$ and by $\kappa(L_i') \geq 2$, L_i must have an $S(1,2)$. If $w_j^i \notin D_2(S(1,2))$, then L_i' has a path from w_j^i to a vertex in $S(1,2)$, internally disjoint from $S(1,2)$. Thus L_i' always has a Y_k ($k \geq 3$) rooted at w_j^i. Therefore, we may assume that $\{w_1^i, w_2^i, w_3^i\} \subset D_2(S(1,2))$, and so for some $z_0 \in D_3(S(1,2))$, $N_{S(1,2)}(z_0) = \{w_1^i, w_2^i, w_3^i\}$. If $L_i' \neq S(1,2)$, then by $\kappa(L_i') \geq 2$ and $c = 5$, J is contained in either an $S(2,2)$ or an $S(1,3)$. This implies that L_i' must have a Y_k with $k \geq 3$ rooted at some w_j^i. This completes the proof when $c = 5$.

Finally, we assume that $c \geq 6$. Since G is essentially 3-edge-connected, if v, v' are two adjacent vertices in C, then either the preimage of one of $\{v, v'\}$ intersects $A_G(L_i)$, or one of $\{v, v'\}$ has degree at least 3 in L_i'.

It follows that if $c \geq 7$, then for any j, L_i' has a Y_2 rooted at w_j^i, and a Y_k with $k \geq 3$ for at least one w_j^i. Hence we may assume that $c = 6$. If for some j, $w_j^i \notin V(C)$, then L_i' has a path Q from w_j^i to $V(C)$, internally disjoint from $V(C)$. Thus, L_i' has a Y_k for some $k \geq 3$ rooted at a $w_{j'}^i$ for some $j' \neq j$. Therefore, we may assume that $\{w_1^i, w_2^i, w_3^i\} \subseteq V(C)$. Since $L_i' \neq C$, either C has a chord or there is a vertex not in C but adjacent to a vertex in C, and so for some w_j^i, L_i' has a Y_3 rooted at w_j^i. □

Lemma 4.14. *Suppose that $v_i, v_j \in V(P(10))$ are distinct vertices such that v_i is of Type* 1A *and $L_i' \notin \{C_4, C_5, C_6\}$, and v_j is not of Type* 3B. *Each of the following holds.*

(i) *If $v_iv_j \notin E(P(10))$, then G has both a P_{14} and a Y_{12}.*

(ii) *If $v_iv_j \in E(P(10))$, then G has both a P_k with $k \geq 14$ and a maximal $Y_{k'}$ with $k' \geq 11$. Furthermore, $k' = 11$ only if both v_j is of Type 3A and $L_i' \in \{K_{2,3}, S(1,2)\}$ with $A_G(L_i)$ satisfying Lemma* 4.13 *(ii).*

Proof. By Definition 4.11(ii) and (iii), and since v_j is not of Type 3B, any path of length k ending at v_j in $P(10)$ can be lifted and extended to a path of length at least $k+1$ by including an additional edge either in L_j (if v_j is not of Types 3A and 3B) or incident with the only vertex in L_j (if v_j is of Type 3A).

(i) Since $v_iv_j \notin E(P(10))$, by Lemma 4.10(i), $P(10)$ has a Hamilton (v_i, v_j)-path Q', which can be lifted to a (w_1^i, w_1^j)-path Q (say) in G of length at least 9. By Lemma 4.13, L_i has a path of length 4 from w_1^i, and L_i has a Y_2 rooted at w_1^i. It follows that Q'' can be further extended to a P_{14} in G. For Y_{12}, we first lift Q' to a path Q'' from L_i to L_j in G with $|E(Q'')| \geq 10$ by including an additional edge either in L_j (if v_j is not of Type 3A) or incident with w_1^j (if v_j is of Type 3A), and then extend it to a Y_{12} in G by including a Y_3 in L_i.

(ii) By the definition of $P(10)$, there is a (v_i, v_j)-path Q' in $P(10)$ with $|E(Q')| = 8$, which can be lifted to a (w_1^i, w_1^j)-path Q (say) of length at least 8 in G. By Lemma 4.13(i), L_i has a path of length at least 4 from w_1^i. If v_j is not of Type 2B or 3A, then by Lemma 4.13 and by $|E(L_j)| \geq 2$, Q can be extended to a path P_k with $k \geq 15$ as well as a $Y_{k'}$ with $k' \geq 13$ (each contains at least 4 edges in L_i and two edges in L_j). Now assume that v_j is of Type 2B or 3A. By a similar argument, Q can be extended to a path P_k with $k \geq 14$ which contains at least 4 edges in L_i and one edge in L_j (if v_j is of Type 2B) or adjacent to w_1^j (if v_j is of

Type 3A). If v_j is of Type 2B, then Q can be extended to a $Y_{k'}$ with $k' \geq 12$ (each contains at least 4 edges in L_i and two edges in L_j). If v_j is of Type 3A, then Q can be extended to maximal $Y_{k'}$ in G with $k' \geq 11$ by including a Y_2 in L_i and one edge in L_j (if v_j is of Type 2B) or adjacent to w_1^j (if v_j is of Type 3A). By Lemma 4.13(ii), $k' = 11$ only if v_j is of 3A and $L_i' \in \{K_{2,3}, S(1,2)\}$ with $A_G(L_i)$ satisfying Lemma 4.13(ii). This proves Lemma 4.14. □

Lemma 4.15. *If $P(10)$ has two Type 1A vertices, then G has both a P_{14} and a Y_{12}.*

Proof. Assume that for $i \neq j$, v_i and v_j are of Type 1A. By Lemma 4.14, if one of L_i' and L_j' is not in $\{C_4, C_5, C_6, K_{2,3}, S(1,2)\}$, then G has a P_{14} and a Y_{12}. Hence we assume that $L_i', L_j' \in \{C_4, C_5, C_6, K_{2,3}, S(1,2)\}$. By Lemma 4.10(v), $P(10)$ has a (v_i, v_k)-Hamilton path Q with $v_jv_k \in E(Q)$. As $L_i', L_j' \in \{C_4, C_5, C_6, K_{2,3}, S(1,2)\}$, the length 8 path $Q[v_i, v_j]$ can be lifted to a P_{15} (including 3 edges in L_i and three edges in L_j) as well as a Y_{12} (including 3 edges in L_i, at least one edge in L_j and the edge v_jv_k) in G. This proves Lemma 4.15. □

Lemma 4.16. *If v_i is of Type 1A and $L_i' \notin \{C_4, C_5, C_6, K_{2,3}, S(1,2)\}$, then either $L(G)$ is hamiltonian, or G has both a P_{14} and a Y_{12}.*

Proof. By Lemma 4.15, we may assume that for any $j \neq i$, v_j is not of Type 1A. By Lemma 4.14, if for some $j \neq i$, v_j is not of Type 3B, G has both a P_{14} and a Y_{12}. Hence we may assume that for any $j \neq i$, v_j is of Type 3B. Let $X = \{e_1^i, e_2^i, e_3^i\}$. Then X is an edge cut of G_0, and $G_0 - X$ has $PI_{G_0}(v_i)$ as a component. Let G_1 denote the other component of $G_0 - X$. Then G_0/G_1 is also a 3-edge-connected graph. By Theorem 4 of [19] (or by Theorem 1.1), either G_0/G_1 has a spanning eulerian subgraph L' or G_0/G_1 has a cycle C' of length at least 9.

If G/G_1 has a spanning eulerian subgraph L', then we may assume that $e_1^i, e_2^i \in E(L')$. Since $G_1 = P(10) - v_i$, by the definition of $P(10)$, $P(10)$ has a cycle L'' of length 9, missing only one vertex of Type 3B. It follows that $G_0[E(L') \cup E(L'')]$ is an eulerian subgraph missing one vertex of Type 3B, and so $G_0[E(L') \cup E(L'')]$ can be lifted to a dominating eulerian subgraph of G. By Theorem 4.1, $L(G)$ is hamiltonian.

Therefore, G_0/G_1 must have a cycle C' of length at least 9. By Lemma 4.10(iv), $P(10)$ has a Y_8 rooted at v_i, which can be lifted to a Y_k (with $k \geq 8$) rooted at w_1^i (say). Since $E(L_i) \cup X = E(G_0/G_1)$ and since $|E(C')| \geq 9$, $G_0[E(L_i) \cup X]$ has a path Q' from w_1^i of length at least 5 (consisting of a path from w_1^i to C' and a path in C'). It follows that $Q' \cup Y_8$ contains both a P_{14} and a Y_{12}. This proves the lemma. □

Lemma 4.17. *Suppose that G does not have a P_k or a Y_{12} as a subgraph, and that $L(G)$ is not hamiltonian. In (i)–(iv) and (iv) below, we assume that $k = 14$. Then each of the following holds.*

(i) *If v_1 is of Type 1A, then for any $i \geq 2$, v_i is not of Type 3B.*

(ii) *For any i, v_i is not of Type 1A.*

(iii) *For any i, v_i is not of Type 3B.*

(iv) *If v_i it of Type 1B, then for any $j \neq i$, v_j cannot be of Type 1B or 2A. Moreover, all Type 2A vertices are independent in G.*

(v) *If $k = 14$, then for any i, v_i is not of Type 1B; and if $k = 13$, then $P(10)$ does not have a vertex of Type 2A.*

(vi) *$P(10)$ does not have two nontrivial edges of G_0.*

Proof. (i) Since v_1 is of Type 1A, by Lemma 4.16, $L_i' = K_1$ for $i \geq 2$; by Lemma 4.15, we have $L_1' \in \{K_1, C_4, C_5, C_6, S(1,2), K_{2,3}\}$ such that if $L_1' \in \{C_4, C_5, C_6\}$, then $A_G(L_i)$ satisfies Remark 4.12(iii) and if $L_1' \in \{S(1,2), K_{2,3}\}$, then $A_G(L_i)$ satisfies Lemma 4.13(ii). Let $X = \{e_1^1, e_2^1, e_3^1\}$ be the three edges in $P(10)$ incident with v_1 such that for $1 \leq j \leq 3$, w_j^1 is incident with e_j^1. If for some $j \geq 2$, v_j is of Type 3B, then by the definition of $P(10)$, $P(10) - v_j$ has a spanning cycle C'. We may assume that $e_1^1, e_2^1 \in E(C')$. Hence by Theorem 2.1(iv), $E(C')$ induces a (w_1^1, w_2^1)-trail in G containing at least one end of every edge in $E(G) - E(PI_G(v_1))$. Since $L_1' \in \{C_4, C_5, C_6, K_1, S(1,2), K_{2,3}\}$, it is routine to verify that L_1' has a (w_1^1, w_2^1)-path Q' such that $E((L_1') - V(Q')) = \emptyset$. It follows that $G_0[E(C') \cup E(Q')]$ can be lifted to a dominating eulerian subgraph of G. By Theorem 4.1, $L(G)$ is hamiltonian, contrary to the assumption of the lemma.

(ii) By contradiction, we may assume that v_1 is of Type 1A. By Lemma 4.14, $L_1' \in \{C_4, C_5, C_6, K_{2,3}, S(1,2)\}$. For any $i \geq 2$, by Lemmas 4.15 and 4.17(i), v_i is not of Types 1A and 3B, and $L_i' = K_1$. Let $v_j \in V(P(10)) - (N_{P(10)(v_1)}) \cup \{v_1\}$. By Lemma 4.10(i), $P(10)$ has a (v_1, v_j)-Hamilton path Q' of length 9. Since v_j is not of Type 3B, Q' can be lifted to a path Q'' from w_1^1 of length at least 10. Since $L_1' \in \{C_4, C_5, C_6, K_{2,3}, S(1,2)\}$, by Remark 4.12(iii) and Lemma 4.14(ii), it is routine to find that L_1' has both a P_4 from w_1^1 and a Y_3 rooted at w_1^1. It follows that $Q' \cup P_4$ and $Q' \cup Y_3$ can be lifted to a P_k ($k \geq 14$) and a $Y_{k'}$ ($k' \geq 12$) in G, contrary to the assumption of Lemma 4.17.

(iii) Suppose that v_1 is of Type 3B. Then $v_1 \in V(G_0)$ and $P(10) - v_1$ has a spanning cycle C'. By Lemma 4.17(ii) and by Theorem 2.1(iv), C' can be lifted to a spanning eulerian subgraph L' of $G_0 - v_1$. By Theorem 4.2(iii), $L(G)$ is hamiltonian, contrary to the assumption of Lemma 4.17.

(iv) By contradiction, we first assume that v_1 is of Type 1B and v_2 is of Type 1B or 2A. By Remark 4.12(iv), L_1 contains a P_k ($k \geq 4$) and a $Y_{k'}$ ($k' \geq 3$) rooted at any vertex of L_1. By Lemma 4.10(i) or (ii), $P(10)$ has a (v_1, v_2)-path of length at least 8, which can be lifted to a P_{14} as well as a Y_{12} of G by including a path of length at least 3 in L_1 and a path of length at least 2 (or a Y_3) in L_2. Hence we cannot have two Type 1B vertices.

Next, we assume that v_1 and v_2 are nonadjacent in $P(10)$ and both of Type 2A. By Lemma 4.10(i), $P(10)$ has a (v_1, v_2)-path Q_1 of length 9, which can be lifted to a P_{14} and a Y_{12} in G by including two edges in L_1 and two edges in L_2.

Remark 4.18. By (ii)–(iv) of Lemma 4.17, we make the following remarks.

(i) $P(10)$ has at most one vertex of Type 1B. Moreover, if $P(10)$ has a vertex of Type 1B, then all other vertices must be of Type 2B or 3A.

(ii) If v_1 if of Type 2A, then any vertex not incident with v_1 must be of Type 2B or 3A. Furthermore, if $v_1v_2 \in E(P(10))$ and both v_1 and v_2 are of Type 2A, any other vertices of $P(10)$ must be of Type 2B or 3A.

(v) We only prove the case when $k = 14$. The case when $k = 13$ is similar and so it will be omitted. Suppose that v_1 is of Type 1B. By Remark 4.18(i), all other vertices must be of Type 2B or 3A. If $P(10)$ has a nontrivial edge e, then by Lemma 4.10(ii), $P(10)$ has a Hamilton (v_1, v_j)-path Q with e not incident with v_j. It follows that Q can be lifted to a (w_1^1, w_1^j)-path Q' with $|E(Q')| \geq 10$, for some $w_1^1 \in A_G(L_1)$ and $w_1^j \in A_G(L_j)$. By Remark 4.12(iv), L_1 has a P_k $(k \geq 4)$ from w_1^1 and a $Y_{k'}$ $(k' \geq 3)$ rooted at w_1^1. Since L_j is of Type 2B or 3A, Q' can be extended to a path of length at least 11. It follows that by including the P_4 or Y_3 in L_1, Q' can be extended to a P_k with $k \geq 14$ and a $Y_{k'}$ with $k' \geq 12$, contrary to the assumption of the lemma. This proves (v).

(vi) Suppose that $P(10)$ has two nontrivial edges e, e' of G_0. By Lemma 4.10(iii), for some vertex v_k, $P(10) - v_k$ has a Hamilton (v_1, v_k)-path Q with $e, e' \in E(Q)$ such that neither e nor e' is incident with a vertex in $\{v_1, v_k\}$. Since Q contains 2 nontrivial edges, it can be lifted to a (w_1^1, w_1^k)-path Q' of length at least 11. By Lemma 4.17(v) and Remark 4.18(ii), v_1 and v_k are of Types 2A, 2B or 3A. It follows that Q can be lifted to a P_k with $k \geq 14$ in G. If $P(10)$ has a vertex (say v_1) of Type 2A or 2B, then by Lemma 4.10(ii), $P(10)$ has a Hamilton (v_1, v_j)-path T containing e and e is not incident with v_j. As $e \in E(T)$ and as v_j is of Types 2A, 2B or 3A, T can be extended to a path from w_1^1 (say) of length at least 11 by including an additional edge incident with a vertex in L_j. Since v_1 is of Type 2A or 2B, L_1 has a Y_2 rooted at w_1^1, and so G has a Y_{12}. Therefore, we have found both a P_{14} and a Y_{12} in G, contrary to the assumption of the lemma. This proves (vi). □

The next lemma can also be verified by similar arguments, whose proofs are then omitted.

Lemma 4.19. *Suppose that $L(G)$ is not hamiltonian. Then each of the following holds.*

(i) *Suppose $P(10)$ has one v_i of Type 2A. Then G has a P_{13} and a Y_{11}. If, in addition, $P(10)$ has a nontrivial edge e, then G has both a Y_{12} and a P_{14}.*

(ii) *Suppose $P(10)$ has one v_i of Type 2B such that $|V(L_i)| \geq 3$, then G has a Y_{11} and a P_{12}; if, in addition, $P(10)$ has a nontrivial edge e, then G has both a Y_{12} and a P_{13}.*

(iii) *Suppose that $P(10)$ has a Type 2A vertex u, and a Type 2A or 2B vertex v. If $uv \notin E(P(10))$, then G has a P_{13} and a Y_{12}, if $uv \in E(P(10))$, then G has a P_{12} and a Y_{11}. If, in addition, $P(10)$ has a nontrivial edge, then G has a P_{14} if $uv \notin E(P(10))$, and a P_{13} and a Y_{12} if $uv \in E(P(10))$.*

(iv) *Suppose that every vertex of $P(10)$ is of Type 2B. If $P(10)$ contains a nontrivial edge, then G has a P_{13}. If, in addition, for some i, $|V(L_i)| \geq 3$, then G has a Y_{12}.*

Proof of Corollaries 4.8 and 4.9. The necessity of both Corollaries 4.8 and 4.9 follows from Lemma 4.7(iv). It remains to prove the sufficiency of the two corollaries. By Theorems 4.4 and 4.5, It suffices to prove the sufficiency of Corollaries 4.8 and 4.9 for line graphs. Let G be a graph such that $L(G)$ is 3-connected and non-hamiltonian, and let G_0 denote the core of G. If $L(G)$ is P_k-free, then G does not have P_{k+1} as a subgraph; If $L(G)$ is Z_k-free, then G does not have Y_{k+2} as a subgraph. By Theorem 4.3, either Theorem 4.3(ii) or Theorem 4.3(iii) must hold.

Case 1. Theorem 4.3(ii) holds and G_0 is contracted to the Petersen graph $P(10)$.

Suppose that $L(G)$ is Z_9-free. Then G does not contain a Y_{11}. By Lemmas 4.17 and 4.19, every vertex of $P(10)$ must be of Type 2B with each $L_i \cong K_2$. Hence $G_0 = P(10)$ and $G = P(10)'$.

Suppose that $L(G)$ is P_{12}-free. Then G does not contain a P_{13}. By Lemmas 4.17 and 4.19, every vertex of $P(10)$ must be of Type 2B. Hence $G_0 = P(10)$ and $G \in \{P(10)', P(10)^{(3)}\}$.

Suppose that $L(G)$ is P_{13}-free and $G \notin \{P(10)', P(10)^{(3)}\}$. Then G does not contain a P_{14}. By Lemmas 4.17 and 4.19, $G_0 = P(10)$ and $G \in \{P(10)'', P(10)^{(4)}, P(10)^{(5)}\}$. This completes the proof for Case 1.

Case 2. Theorem 4.3(iii) holds and $c(G_0) \geq 12$ and G_0 does not have a dominating eulerian subgraph. Let C' be a longest cycle of G_0 with $|E(C')| = c(G_0) \geq 12$. Since G_0 is a contraction of G, there must be an edge subset X such that $G_0 = G/X$. Therefore, there must be a cycle C of G such that $C' = G/(E(C) \cap X)$. Let $C = u_1 \ldots u_h u_1$ for some $h \geq 12$. Since C is not dominating in G, there must be an edge $e = w_1w_2 \in E(G - C)$. Since G is connected, we may assume that $w_1u_1 \in E(G)$, and so G has a path $P_{h+2} = w_2w_1u_1 \ldots u_h$. Since $h \geq 12$, G has a P_{14} and a Y_{11}. This implies that in Corollaries 4.8 and 4.9, this case cannot occur, and so it completes the proof for both corollaries. □

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