Edge-Disjoint Spanning Trees, Edge Connectivity, and Eigenvalues in Graphs

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Abstract: Let $\lambda_2(G)$ and $\tau(G)$ denote the second largest eigenvalue and the maximum number of edge-disjoint spanning trees of a graph G, respectively. Motivated by a question of Seymour on the relationship between eigenvalues of a graph G and bounds of $\tau(G)$, Cioabă and Wong conjectured that for any integers $d, k \geq 2$ and a d-regular graph G, if $\lambda_2(G) < d - \frac{2k-1}{d+1}$, then $\tau(G) \geq k$. They proved the conjecture for $k = 2, 3$, and presented evidence for the cases when $k \geq 4$. Thus the conjecture remains open for $k \geq 4$. We propose a more general conjecture that for a graph G with minimum degree $\delta \geq 2k \geq 4$, if $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$. In this article, we prove that for a graph G with minimum degree δ, each of the following holds.

(i) For $k \in \{2, 3\}$, if $\delta \geq 2k$ and $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.
(ii) For $k \geq 4$, if $\delta \geq 2k$ and $\lambda_2(G) < \delta - \frac{3k-1}{\delta+1}$, then $\tau(G) \geq k$.

Our results sharpen theorems of Cioabă and Wong and give a partial solution to Cioabă and Wong's conjecture and Seymour's problem. We also prove that for a graph G with minimum degree $\delta \geq k \geq 2$, if $\lambda_2(G) < \delta - \frac{2(k-1)}{\delta+1}$, then the edge connectivity is at least k, which generalizes a former result of Cioabă. As corollaries, we investigate the Laplacian and signless Laplacian eigenvalue conditions on $\tau(G)$ and edge connectivity. J. Graph Theory 81: 16–29, 2016

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1. INTRODUCTION

In this article, we consider finite undirected simple graphs. We follow notation of Bondy and Murty [1] for graphs, unless otherwise defined. Thus for a graph G, $c(G)$ denotes the number of components of G, and $\kappa'(G)$ denotes the edge connectivity of G. A graph G is **nontrivial** if $E(G) \neq \emptyset$. For a graph G, we use $\bar{d}(G)$ to denote the average degree of G. Let $U \subseteq V(G)$, $\bar{d}_G(U)$ or simply $\bar{d}(U)$ denotes the average degree of all vertices of U in G. Thus $\bar{d}(G[U])$ and $\bar{d}(U)$ are different. The former means the average degree of the induced subgraph $G[U]$, while the latter is the average degree of all vertices of U in G. For a connected graph G, $\tau(G)$ denotes the maximum number of edge-disjoint spanning trees in G. A survey on $\tau(G)$ can be found in [11]. By definition, $\tau(K_1) = \infty$.

Let G be an undirected graph on n vertices with vertex set $\{v_1, v_2, \cdots, v_n\}$. The **adjacency matrix** of G is an n by n matrix $A(G) = (a_{ij})$ given by a_{ij} equals the number of edges between v_i and v_j for $1 \leq i, j \leq n$. By definition, if G is simple, then $A(G)$ is a symmetric (0, 1)-matrix. Eigenvalues of G are the eigenvalues of $A(G)$. We use $\lambda_i(G)$ to denote the i-th largest eigenvalue of G; and when the graph G is understood from the context, we often use λ_i for $\lambda_i(G)$. With this notation, we always have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Let $A(G)$ be the adjacency matrix of a graph G and $D(G)$ be the diagonal matrix of row sums of $A(G)$ (i.e. the degrees of G), which is the **degree matrix** of G. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the **Laplacian matrix** and the **signless Laplacian matrix** of G, respectively. We use $\mu_i(G)$ and $q_i(G)$ to denote the i-th largest eigenvalue of $L(G)$ and $Q(G)$, respectively. It is not difficult to see that $\mu_n(G) = 0$. The second smallest eigenvalue of $L(G)$, $\mu_{n-1}(G)$, is known as the **algebraic connectivity** of G.

Seymour proposed the following problem on predicting $\tau(G)$ by means of the eigenvalues.

Problem 1. *([4]) Let G be a connected graph. Determine the relationship between $\tau(G)$ and eigenvalues of G.*

Motivated by this problem of Seymour, Cioabă, and Wong proposed the following conjecture.

Conjecture 1.1 (Cioabă and Wong [4]). *Let k and d be two integers with $d \geq 2k \geq 4$. If G is a d-regular graph with $\lambda_2(G) < d - \frac{2k-1}{d+1}$, then $\tau(G) \geq k$.*

A fundamental theorem of Nash-Williams and Tutte characterizes graphs with at least k edge-disjoint spanning trees.

Theorem 1.1 (Nash-Williams [10] and Tutte [13]). *Let G be a connected graph with $E(G) \neq \emptyset$, and let $k > 0$ be an integer. Then $\tau(G) \geq k$ if and only if for any $X \subseteq E(G)$, $|X| \geq k(c(G - X) - 1)$.*

Utilizing Theorem 1.1, Cioabă [3], Cioabă and Wong [4] proved a number of theorems in this direction, settling Conjecture 1.1 for the cases when $k \in \{2, 3\}$ and obtaining partial results towards the conjecture for other values of k.

Theorem 1.2 (Cioabă, Theorem 1.3 in [3]). *Let k and d be two integers with $d \geq k \geq 2$. If G is a d-regular graph with $\lambda_2(G) < d - \frac{2(k-1)}{d+1}$, then $\kappa'(G) \geq k$.*

Theorem 1.3 (Cioabă and Wong, Theorem 1.1 in [4]). *Let d be an integer with $d \geq 4$. If G is a d-regular graph with $\lambda_2(G) < d - \frac{3}{d+1}$, then $\tau(G) \geq 2$.*

Theorem 1.4 (Cioabă and Wong, Theorem 1.2 in [4]). *Let d be an integer with $d \geq 6$. If G is a d-regular graph with $\lambda_2(G) < d - \frac{5}{d+1}$, then $\tau(G) \geq 3$.*

Theorem 1.5 (Cioabă and Wong [4]). *Let k and d be two integers with $d \geq 2k \geq 4$. If G is a d-regular graph with $\lambda_2(G) < d - \frac{2(2k-1)}{d+1}$, then $\tau(G) \geq k$.*

The main purpose of this article is to continue the investigation between eigenvalues of a simple graph (not necessarily regular) and the number of edge-disjoint spanning trees. As suggested by Theorem 1.1, high edge connectivity also implies more edge-disjoint spanning trees packing in a graph (see [7] for an example), we also investigate the relationship between edge connectivity of a simple graph and its second largest eigenvalue. Firstly, we present a more general conjecture, stated below.

Conjecture 1.2. *Let k be an integer with $k \geq 2$ and G be a graph with minimum degree $\delta \geq 2k$. If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.*

The following are the main results in this article. Theorem 1.6 generalizes Theorem 1.2. While Theorems 1.7 (i) and (ii) settle two special cases of Conjecture 1.2, Theorem 1.7 (iii) sheds some light to support Conjecture 1.2. Theorem 1.7 generalizes Theorems 1.3, 1.4, and 1.5, provides further evidence to support Conjectures 1.1 and 1.2, and sharpens Theorem 1.5.

Theorem 1.6. *Let k be an integer with $k \geq 2$ and G be a graph with minimum degree $\delta \geq k$. If $\lambda_2(G) < \delta - \frac{2(k-1)}{\delta+1}$, then $\kappa'(G) \geq k$.*

Theorem 1.7. *Let $k \geq 2$ be an integer, G be a graph with minimum degree δ.*

(i) *If $\delta \geq 4$ and $\lambda_2(G) < \delta - \frac{3}{\delta+1}$, then $\tau(G) \geq 2$.*
(ii) *If $\delta \geq 6$ and $\lambda_2(G) < \delta - \frac{5}{\delta+1}$, then $\tau(G) \geq 3$.*
(iii) *For $k \geq 4$, if $\delta \geq 2k$ and $\lambda_2(G) < \delta - \frac{3k-1}{\delta+1}$, then $\tau(G) \geq k$.*

As applications of Theorem 1.6 and Theorem 1.7, we investigate the relationship between algebraic connectivity, the second largest eigenvalue of signless Laplacian matrix and edge connectivity, the number of edge-disjoint spanning trees of a simple graph.

In Section 2, we display some preliminaries and mechanisms, including eigenvalue interlacing properties and quotient matrices. These will be applied in the proofs of the main results, to be presented in Section 3 and 4. As corollaries, Laplacian and signless Laplacian eigenvalue conditions on $\tau(G)$ and edge connectivity are presented in the last section.

2. PRELIMINARIES

In this section, we present some of the preliminaries and former results to be used in our arguments. Throughout this section, G always denotes a simple graph.

Let $\mathbb{E}^n = \{(x_1, x_2, \cdots, x_n)^T \mid \sum_{i=1}^n x_i = 1 \text{ and } x_i \geq 0 \text{ for } i = 1, 2, \cdots, n\}$.

Theorem 2.1 (Page 17 in [9]). *Let A be an irreducible nonnegative $n \times n$ matrix with the largest eigenvalue λ_1. Then*

$$\lambda_1 = \min_{x \in \mathbb{E}^n} \left\{ \max_{x_i \neq 0} \frac{(Ax)_i}{x_i} \right\} = \max_{x \in \mathbb{E}^n} \left\{ \min_{x_i \neq 0} \frac{(Ax)_i}{x_i} \right\}.$$

Theorem 2.2 (Proposition 2.2 in [2]). *Let G be a graph with largest eigenvalue λ_1, maximum degree Δ and average degree $\bar{d}$. Then $\bar{d} \leq \lambda_1 \leq \Delta$.*

Given two real sequences $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $n > m$, the second sequence is said to **interlace** the first one if $\theta_i \geq \eta_i \geq \theta_{n-m+i}$, for $i = 1, 2, \cdots, m$. When we say the eigenvalues of a matrix B interlace the eigenvalues of a matrix A, it means the non-increasing eigenvalue sequence of B interlaces that of A. The following interlace results are well-known, and can be found in many textbooks.

Theorem 2.3 (Cauchy Interlacing). *Let A be a real symmetric matrix and B be a principal submatrix of A. Then the eigenvalues of B interlace the eigenvalues of A.*

Corollary 2.4 ([8]). *If H is an induced subgraph of G, then the eigenvalues of H interlace the eigenvalues of G.*

Let S and T be disjoint subsets of $V(G)$. We denote by $E(S, T)$ the set of edges each of which has one vertex in S and the other vertex in T and let $e(S, T) = |E(S, T)|$. The next useful lemma follows immediately from Theorem 2.2 and Corollary 2.4.

Lemma 2.5 ([4]). *Let S and T be disjoint subsets of $V(G)$ and $e(S, T) = 0$. Then*

$$\lambda_2(G) \geq \lambda_2(G[S \cup T]) \geq \min\{\lambda_1(G[S]), \lambda_1(G[T])\} \geq \min\{\bar{d}(G[S]), \bar{d}(G[T])\},$$

where $\bar{d}$ denotes the average degree of a graph.

Suppose that we partition $V(G)$ into s nonempty subsets $V_1, V_2, \cdots, V_s$. We denote this partition by π. The **quotient matrix** $A_\pi(G) = A(V_1, V_2, \cdots, V_s)$ of G with respect to π, is an s by s matrix (b_{ij}) such that b_{ij} is the average number of neighbors in V_j of the vertices in V_i for $1 \leq i, j \leq s$. If the partition π is not specified, we often use A_s to denote the quotient matrix. As A_s is an s by s square real matrix, the following is well known from linear algebra (for example, see Page 289 in [12]).

$$\lambda_1(A_s) + \lambda_2(A_s) + \cdots\cdots + \lambda_s(A_s) = tr(A_s). \tag{1}$$

We denote the average degree of V_i by $\overline{d}_i$ for $1 \le i \le s$. By the definition of the quotient matrix, the sum of all entries in the i-th row is exactly $\overline{d}_i$. Let $\Delta_\pi(G) = \max_{1\le i\le s}\{\overline{d}_i\}$ and $\delta_\pi(G) = \min_{1\le i\le s}\{\overline{d}_i\}$. The following theorem is an analogue of Theorem 2.2.

Theorem 2.6. *Let G be a connected graph and π be a partition of $V(G)$. Then*

$$\delta_\pi \le \lambda_1(A_\pi) \le \Delta_\pi.$$

Proof. Suppose that the partition π has s parts. Let $x = (\frac{1}{s}, \frac{1}{s}, \cdots, \frac{1}{s})^T \in \mathbb{E}^s$. By Theorem 2.1,

$$\lambda_1(A_\pi) \le \max_{1\le i\le s} \frac{(Ax)_i}{x_i} = \max_{1\le i\le s} \frac{\frac{1}{s}\cdot \overline{d}_i}{\frac{1}{s}} = \max_{1\le i\le s} \overline{d}_i = \Delta_\pi.$$

Similarly, by Theorem 2.1,

$$\lambda_1(A_\pi) \ge \min_{1\le i\le s} \frac{(Ax)_i}{x_i} = \min_{1\le i\le s} \frac{\frac{1}{s}\cdot \overline{d}_i}{\frac{1}{s}} = \min_{1\le i\le s} \overline{d}_i = \delta_\pi.$$ ■

Theorem 2.7 (Corollary 2.3 in [8]. See also [2], [5])**.** *Let G be a graph. The eigenvalues of any quotient matrix of G interlace the eigenvalues of G.*

Lemma 2.8. *Let G be a graph with minimum degree δ and U be a non-empty proper subset of $V(G)$. If $e(U, V\backslash U) \le \delta - 1$, then $|U| \ge \delta + 1$.*

Proof. We argue by contradiction and assume that $|U| \le \delta$. Then $|U|(|U|-1) + e(U, V\backslash U) \ge |U|\delta$ by counting the total degrees of vertices in U. But $|U|(|U|-1) + e(U, V\backslash U) \le \delta(|U|-1) + (\delta - 1) \le |U|\delta - 1$, contrary to the fact that $|U|(|U|-1) + e(U, V\backslash U) \ge |U|\delta$. Thus $|U| \ge \delta + 1$. ■

3. EIGENVALUES AND EDGE CONNECTIVITY IN GRAPHS

In this section, we present the proof of Theorem 1.6.

Proof of Theorem 1.6. We argue by contradiction and assume that $\kappa'(G) \le k - 1$. Then there exists a non-empty proper subset $V_1 \subseteq V(G)$ such that $e(V_1, V\backslash V_1) \le k - 1$. Let $r = e(V_1, V\backslash V_1)$ and $V_2 = V\backslash V_1$. By Lemma 2.8, $|V_1| \ge \delta + 1$ and $|V_2| \ge \delta + 1$. The quotient matrix of G with respect to the partition (V_1, V_2) is

$$A_2 = \begin{bmatrix} \bar{d}_1 - \frac{r}{|V_1|} & \frac{r}{|V_1|} \\ \frac{r}{|V_2|} & \bar{d}_2 - \frac{r}{|V_2|} \end{bmatrix},$$

where $\bar{d}_i$ denotes the average degree of V_i in G for $i = 1, 2$. By (1), $\lambda_2(A_2) = tr(A_2) - \lambda_1(A_2)$. By Theorem 2.6, $\lambda_1(A_2) \le \max\{\bar{d}_1, \bar{d}_2\}$ and by Theorem 2.7, $\lambda_2(A_2) \le \lambda_2(G)$. Thus $\lambda_2(G) \ge \lambda_2(A_2) \ge tr(A_2) - \max\{\bar{d}_1, \bar{d}_2\}$, which implies that

$$\lambda_2(G) \ge tr(A_2) - \max\{\bar{d}_1, \bar{d}_2\} = \bar{d}_1 + \bar{d}_2 - \left(\frac{r}{|V_1|} + \frac{r}{|V_2|}\right) - \max\{\bar{d}_1, \bar{d}_2\} \ge \delta - \frac{2(k-1)}{\delta+1},$$

contrary to the fact that $\lambda_2(G) < \delta - \frac{2(k-1)}{\delta+1}$. This completes the proof of the theorem. ■

4. EIGENVALUES AND EDGE-DISJOINT SPANNING TREES

The proof for Theorem 1.7 will be given in this section. We shall argue by contradiction and assume that $\tau(G) \leq k-1$. By Theorem 1.1, there exists an edge subset $X \subseteq E(G)$ such that $|X| \leq k(c(G-X)-1)-1$. Let $c(G-X)=t$ and $G_1, G_2, \ldots, G_t$ be the components of $G-X$. For $1 \leq i \leq t$, let $V_i = V(G_i)$, $E_i = E(G_i)$, and $r_i = e(V_i, V \backslash V_i)$. Without lose of generality, we always assume that

$$r_1 \leq r_2 \leq \cdots \leq r_t. \tag{2}$$

With these notations and by $|X| \leq k(c(G-X)-1)-1$, we have

$$\sum_{1 \leq i < j \leq t} e(V_i, V_j) \leq k(t-1)-1 = kt-k-1. \tag{3}$$

Claim 4.1. *For $k \geq 2$, if $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then there exist no indices p and q with $1 \leq p \neq q \leq t$ such that $e(V_p, V_q) = 0$ and $r_p, r_q \leq 2k-1$.*

Proof of Claim 1. We argue by contradiction. By Lemma 2.8, $|V_p| \geq \delta+1$ and $|V_q| \geq \delta+1$. It follows that $\bar{d}(G[V_p]) \geq \delta - \frac{2k-1}{|V_p|} \geq \delta - \frac{2k-1}{\delta+1}$ and $\bar{d}(G[V_q]) \geq \delta - \frac{2k-1}{|V_q|} \geq \delta - \frac{2k-1}{\delta+1}$. By Lemma 2.5, $\lambda_2(G) \geq \min\{\bar{d}(G[V_p]), \bar{d}(G[V_q])\} \geq \delta - \frac{2k-1}{\delta+1}$, contrary to the assumption that $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$. Thus the proof for Claim 4.1 is done.

Claim 4.2. *For $k \geq 2$, if $\delta \geq 2k$ and if $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then for any i with $1 \leq i \leq t$, $r_i \geq k$.*

Proof of Claim 2. We argue by contradiction and assume that for some i, $r_i < k$. Then $\kappa'(G) < k$. By Theorem 1.6, $\lambda_2(G) \geq \delta - \frac{2(k-1)}{\delta+1}$, contrary to the assumption that $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$. Therefore, we must have $r_i \geq k$. This proves Claim 2.

4.1 The Case when $k = 2$

In this subsection, we shall prove Theorem 1.7(i). By (3) with $k=2$, we have

$$\sum_{i=1}^{t} r_i = 2 \sum_{1 \leq i < j \leq t} e(V_i, V_j) \leq 4t-6.$$

Let x_l denote the multiplicity of l in $\{r_1, r_2, \cdots, r_t\}$ for $l = 1, 2, 3$. By Claim 2, $r_t \geq \cdots \geq r_2 \geq r_1 \geq 2$. Thus $x_1 = 0$. It follows by (3) with $k=2$ that

$$2x_2 + 3x_3 + 4(t - x_2 - x_3) \leq \sum_{i=1}^{t} r_i \leq 4t-6,$$

which implies that $2x_2 + x_3 \geq 6$. Thus if $x_2 = 0$, then $x_3 \geq 6$; and if $x_2 = 1$, then $x_3 \geq 4$. It follows that when $0 \leq x_2 \leq 1$, there always exist p and q with $1 \leq p \neq q \leq t$ such that $e(V_p, V_q) = 0$ and $r_p \leq 3$ and $r_q = 3$. But such indices p and q are forbidden by Claim 4.1, a contradiction.

Hence we must have $x_2 \geq 2$, and so we may assume, by (2), that $r_1, r_2 = 2$ and $2 \leq r_3 \leq 3$. Let $V' = V \backslash (V_1 \cup V_2)$. Then $V_3 \subseteq V'$. By Lemma 2.8, $|V_i| \geq \delta+1$ for $i =$

$1, 2, 3$, and so $|V'| \geq |V_3| \geq \delta + 1$. The quotient matrix of G with respect to the partition (V_1, V_2, V') is

$$A_3 = \begin{bmatrix} \bar{d_1} - \frac{2}{|V_1|} & \frac{1}{|V_1|} & \frac{1}{|V_1|} \\ \frac{1}{|V_2|} & \bar{d_2} - \frac{2}{|V_2|} & \frac{1}{|V_2|} \\ \frac{1}{|V'|} & \frac{1}{|V'|} & \bar{d'} - \frac{2}{|V'|} \end{bmatrix},$$

where $\bar{d_i}$ denotes the average degree of V_i in G for $i = 1, 2$ and $\bar{d'}$ denotes the average degree of V' in G.

By (1), $\lambda_2(A_3) + \lambda_3(A_3) = tr(A_3) - \lambda_1(A_3)$. By Theorem 2.7, $\lambda_2(G) \geq \lambda_2(A_3)$, $\lambda_3(G) \geq \lambda_3(A_3)$ and by Theorem 2.6, $\lambda_1(A_3) \leq \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. Thus $\lambda_2(G) + \lambda_3(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. As $\lambda_2(G) \geq \lambda_3(G)$, we have

$$2\lambda_2(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} = \bar{d_1} + \bar{d_2} + \bar{d'} - \left(\frac{2}{|V_1|} + \frac{2}{|V_2|} + \frac{2}{|V'|}\right)$$
$$- \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} \geq 2\delta - \frac{6}{\delta + 1},$$

contrary to the assumption in Theorem 1.7 (i) that $\lambda_2(G) < \delta - \frac{3}{\delta+1}$. This completes the proof of Theorem 1.7 (i).

4.2 The Case when $k = 3$

In this subsection, we shall prove Theorem 1.7(ii). By (3) with $k = 3$, we have

$$\sum_{i=1}^{t} r_i = 2 \sum_{1 \leq i < j \leq t} e(V_i, V_j) \leq 6t - 8.$$

Let x_l denote the multiplicity of l in $\{r_1, r_2, \cdots, r_t\}$ for $1 \leq l \leq 5$. By Claim 2, $r_t \geq \cdots \geq r_2 \geq r_1 \geq 3$. Thus $x_1 = x_2 = 0$. It follows that

$$3x_3 + 4x_4 + 5x_5 + 6(t - x_3 - x_4 - x_5) \leq \sum_{i=1}^{t} r_i \leq 6t - 8,$$

which implies that

$$3x_3 + 2x_4 + x_5 \geq 8. \tag{4}$$

Case 4.1. $x_3 \geq 2$.

Then by (2), $r_1 = r_2 = 3$ and $r_3 \leq 5$. Let $V' = V \backslash (V_1 \cup V_2)$. Then it must be the case that

(i) there is exactly one edge between V_1 and V_2;
(ii) there are exactly two edges between V_i and V' for each $i = 1, 2$.

Thus the structure is unique. By Lemma 2.8, $|V_i| \geq \delta + 1$ for $i = 1, 2, 3$. Then $|V'| \geq |V_3| \geq \delta + 1$. The quotient matrix of G with respect to the partition (V_1, V_2, V') is

$$A_3 = \begin{bmatrix} \bar{d_1} - \frac{3}{|V_1|} & \frac{1}{|V_1|} & \frac{2}{|V_1|} \\ \frac{1}{|V_2|} & \bar{d_2} - \frac{3}{|V_2|} & \frac{2}{|V_2|} \\ \frac{2}{|V'|} & \frac{2}{|V'|} & \bar{d'} - \frac{4}{|V'|} \end{bmatrix},$$

where $\bar{d_i}$ denotes the average degree of V_i in G for $i = 1, 2$ and $\bar{d'}$ denotes the average degree of V' in G.

By (1), $\lambda_2(A_3) + \lambda_3(A_3) = tr(A_3) - \lambda_1(A_3)$. By Theorem 2.7, $\lambda_2(G) \geq \lambda_2(A_3)$, $\lambda_3(G) \geq \lambda_3(A_3)$ and by Theorem 2.6, $\lambda_1(A_3) \leq \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. Thus $\lambda_2(G) + \lambda_3(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. As $\lambda_2(G) \geq \lambda_3(G)$, we have

$$\begin{aligned} 2\lambda_2(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} &= \bar{d_1} + \bar{d_2} + \bar{d'} - \left(\frac{3}{|V_1|} + \frac{3}{|V_2|} + \frac{4}{|V'|}\right) \\ &- \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} \geq 2\delta - \frac{10}{\delta+1}, \end{aligned}$$

contrary to the assumption in Theorem 1.7 (ii) that $\lambda_2(G) < \delta - \frac{5}{\delta+1}$.

Case 4.2. $x_3 = 1$.

By (4), $2x_4 + x_5 \geq 5$. If $x_4 = 0$, then $x_5 \geq 5$, and so there exist p and q with $1 \leq p \neq q \leq t$ such that $e(V_p, V_q) = 0$ and $r_p = 3$ and $r_q = 5$. This is prohibited by Claim 4.1. Therefore we must have $x_4 \geq 1$, and so by (2), $r_1 = 3$, $r_2 = 4$, and $r_3, r_4 \leq 5$. By Lemma 2.8, $|V_i| \geq \delta + 1$ for $i = 1, 2, 3, 4$. Let $V' = V \backslash (V_1 \cup V_2)$. Thus $V_3, V_4 \subseteq V'$, whence $|V'| \geq 2(\delta + 1)$. The quotient matrix of G with respect to the partition (V_1, V_2, V') is

$$A_3 = \begin{bmatrix} \bar{d_1} - \frac{3}{|V_1|} & \frac{1}{|V_1|} & \frac{2}{|V_1|} \\ \frac{1}{|V_2|} & \bar{d_2} - \frac{4}{|V_2|} & \frac{3}{|V_2|} \\ \frac{2}{|V'|} & \frac{3}{|V'|} & \bar{d'} - \frac{5}{|V'|} \end{bmatrix},$$

where $\bar{d_i}$ denotes the average degree of V_i in G for $i = 1, 2$ and $\bar{d'}$ denotes the average degree of V' in G.

By (1), $\lambda_2(A_3) + \lambda_3(A_3) = tr(A_3) - \lambda_1(A_3)$. By Theorem 2.7, $\lambda_2(G) \geq \lambda_2(A_3)$, $\lambda_3(G) \geq \lambda_3(A_3)$ and by Theorem 2.6, $\lambda_1(A_3) \leq \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. Thus $\lambda_2(G) + \lambda_3(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. As $\lambda_2(G) \geq \lambda_3(G)$, we have

$$\begin{aligned} 2\lambda_2(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} &= \bar{d_1} + \bar{d_2} + \bar{d'} - \left(\frac{3}{|V_1|} + \frac{4}{|V_2|} + \frac{5}{|V'|}\right) \\ &- \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} \geq 2\delta - \frac{19/2}{\delta+1}, \end{aligned}$$

contrary to the assumption in Theorem 1.7 (ii) that $\lambda_2(G) < \delta - \frac{5}{\delta+1}$.

Case 4.3. $x_3 = 0$.

By (4), $2x_4 + x_5 \geq 8$. If $x_4 < 2$, then either $x_4 = 1$ and $x_5 \geq 6$, or $x_4 = 0$ and $x_5 \geq 8$. In either case, there exist p and q with $1 \leq p \neq q \leq t$ such that $e(V_p, V_q) = 0$ and $r_p, r_q \leq 5$, violating Claim 4.1. Hence, by (2), we may assume that $r_1 = r_2 = 4$. Since $2x_4 + x_5 \geq 8$, $r_3, r_4 \leq 5$.

Subcase 3.1: $t = 4$. Then (V_1, V_2, V_3, V_4) is a partition of $V(G)$. As $2x_4 + x_5 \geq 8$, we must have $x_4 = 4$ and $x_5 = 0$. Thus $r_i = 4$ for $i = 1, 2, 3, 4$. By Claim 1, and since $r_1 = 4$, there exists V_j (say $j = 2$) such that $e(V_1, V_j) = 2$. Let $V' = V \backslash (V_1 \cup V_2)$. Then

$|V'| = |V_3| + |V_4| \geq 2(\delta + 1)$. The quotient matrix of G with respect to the partition (V_1, V_2, V') is

$$A_3 = \begin{bmatrix} \bar{d_1} - \frac{4}{|V_1|} & \frac{2}{|V_1|} & \frac{2}{|V_1|} \\ \frac{2}{|V_2|} & \bar{d_2} - \frac{4}{|V_2|} & \frac{2}{|V_2|} \\ \frac{2}{|V'|} & \frac{2}{|V'|} & \bar{d'} - \frac{4}{|V'|} \end{bmatrix},$$

where $\bar{d_i}$ denotes the average degree of V_i in G for $i = 1, 2$ and $\bar{d'}$ denotes the average degree of V' in G.

By (1), $\lambda_2(A_3) + \lambda_3(A_3) = tr(A_3) - \lambda_1(A_3)$. By Theorem 2.7, $\lambda_2(G) \geq \lambda_2(A_3)$, $\lambda_3(G) \geq \lambda_3(A_3)$ and by Theorem 2.6, $\lambda_1(A_3) \leq \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. Thus $\lambda_2(G) + \lambda_3(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. As $\lambda_2(G) \geq \lambda_3(G)$, we have

$$2\lambda_2(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} = \bar{d_1} + \bar{d_2} + \bar{d'} - \left(\frac{4}{|V_1|} + \frac{4}{|V_2|} + \frac{4}{|V'|}\right) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} \geq 2\delta - \frac{10}{\delta + 1},$$

contrary to the assumption in Theorem 1.7 (ii) that $\lambda_2(G) < \delta - \frac{5}{\delta+1}$.

Subcase 3.2: $t \geq 5$.

Subcase 3.2.1: $r_5 \leq 5$. Let $V' = V\backslash(V_1 \cup V_2)$. By Lemma 2.8, $|V_i| \geq \delta + 1$ for $i = 1, 2, 3, 4, 5$. Then $|V'| \geq |V_3| + |V_4| + |V_5| \geq 3(\delta + 1)$. The quotient matrix of G with respect to the partition (V_1, V_2, V') is

$$A_3 = \begin{bmatrix} \bar{d_1} - \frac{4}{|V_1|} & \frac{1}{|V_1|} & \frac{3}{|V_1|} \\ \frac{1}{|V_2|} & \bar{d_2} - \frac{4}{|V_2|} & \frac{3}{|V_2|} \\ \frac{3}{|V'|} & \frac{3}{|V'|} & \bar{d'} - \frac{6}{|V'|} \end{bmatrix},$$

where $\bar{d_i}$ denotes the average degree of V_i in G for $i = 1, 2$ and $\bar{d'}$ denotes the average degree of V' in G.

By (1), $\lambda_2(A_3) + \lambda_3(A_3) = tr(A_3) - \lambda_1(A_3)$. By Theorem 2.7, $\lambda_2(G) \geq \lambda_2(A_3)$, $\lambda_3(G) \geq \lambda_3(A_3)$ and by Theorem 2.6, $\lambda_1(A_3) \leq \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. Thus $\lambda_2(G) + \lambda_3(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. As $\lambda_2(G) \geq \lambda_3(G)$, we have

$$2\lambda_2(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} = \bar{d_1} + \bar{d_2} + \bar{d'} - \left(\frac{4}{|V_1|} + \frac{4}{|V_2|} + \frac{6}{|V'|}\right) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} \geq 2\delta - \frac{10}{\delta + 1},$$

contrary to the assumption in Theorem 1.7 (ii) that $\lambda_2(G) < \delta - \frac{5}{\delta+1}$.

Subcase 3.2.2: $r_5 > 5$. As $2x_4 + x_5 \geq 8$, we must have $r_i = 4$ for $i = 1, 2, 3, 4$. Let $V'' = V \setminus (V_1 \cup V_2 \cup V_3 \cup V_4)$, and so $(V_1, V_2, V_3, V_4, V'')$ is a partition of $V(G)$. By Claim 1, $e(V_i, V_j) \geq 1$ for $1 \leq i, j \leq 4$. Since $r_i = 4$ for $i = 1, 2, 3, 4.$, we must have $e(V_i, V'') \leq 1$ for $i = 1, 2, 3, 4$. Thus $e(V'', V\backslash V'') = \sum_{i=1}^{4} e(V_i, V'') \leq 4 \leq \delta - 1$. By Lemma 2.8, $|V''| \geq \delta + 1$. Let $V' = V\backslash(V_1 \cup V_2)$. Then $|V'| = |V_3| + |V_4| + |V''| \geq$

$3(\delta+1)$. Let $e(V_1, V_2) = y$. Then $y \geq 1$. The quotient matrix of G with respect to the partition (V_1, V_2, V') is

$$A_3 = \begin{bmatrix} \bar{d_1} - \frac{4}{|V_1|} & \frac{y}{|V_1|} & \frac{4-y}{|V_1|} \\ \frac{y}{|V_2|} & \bar{d_2} - \frac{4}{|V_2|} & \frac{4-y}{|V_2|} \\ \frac{4-y}{|V'|} & \frac{4-y}{|V'|} & \bar{d'} - \frac{2(4-y)}{|V'|} \end{bmatrix},$$

where $\bar{d_i}$ denotes the average degree of V_i in G for $i = 1, 2$ and $\bar{d'}$ denotes the average degree of V' in G.

By (1), $\lambda_2(A_3) + \lambda_3(A_3) = tr(A_3) - \lambda_1(A_3)$. By Theorem 2.7, $\lambda_2(G) \geq \lambda_2(A_3)$, $\lambda_3(G) \geq \lambda_3(A_3)$ and by Theorem 2.6, $\lambda_1(A_3) \leq \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. Thus $\lambda_2(G) + \lambda_3(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. As $\lambda_2(G) \geq \lambda_3(G)$, we have

$$\begin{aligned} 2\lambda_2(G) &\geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} \\ &= \bar{d_1} + \bar{d_2} + \bar{d'} - \left(\frac{4}{|V_1|} + \frac{4}{|V_2|} + \frac{2(4-y)}{|V'|}\right) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} \\ &\geq 2\delta - \frac{10}{\delta+1}, \end{aligned}$$

contrary to the assumption in Theorem 1.7 (ii) that $\lambda_2(G) < \delta - \frac{5}{\delta+1}$. This completes the proof.

4.3 The Case when $k \geq 4$

In this subsection, we shall prove Theorem 1.7(iii). Let x_l denote the multiplicity of l in $\{r_1, r_2, \cdots, r_t\}$ for $1 \leq l \leq 2k-1$. By Claim 2, $r_t \geq \cdots \geq r_2 \geq r_1 \geq k$. Thus $x_j = 0$ for $j = 1, 2, \cdots, k-1$. By (3), we have

$$\begin{aligned} kx_k \ &+ (k+1)x_{k+1} + \cdots + (2k-1)x_{2k-1} + 2k(t - (x_k + x_{k+1} + \cdots + x_{2k-1})) \\ &\leq \sum_{i=1}^{t} r_i \leq 2kt - 2(k+1), \end{aligned}$$

which implies that

$$kx_k + (k-1)x_{k+1} + \cdots + 2x_{2k-2} + x_{2k-1} \geq 2(k+1).$$

Let h be the smallest index such that $x_h \neq 0$. Then we have

$$(2k-h)x_h + (2k-h-1)x_{h+1} + \cdots + 2x_{2k-2} + x_{2k-1} \geq 2(k+1). \tag{5}$$

Since $h \geq k$, we have $2(k+1) > 2(2k-h)$.

Case 4.4. $x_h \geq 2$.

Since $2(k+1) > 2(2k-h)$, there exists an integer $b \geq 3$ such that $(b-1)(2k-h) < 2(k+1) \leq b(2k-h)$. By $2(k+1) \leq b(2k-h)$, we have $h \leq \frac{(2b-2)k-2}{b}$. It follows by $(b-1)(2k-h) < 2(k+1)$ and by (5) that $x_h + x_{h+1} + \cdots + x_{2k-2} + x_{2k-1} \geq b$, and so by (2), we have $r_1 \leq r_2 \leq \cdots \leq r_b \leq 2k-1$. By Lemma 2.8, $|V_i| \geq \delta + 1$ with $1 \leq i \leq b$.

Let $V' = V\backslash(V_1 \cup V_2)$. Then $|V'| \geq |V_3| + \cdots + |V_b| \geq (b-2)(\delta+1)$. Let $e(V_1, V_2) = y$. Then $y \geq 1$. The quotient matrix of G with respect to the partition (V_1, V_2, V') is

$$A_3 = \begin{bmatrix} \bar{d_1} - \frac{h}{|V_1|} & \frac{y}{|V_1|} & \frac{h-y}{|V_1|} \\ \frac{y}{|V_2|} & \bar{d_2} - \frac{h}{|V_2|} & \frac{h-y}{|V_2|} \\ \frac{h-y}{|V'|} & \frac{h-y}{|V'|} & \bar{d'} - \frac{2(h-y)}{|V'|} \end{bmatrix},$$

where $\bar{d_i}$ denotes the average degree of V_i in G for $i = 1, 2$ and $\bar{d'}$ denotes the average degree of V' in G.

By (1), $\lambda_2(A_3) + \lambda_3(A_3) = tr(A_3) - \lambda_1(A_3)$. By Theorem 2.7, $\lambda_2(G) \geq \lambda_2(A_3)$, $\lambda_3(G) \geq \lambda_3(A_3)$ and by Theorem 2.6, $\lambda_1(A_3) \leq \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. Thus $\lambda_2(G) + \lambda_3(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. As $\lambda_2(G) \geq \lambda_3(G)$, we have

$$\begin{aligned} 2\lambda_2(G) &\geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} \\ &= \bar{d_1} + \bar{d_2} + \bar{d'} - \left(\frac{h}{|V_1|} + \frac{h}{|V_2|} + \frac{2(h-y)}{|V'|}\right) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\} \\ &\geq 2\delta - \frac{2(\frac{b-1}{b-2}h - \frac{y}{b-2})}{\delta+1} \\ &\geq 2\left(\delta - \frac{\frac{2(b-1)^2}{b(b-2)}k - \frac{3b-2}{b(b-2)}}{\delta+1}\right) > 2\left(\delta - \frac{3k-1}{\delta+1}\right), \end{aligned} \tag{6}$$

contrary to the assumption in Theorem 1.7 (iii) that $\lambda_2(G) < \delta - \frac{3k-1}{\delta+1}$. (See Appendix A for the proof of (6)). This proves Case 1.

Case 4.5. $x_h = 1$.

Then (5) becomes $(2k-h-1)x_{h+1} + \cdots + 2x_{2k-2} + x_{2k-1} \geq 2(k+1) - (2k-h) = h+2 \geq k+2$. Let h' be the smallest index such that $x_{h'} > 0$ with $h' > h$. Then

$$(2k-h')x_{h'} + \cdots + 2x_{2k-2} + x_{2k-1} \geq h+2 \geq k+2. \tag{7}$$

As $h' \geq h \geq k$, we have $h'+2 > k$ and so $k+2 > 2k-h'$. Thus there must be an integer $b' \geq 2$ such that $(b'-1)(2k-h') < k+2 \leq b'(2k-h')$. It follows by $k+2 \leq b'(2k-h')$ that $h' \leq \frac{(2b'-1)k-2}{b'}$. By $(b'-1)(2k-h') < k+2$ and by (7), we have $x_{h'} + \cdots + x_{2k-2} + x_{2k-1} \geq b'$, and so by (2), $r_1 \leq r_2 \leq \cdots \leq r_{b'} \leq r_{b'+1} \leq 2k-1$. By Lemma 2.8, $|V_i| \geq \delta+1$ for $i = 1, 2, \cdots, b'+1$. Let $V' = V\backslash(V_1 \cup V_2)$. Then $|V'| \geq |V_3| + \cdots + |V_{b'+1}| \geq (b'-1)(\delta+1)$. Let $e(V_1, V_2) = y$. Then $y \geq 1$. The quotient matrix of G with respect to the partition (V_1, V_2, V') is

$$A_3 = \begin{bmatrix} \bar{d_1} - \frac{h}{|V_1|} & \frac{y}{|V_1|} & \frac{h-y}{|V_1|} \\ \frac{y}{|V_2|} & \bar{d_2} - \frac{h'}{|V_2|} & \frac{h'-y}{|V_2|} \\ \frac{h-y}{|V'|} & \frac{h'-y}{|V'|} & \bar{d'} - \frac{h+h'-2y}{|V'|} \end{bmatrix},$$

where $\bar{d_i}$ denotes the average degree of V_i in G for $i = 1, 2$ and $\bar{d'}$ denotes the average degree of V' in G.

By (1), $\lambda_2(A_3) + \lambda_3(A_3) = tr(A_3) - \lambda_1(A_3)$. By Theorem 2.7, $\lambda_2(G) \geq \lambda_2(A_3)$, $\lambda_3(G) \geq \lambda_3(A_3)$ and by Theorem 2.6, $\lambda_1(A_3) \leq \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. Thus $\lambda_2(G) + \lambda_3(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$. As $\lambda_2(G) \geq \lambda_3(G)$, we have

$$2\lambda_2(G) \geq tr(A_3) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$$

$$= \bar{d_1} + \bar{d_2} + \bar{d'} - \left(\frac{h}{|V_1|} + \frac{h'}{|V_2|} + \frac{h+h'-2y}{|V'|} \right) - \max\{\bar{d_1}, \bar{d_2}, \bar{d'}\}$$
$$\geq 2\delta - \frac{b'h + b'h' - 2y}{(b'-1)(\delta+1)} \geq 2\delta - \frac{2(b'h'-y)}{(b'-1)(\delta+1)}$$
$$\geq 2\left(\delta - \frac{\frac{2b'-1}{b'-1}k - \frac{3}{b'-1}}{\delta+1}\right) > 2\left(\delta - \frac{3k-1}{\delta+1}\right), \tag{8}$$

contrary to the assumption in Theorem 1.7 (iii) that $\lambda_2(G) < \delta - \frac{3k-1}{\delta+1}$. (See Appendix B for the proof of (8)). This completes the proof.

5. LAPLACIAN AND SIGNLESS LAPLACIAN EIGENVALUE CONDITIONS

In this section, we will investigate the relationship between $\mu_{n-1}(G)$, $q_2(G)$ and $\tau(G)$, $\kappa'(G)$ of a simple graph G. Theorem 5.3 and 5.4 are main results, which are analogues of Theorem 1.6 and Theorem 1.7. We present a useful theorem first.

Theorem 5.1 (Weyl Inequalities). *Let B and C be Hermitian matrices of order n, and let $1 \leq i, j \leq n$. Then*

(i) $\lambda_i(B) + \lambda_j(C) \leq \lambda_{i+j-n}(B+C)$ *if* $i + j \geq n+1$.
(ii) $\lambda_i(B) + \lambda_j(C) \geq \lambda_{i+j-1}(B+C)$ *if* $i + j \leq n+1$.

Corollary 5.2. *Let δ, Δ, λ_2, μ_{n-1}, and q_2 be the minimum degree, maximum degree, second largest eigenvalue, second smallest Laplacian eigenvalue and second largest signless Laplacian eigenvalue of a graph G. Then*

(i) $\mu_{n-1} + \lambda_2 \leq \Delta$.
(ii) $\delta + \lambda_2 \leq q_2$.

Proof. Let A, D, L, Q be the adjacency matrix, diagonal matrix, Laplacian matrix and signless Laplacian matrix.

(i) : Since $L = D - A$, we have $D = L + A$. By Theorem 5.1 (i), $\lambda_{n-1}(L) + \lambda_2(A) \leq \lambda_1(D)$. Thus $\mu_{n-1} + \lambda_2 \leq \Delta$.
(ii) : Since $Q = D + A$, by Theorem 5.1 (i), $\lambda_n(D) + \lambda_2(A) \leq \lambda_2(Q)$. Thus $\delta + \lambda_2 \leq q_2$. ■

Theorem 5.3. *Let $k \geq 2$ be an integer, G be a graph with minimum degree δ.*

(1) *(i) If $\delta \geq 4$ and $\mu_{n-1}(G) > \Delta - \delta + \frac{3}{\delta+1}$, then $\tau(G) \geq 2$.*
(ii) *If $\delta \geq 6$ and $\mu_{n-1}(G) > \Delta - \delta + \frac{5}{\delta+1}$, then $\tau(G) \geq 3$.*
(iii) *For $k \geq 4$, if $\delta \geq 2k$ and $\mu_{n-1}(G) > \Delta - \delta + \frac{3k-1}{\delta+1}$, then $\tau(G) \geq k$.*
(iv) *For $k \geq 2$ and $\delta \geq k$, if $\mu_{n-1}(G) > \Delta - \delta + \frac{2(k-1)}{\delta+1}$, then $\kappa'(G) \geq k$.*

Proof. By Corollary 5.2 and Theorems 1.6-1.7. ■

Theorem 5.4. *Let $k \geq 2$ be an integer, G be a graph with minimum degree δ.*

(i) *If $\delta \geq 4$ and $q_2(G) < 2\delta - \frac{3}{\delta+1}$, then $\tau(G) \geq 2$.*
(ii) *If $\delta \geq 6$ and $q_2(G) < 2\delta - \frac{5}{\delta+1}$, then $\tau(G) \geq 3$.*

(iii) *For* $k \geq 4$, *if* $\delta \geq 2k$ *and* $q_2(G) < 2\delta - \frac{3k-1}{\delta+1}$, *then* $\tau(G) \geq k$.
(iv) *For* $k \geq 2$ *and* $\delta \geq k$, *if* $q_2(G) < 2\delta - \frac{2(k-1)}{\delta+1}$, *then* $\kappa'(G) \geq k$.

Proof. By Corollary 5.2 and Theorems 1.6-1.7.

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APPENDIX A: THE PROOF OF (6)

It suffices to show that $\frac{2(b-1)^2}{b(b-2)}k - \frac{3b-2}{b(b-2)} < 3k - 1$, which can be seen from the follow equivalences:

$$\frac{2(b-1)^2}{b(b-2)}k - \frac{3b-2}{b(b-2)} < 3k - 1$$

$$\iff 1 - \frac{3b-2}{b(b-2)} < 3k - \frac{2(b-1)^2}{b(b-2)}k$$
$$\iff \frac{b^2-5b+2}{b(b-2)} < \frac{b^2-2b-2}{b(b-2)}k.$$

As $b \geq 3$ and $k \geq 1$, it suffices to show that $b^2 - 5b + 2 < b^2 - 2b - 2$, which is equivalent to $4 < 3b$, which is correct since $b \geq 3$. This completes the proof.

APPENDIX B: THE PROOF OF (8)

It suffices to show that $\frac{2b'-1}{b'-1}k - \frac{3}{b'-1} < 3k - 1$, which can be seen from the follow equivalences:

$$\frac{2b'-1}{b'-1}k - \frac{3}{b'-1} < 3k - 1$$
$$\iff 1 - \frac{3}{b'-1} < 3k - \frac{2b'-1}{b'-1}k$$
$$\iff \frac{b'-4}{b'-1} < \frac{b'-2}{b'-1}k,$$

which is obviously correct when $b' \geq 2$ and $k \geq 1$. It completes the proof.