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Element deletion changes in dynamic coloring of graphs

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ABSTRACT

A proper vertex k-coloring of a graph G is dynamic if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors. The smallest integer k such that G has a dynamic k-coloring is the dynamic chromatic number $\chi_d(G)$. In this paper the differences between $\chi_d(G)$ and $\chi_d(G-e)$, and between $\chi_d(G)$ and $\chi_d(G-v)$ are investigated respectively.

1. Introduction

In this paper, all graphs $G = (V, E)$ are finite, simple and undirected. For $v \in V$, $N_G(v)$ is the set of vertices adjacent to v, and the degree of v, denoted by $d_G(v)$, is $|N_G(v)|$. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and minimum degree of G, respectively. When the graph G is understood from the context, we often omit the subscript G, and use δ, Δ for $\delta(G)$, $\Delta(G)$, respectively. If $uv \in E$, then u is a **neighbor** of v. For $W \subseteq V$, $G - W$ denotes the graph obtained from G by deleting the vertices in W together with their incident edges. If $W = \{w\}$, we often write $G - w$ for $G - \{w\}$. If $U \subseteq V$, then $G[U]$ denotes the graph on U whose edges are precisely the edges of G with both ends in U. Let C_n and P_n denote a cycle and a path on n vertices, respectively. In a graph G, an **elementary subdivision** of an edge $e = uv \in E(G)$ is the operation of replacing e with a path uv_ev through a new vertex v_e. A graph H is a **subdivision** of a graph G if H can be obtained from G by a sequence of elementary subdivisions. For a real number x, we use $\lceil x \rceil$ to denote the least integer no less than x.

For an integer $k > 0$, let $\overline{k} = \{1, 2, \ldots, k\}$. If $S \subseteq V(G)$ is a subset and $c : V(G) \mapsto \overline{k}$ is a mapping, then define $c(S) = \{c(x) : x \in S\}$. A **dynamic k-coloring** of a graph G is a mapping $c : V(G) \mapsto \overline{k}$ satisfying both of the following:

(C1) If $uv \in E(G)$, then $\varphi(u) \neq \varphi(v)$, and

(C2) for each vertex $v \in V(G)$, $|c(N(v))| \geq \min\{2, d_G(v)\}$.

The **dynamic chromatic number** $\chi_d(G)$ is the smallest integer k such that G has a dynamic k-coloring. Dynamic coloring was first introduced in [12,9], and is a special case of the r-hued colorings [8,7,13] when $r = 2$. The study of dynamic coloring has drawn lots of attention, as seen in [1–6,8,9,12,10,11,13,14], among others.

Unlike classic colorings, a subgraph of a graph G may have a bigger dynamic chromatic number than G. A natural problem is to investigate the differences between $\chi_d(G)$ and $\chi_d(G-e)$, and between $\chi_d(G)$ and $\chi_d(G-v)$. This motivates the current study. In Section 2, we will investigate the best possible bounds for the differences between $\chi_d(G-e)$ and $\chi_d(G)$, and between $\chi_d(G-v)$ and $\chi_d(G)$.

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2. Comparisons between $\chi_d(G)$ and $\chi_d(G-e)$, and between $\chi_d(G)$ and $\chi_d(G-v)$

It is well known that if H is a subgraph of a graph G, then $\chi(G) \geq \chi(H)$. However, there exist graphs G with a subgraph H such that $\chi_d(H) > \chi_d(G)$. For example, let G be the 5-cycle with one chord, and let H be the 5-cycle, then it is routine to verify that $\chi_d(G) = 4$ but $\chi_d(H) = 5$.

In this section, we investigate tight bounds for the change of the dynamic chromatic number when an edge or a vertex is being removed. We start with a lemma, which follows from definition immediately.

Lemma 2.1. *If G is a connected graph on at least 2 vertices, then $\chi_d(G) \leq 2$ is and only if $G \in \{K_1, K_2\}$.*

Theorem 2.1. *Each of the following holds.*

(i) *Let G be a connected graph with $|V(G)| \geq 3$. Then for any edge $e = uv \in E(G)$,*

$$\chi_d(G) - 2 \leq \chi_d(G-e) \leq \chi_d(G) + 2. \tag{1}$$

(ii) *There exists a graph G such that $\chi_d(G-e) = \chi_d(G) + 2$ for at least one edge $e \in E(G)$.*

(iii) *If a connected graph G satisfies that $\chi_d(G-e) = \chi_d(G) - 2$ for at least one edge e in G, then $G = C_5$.*

Proof. (i) Let $k_1 = \chi_d(G-e)$, and let $c_1 : V(G-e) \mapsto \overline{k}_1$ be a dynamic k_1-coloring of $G-e$. Obtain a new coloring c_1' from c_1 by defining

$$c_1'(z) = \begin{cases} c_1(z) & \text{if } z \notin \{u, v\} \\ k_1 + 1 & \text{if } z = u \\ k_1 + 2 & \text{if } z = v. \end{cases}$$

By definition, $c_1' : V(G) \mapsto \overline{k_1 + 2}$ is a dynamic $(k_1 + 2)$-coloring of G, and so $\chi_d(G) \leq \chi_d(G-e) + 2$.

Now let $k_2 = \chi_d(G)$ and $c_2 : V(G) \mapsto \overline{k}_2$ be a dynamic k_2-coloring of G. Since $|V(G)| \geq 3$ and since G is connected, there exists $x \in N_G(u) - \{v\}$ or $y \in N_G(v) - \{u\}$. Choose such x and y so that $|\{x, y\}|$ is maximized. If $|\{x, y\}| = 1$, then by the maximality of $|\{x, y\}|$, and since G is connected, we must have $d_G(u) \leq 2$ and $d_G(v) \leq 2$. In this case, we have $\chi_d(G) = \chi_d(G-e)$, and so $\chi_d(G) \leq \chi_d(G-e)+2$. Hence we assume that $x \neq y$. Obtain a new coloring c_2' from c_2 by defining

$$c_2'(z) = \begin{cases} c_2(z) & \text{if } z \notin \{x, y\} \\ k_2 + 1 & \text{if } z = x \\ k_2 + 2 & \text{if } z = y. \end{cases}$$

By definition, $c_2' : V(G-e) \mapsto \overline{k_2 + 2}$ is a dynamic (k_2+2)-coloring of $G-e$, and so $\chi_d(G-e) \leq \chi_d(G)+2$. This proves (i).

(ii) For an integer $r \geq 4$, let H be a complete r-partite graph with partite sets $V_1, V_2, \ldots, V_r$, such that $|V_i| \geq 2$ for each i with $1 \leq i \leq r$, and let u and v be two new vertices. Let G be the graph obtained from H by adding a new edge uv to H and by joining u to every vertex in V_1 and joining v to every vertex in V_2. It is routine to verify that $\chi_d(G) = \chi(G) = r$, and that $\chi_d(G-uv) = r+2$, since the vertices in each of V_1 and V_2 must be colored with at least two colors.

(iii) Let G be a connected graph with at least one edge such that $\chi_d(G-e) = \chi_d(G) - 2$ for some edge $e = uv \in E(G)$, and let $k = \chi_d(G-e)$. If $\chi_d(G-e) \leq 2$, then by Lemma 2.1, $G \in \{K_2, P_3\}$, contrary to the assumption that $\chi_d(G-e) = \chi_d(G)-2$ for some $e \in E(G)$. Hence we assume that $k = \chi_d(G-e) \geq 3$.

Let $c : V(G-e) \mapsto \overline{k}$ be a dynamic k-coloring. Assume without loss of generality, that $d_G(u) \geq d_G(v)$. If $d_G(v) = 1$, then v is an isolated vertex of $G-e$. As $k \geq 3$, we can pick a vertex $u' \in N_G(u) - \{v\}$ and redefine $c(v) \in \overline{k} - \{c(u), c(u')\}$ to obtain a k-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G) - 2$. If $d_G(u) \geq 3$, then by $k \geq 3$, we can redefine $c(u) = k+1$ to obtain a $(k+1)$-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G) - 2$. Hence we may assume that $d_G(u) = d_G(v) = 2$. Let $N_G(u) = \{v, u'\}$, $N_G(v) = \{u, v'\}$. We have the following claims.

Claim 1. $u' \neq v'$.

If $u' = v'$, then obtain a new coloring c' from c by defining

$$c'(z) = \begin{cases} c(z) & \text{if } z \neq u \\ k+1 & \text{if } z = u. \end{cases}$$

By definition, $c' : V(G) \mapsto \overline{k+1}$ is a dynamic $(k+1)$-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G)-2$. Thus Claim 1 must hold.

Claim 2. $c(u) = c(v') \neq c(u') = c(v)$.

If $c(u) \neq c(v')$, then obtain a new coloring c'' from c by defining

$$c''(z) = \begin{cases} c(z) & \text{if } z \neq v \\ k+1 & \text{if } z = v. \end{cases}$$

By definition, $c'' : V(G) \mapsto \overline{k+1}$ is a dynamic $(k+1)$-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G) - 2$. Thus we must have that $c(u) = c(v')$. By a similar argument, we also have that $c(u') = c(v)$. Since $uu' \in E(G-e)$, we conclude that $c(u) \neq c(u')$.

Claim 3. $\min\{d_G(u'), d_G(v')\} \geq 2$.

By contradiction, assume without loss of generality that $d_G(v') = 1$, and so $N_G(v') = \{v\}$. Obtain a new coloring $c^{(3)}$ from c by defining

$$c^{(3)}(z) = \begin{cases} c(z) & \text{if } z \notin \{v, v'\} \\ k+1 & \text{if } z = v \\ a & \text{where } a \in \overline{k} - \{c(u)\}, \text{if } z = v'. \end{cases}$$

By definition, $c^{(3)} : V(G) \mapsto \overline{k+1}$ is a dynamic $(k+1)$-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G) - 2$. This proves Claim 3.

Claim 4. $d_G(u') = d_G(v') = 2$.

By contradiction and by symmetry, assume that $d_G(u') \geq 3$. Pick a color $a' \in c(N_G(u')) - \{c(u), c(v)\}$ if $c(N_G(u')) - \{c(u), c(v)\} \neq \emptyset$, and define $\{a'\} = \emptyset$ if $c(N_G(u')) - \{c(u), c(v)\} = \emptyset$.

If $k = \chi_d(G-e) \geq 4$, then obtain a new coloring $c^{(4)}$ from c by defining

$$c^{(4)}(z) = \begin{cases} c(z) & \text{if } z \notin \{u, v\} \\ a & \text{where } a \in \overline{k} - (\{c(u), c(v)\} \cup \{a'\}), \text{if } z = u \\ k+1 & \text{if } z = v. \end{cases}$$

By definition, $c^{(4)} : V(G) \mapsto \overline{k+1}$ is a dynamic $(k+1)$-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G) - 2$. Thus we assume that $k = \chi_d(G-e) = 3$. By Claim 2, we may assume that $c(u) = c(v') = 1$ and $c(u') = c(v) = 2$ in the rest of the proof.

If $c(N_G(u') - \{u\}) = \{3\}$, then $N_G(u')$ is an independent set, and as $c(v') = 1$, $v' \notin N_G(u') - \{u\}$. Pick a vertex $u'' \in N_G(u') - \{u\}$. Obtain a new coloring $c^{(5)}$ from c by defining

$$c^{(5)}(z) = \begin{cases} c(z) & \text{if } z \notin \{u, v, u''\} \\ 3 & \text{if } z = u \\ k+1 & \text{if } z \in \{v, u''\}. \end{cases}$$

By definition, $c^{(5)} : V(G) \mapsto \overline{k+1}$ is a dynamic $(k+1)$-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G) - 2$.

If there exists an $u''' \in N_G(u') - \{u\}$ with $c(u''') = 1$, then obtain a new coloring $c^{(6)}$ from c by defining

$$c^{(6)}(z) = \begin{cases} c(z) & \text{if } z \notin \{u, v\} \\ 3 & \text{if } z = u \\ 4 & \text{if } z = v. \end{cases}$$

By definition, $c^{(6)} : V(G) \mapsto \overline{4}$ is a dynamic 4-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G) - 2$. Therefore, we must have $d_G(u') = d_G(v') = 2$.

Denote $N_G(u') = \{u, u''\}$, $N_G(v') = \{v, v''\}$. Assume first that $u'' \neq v''$ or both $u'' = v''$ and $d_G(u'') \geq 3$. Obtain a new coloring $c^{(7)}$ from c by defining

$$c^{(7)}(z) = \begin{cases} c(z) & \text{if } z \notin \{u', v'\} \\ k+1 & \text{if } z \in \{u', v'\}. \end{cases}$$

By definition, $c^{(7)} : V(G) \mapsto \overline{k+1}$ is a dynamic $(k+1)$-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G) - 2$. It follows that we must have $u'' = v''$ and $d_G(u'') = d_G(v'') = 2$, and so $G = C_5$. This completes the proof of Theorem 2.4. ■

The corollary below follows from Theorem 2.1(iii) and from Theorem 2.2 of [8].

Corollary 2.1. *Let G be a connected graph. The following are equivalent.*

(i) $G = C_5$.

(ii) *For any edge* $e \in E(G)$, $\chi_d(G-e) = \chi_d(G) - 2$.

In view of Theorem 2.1(ii) and Corollary 2.1, it is natural to investigate conditions on a graph G such that $\chi_d(G-e) \leq \chi_d(G) + 1$ for any $e \in E(G)$. The next result is an attempt in this direction.

Theorem 2.2. *Let G be a connected graph with* $n = |V(G)| \geq 2$. *If G does not contain a subdivision of* $K_{3,3}$, *then* $\chi_d(G-e) \leq \chi_d(G) + 1$ *for any* $e \in E(G)$.

Proof. To the contrary, we assume that there exists a $K_{3,3}$-minor free graph G such that $\chi_d(G-e) \geq \chi_d(G)+2$ for some $e = uv \in E(G)$. By Theorem 2.1(i), we have $\chi_d(G-e) = \chi_d(G)+2$. As the theorem holds trivially if $n \leq 5$, we assume that $n \geq 6$. Without loss of generality, we assume that $d_G(u) \geq d_G(v)$. Let $k = \chi_d(G)$ and let $c : V(G) \mapsto \overline{k}$ be a dynamic k- coloring of G. We make the following claims.

Claim 1. $d(u) \geq d(v) \geq 3$.

If $d(v) \leq 2$, then since $n \geq 6$ and since G is connected, we have $d_G(u) \geq 2$. Pick a vertex $x \in N_G(u) - \{v\}$ and obtain a new coloring $c' : V(G-e) \mapsto \overline{k+1}$ as follows:

$$c'(z) = \begin{cases} c(z) & \text{if } z \neq x \\ k+1 & \text{if } z = x. \end{cases}$$

By definition, c' is a dynamic $(k+1)$-coloring of $G-e$, contrary to the assumption that $\chi_d(G-e) = \chi_d(G)+2$. This justifies Claim 1.

Claim 2. $|c(N(u)-\{v\})| = |c(N(v)-\{u\})| = 1$.

If $|c(N(u)-\{v\})| \geq 2$ and $|c(N(v)-\{u\})| \geq 2$, then c is also a dynamic k-coloring of $G-uv$, and so $\chi_d(G-e) \leq k = \chi_d(G)$. Thus $\min\{|c(N(u)-\{v\})|, |c(N(v)-\{u\})|\} = 1$. By symmetry, we may assume that $|c(N(u)-\{v\})| = 1$.

If $|c(N(v)-\{u\})| \geq 2$, then by Claim 1, there exists a vertex $x \in N(u)-\{v\}$. Define a coloring $c'' : V(G-e) \mapsto \overline{k+1}$ as follows:

$$c''(z) = \begin{cases} c(z) & \text{if } z \neq x \\ k+1 & \text{if } z = x. \end{cases}$$

By definition, c'' is a dynamic $(k+1)$-coloring of $G-e$, contrary to the assumption that $\chi_d(G-e) = \chi_d(G)+2$. This proves Claim 2.

Claim 3. $c(N(u)-\{v\}) \neq c(N(v)-\{u\})$.

By contradiction, assume that $c(N(u)-\{v\}) = c(N(v)-\{u\})$. By Claim 1, there exist $x \in N(u)-\{v\}$ and $y \in N(v)-\{u,x\}$. Obtain a coloring $c^{(3)} : V(G-e) \mapsto \overline{k+1}$ as follows:

$$c^{(3)}(z) = \begin{cases} c(z) & \text{if } z \notin \{x,y\} \\ k+1 & \text{if } z \in \{x,y\}. \end{cases} \quad (2)$$

By definition, $c^{(3)}$ is a dynamic $(k+1)$-coloring of $G-e$, contrary to the assumption that $\chi_d(G-e) = \chi_d(G)+2$. This proves Claim 3.

Claim 4. For every $x \in N(u)-\{v\}$ and for every $y \in N(v)-\{u\}$, either $xy \in E(G)$ or $N_G(x) \cap N_G(y)$ contains a vertex of degree 2 in G.

Suppose that there exist a vertex $x \in N(u)-\{v\}$ and a vertex $y \in N(v)-\{u\}$ such that $xy \notin E(G)$ and $N_G(x) \cap N_G(y)$ contains no vertices of degree 2 in G. Obtain a coloring $c^{(3)} : V(G-e) \mapsto \overline{k+1}$ as defined in (2). By definition, $c^{(3)}$ is a dynamic $(k+1)$-coloring of $G-e$, contrary to the assumption that $\chi_d(G-e) = \chi_d(G)+2$. Hence Claim 4 must hold.

By Claims 1–4, $G[N(u) \cup N(v) \cup N(N(u)) \cup N(N(v))]$ contains a subdivision of $K_{3,3}$, contrary to the assumption of the theorem. This completes the proof of Theorem 2.2. ■

To investigate the corresponding problem using vertex removal instead of edge removal, we quote a theorem of Montgomery.

Theorem 2.3 (*Montgomery, [12]*). *For any graph G, $\chi_d(G-v) \geq \chi_d(G)-2$ for any vertex $v \in V(G)$. The only graphs for which $\chi_d(G-v) \geq \chi_d(G)-2$ for at least one vertex are $K_{1,n-1}$, $n \geq 3$ and C_5.*

A natural question is to see if there exists a constant $M > 0$ such that $\chi_d(G-v) \leq \chi_d(G)+M$ for any vertex $v \in V(G)$. The next example addresses to this problem, and indicates that the difference $\chi_d(G-v) - \chi_d(G)$ could be unbounded.

Example 2.1. For any integer $M \geq 1$, there exists a graph G such that $\chi_d(G-v) \geq \chi_d(G)+M$ for at least one vertex $v \in V(G)$.

Let $k \geq 4$ and $M = k-3$ be integers. Let SK_k be the bipartite graph with vertex bipartition X and Y, where $X = \overline{k}$ and $Y = X^{[2]}$, which is the set of all 2-element subsets of $\overline{k}$, such that a vertex $x \in X$ is adjacent to a vertex $\{i,j\} \in Y$ if and only if $x \in \{i,j\}$. Thus SK_k is the graph obtained from K_k by subdividing every edge of K_k exactly once. (See [9] and [12].) As shown in [9] and [12], we know that $\chi_d(SK_k) = k$. Let G_k be the graph obtained from SK_k by adding a new vertex w to SK_k and joining w to every vertex of SK_k. Obtain a dynamic 3-coloring c of G_k by defining $c(w) = 3$, $c(x) = 1$ if $x \in X$ and $c(y) = 2$ if $y \in Y$. It follows that $\chi_d(G_k) = 3$. Since $G_k - w = SK_k$, we have $\chi_d(G-v) = \chi_d(G)+M$.

Remark 2.1. Given the main results in this note, it is natural to seek possible characterization of graphs G such that $\chi_d(G-e) = \chi_d(G) - c$ for some edge $e \in E(G)$ (or for any $e \in E(G)$), where $c \in \{-1, 0, 1\}$. This seems to be a difficult task, and remains to be investigated.

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References

[1] S. Akbari, M. Ghanbari, S. Jahanbekam, On the list dynamic coloring of graphs, Discrete Appl. Math. 157 (2009) 3005–3007.
[2] M. Alishahi, On the dynamic coloring of graphs, Discrete Appl. Math. 159 (2011) 152–156.
[3] Y. Chen, S.H. Fan, H.-J. Lai, H.M. Song, L. Sun, On dynamic coloring for planar graphs and graphs of higher genus, Discrete Appl. Math. 160 (2012) 1064–1071.
[4] C. Ding, S.H. Fan, H.-J. Lai, Upper bound on conditional chromatic number of graphs, J. Jinan Univ. Nat. Sci. Med. 29((2008) 7–14.
[5] L. Esperet, Dynamic list coloring of bipartite graphs, Discrete Appl. Math. 158 (2010) 1963–1965.
[6] S.-J. Kim, S.-J. Lee, W.-J. Park, Dynamic coloring and list dynamic coloring of planar graph, Discrete Appl. Math. 161 (2013) 2207–2212.
[7] H.-J. Lai, X. Li, G. Yu, An inequality for the group chromatic number of a graph, Discrete Math. 307 (2007) 3076–3080.
[8] H.-J. Lai, J. Lin, B. Montgomery, Z. Tao, S. Fan, Conditional colorings of graphs, Discrete Math. 306 (2006) 1997–2004.
[9] H.-J. Lai, B. Montgomery, H. Poon, Upper bounds of dynamic chromatic number, Ars Combin. 68 (2003) 193–201.
[10] X. Li, X. Yao, W. Zhou, H.J. Broersma, Complexity of conditional colorability of graphs, Appl. Math. Lett. 22 (2009) 320–324.
[11] Y. Lin, Upper bounds of conditional chromatics number (Master thesis), Jinan University, 2008.
[12] B. Montgomery, Dynamic coloring (Ph.D. dissertation), West Virginia University, 2001.
[13] H.M. Song, S.H. Fan, Y. Chen, L. Sun, H.-J. Lai, On r-hued coloring of K_4-minor free graphs, Discrete Appl. Math. 315-316 (2014) 47–52.
[14] H.M. Song, H.-J. Lai, J.L. Wu, On r-hued coloring of planar graphs with girth at least 6, Discrete Applied Mathematics 198 (2016) 251–263.