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Research Article

Dicycle Cover of Hamiltonian Oriented Graphs

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A dicycle cover of a digraph D is a family $\mathcal{F}$ of dicycles of D such that each arc of D lies in at least one dicycle in $\mathcal{F}$. We investigate the problem of determining the upper bounds for the minimum number of dicycles which cover all arcs in a strong digraph. Best possible upper bounds of dicycle covers are obtained in a number of classes of digraphs including strong tournaments, Hamiltonian oriented graphs, Hamiltonian oriented complete bipartite graphs, and families of possibly non-Hamiltonian digraphs obtained from these digraphs via a sequence of 2-sum operations.

1. The Problem

We consider finite loopless graphs and digraphs, and undefined notations and terms will follow [1] for graphs and [2] for digraphs. In particular, a cycle is a 2-regular connected nontrivial graph. A cycle cover of a graph G is a collection $\mathcal{C}$ of cycles of G such that $E(G) = \bigcup_{C\in\mathcal{C}} E(C)$. Bondy [3] conjectured that if G is a 2-connected simple graph with $n \geq 3$ vertices, then G has a cycle cover $\mathcal{C}$ with $|\mathcal{C}| \leq (2n-3)/3$. Bondy [3] showed that this conjecture, if proved, would be best possible. Luo and Chen [4] proved that this conjecture holds for 2-connected simple cubic graphs. It has been shown that, for plane triangulations, serial-parallel graphs, or planar graphs in general, one can have a better bound for the number of cycles used in a cover [5–8]. Barnette [9] proved that if G is a 3-connected simple planar graph of order n, then the edges of G can be covered by at most $(n+1)/2$ cycles. Fan [10] settled this conjecture by showing that it holds for all simple 2-connected graphs. The best possible number of cycles needed to cover cubic graphs has been obtained in [11, 12].

A directed path in a digraph D from a vertex u to a vertex v is called a (u,v)-dipath. To emphasize the distinction between graphs and digraphs, a directed cycle or path in a digraph is often referred to as a dicycle or dipath. It is natural to consider the number of dicycles needed to cover a digraph. Following [2], for a digraph $D, V(D)$ and $A(D)$ denote the vertex set and arc set of D, respectively. If $A' \subseteq A(D)$, then $D[A']$ is the subdigraph induced by A'. Let K_n^* denote the complete digraph on n vertices. Any simple digraph D on n vertices can be viewed as a subdigraph of K_n^*. If W is an arc subset of $A(K_n^*)$, then $D+W$ denotes the digraph $K_n^*[A(D)\cup W]$.

A digraph D is strong if, for any distinct $u, v \in V(D)$, D has a (u,v)-dipath. As in [2], $\lambda(D)$ denotes the arc-strong-connectivity of D. Thus a digraph D is strong if and only if $\lambda(D) \geq 1$. We use (u,v) denoting an arc with tail u and head v. For $X, Y \subseteq V(D)$, we define

$$(X,Y)_D = \{(x,y) \in A(D) : x \in X,\ y \in Y\};$$
$$\partial_D^+(X) = (X, V(D)-X)_D. \tag{1}$$

Let

$$d_D^+(X) = \left|\partial_D^+(X)\right|,$$
$$d_D^-(X) = \left|\partial_D^-(X)\right|. \tag{2}$$

When $X=\{v\}$, we write $d_D^+(v) = |\partial_D^+\{v\}|$ and $d_D^-(v) = |\partial_D^-\{v\}|$. Let $N_D^+(v) = \{u \in V(D)-v : (v,u) \in A(D)\}$ and $N_D^-(v) = \{u \in V(D)-v : (u,v) \in A(D)\}$ denote the *out-neighbourhood* and *in-neighbourhood* of v in D, respectively. We call the vertices in $N_D^+(v)$ and $N_D^-(v)$ the *out-neighbours* and the *in-neighbours* of v. Thus, for a digraph D, $\lambda(D) \geq 1$ if and only if, for any proper nonempty subset $\emptyset \neq X \subset V(D)$, $|\partial_D^+(X)| \geq 1$.

A *dicycle cover* of a digraph D is a collection $\mathscr{C}$ of dicycles of D such that $\bigcup_{C\in\mathscr{C}} A(C) = A(D)$. If D is obtained from a simple undirected graph G by assigning an orientation to the edges of G, then D is an *oriented graph*. The main purpose is to investigate the number of dicycles needed to cover a Hamiltonian oriented graph. We prove the following.

Theorem 1. *Let D be an oriented graph on n vertices and m arcs. If D has a Hamiltonian dicycle, then D has a dicycle cover $\mathscr{C}$ with $|\mathscr{C}| \le m - n + 1$. This bound is best possible.*

In the next section, we will first show that every Hamiltonian oriented graph with n vertices and m arcs can be covered by at most $m - n + 1$ dicycles. Then we show that, for every Hamiltonian graph G with n vertices and m edges, there exists an orientation $D = D(G)$ of G such that any dicycle cover of D must have at least $m - n + 1$ dicycles.

2. Proof of the Main Result

In this section, all graphs are assumed to be simple. We start with an observation, stated as lemma below. A digraph D is *weakly connected* if the underlying graph of D is connected.

Lemma 2. *A weakly connected digraph D has a dicycle cover if and only if $\lambda(D) \ge 1$.*

Proof. Suppose that D has a dicycle cover $\mathscr{C}$. If D is not strong, then there exists a proper nonempty subset $\emptyset \ne X \subset V(D)$ such that $|\partial_D^+(X)| = 0$. Since D is weakly connected, D contains an arc $(u, v) \in (V(D) - X), X)_D$. Since $\mathscr{C}$ is a dicycle cover of D, there exists a dicycle $C \in \mathscr{C}$ with $(u, v) \in A(C)$. Since $(u, v) \in (V(D) - X), X)_D$, we conclude that $\emptyset \ne A(C) \cap (X, V(D) - X))_D \subseteq \partial_D^+(X)$, contrary to the assumption that $|\partial_D^+(X)| = 0$. This proves that D must be strong.

Conversely, assume that D is strong. For any arc $a = (u, v) \in A(D)$, since D is strong, there must be a directed (v, u)-path P in D. It follows that $C_a = P + a$ is a dicycle of D containing a, and so $\{C_a : a \in A(D)\}$ is a dicycle cover of D. □

Let C be a dicycle and let $a = (u, v)$ be an arc not in $A(C)$ but with $u, v \in V(C)$. Then $C + a$ contains a unique dicycle C_a containing a. In the following, we call C_a the fundamental dicycle of a with respect to C.

Lemma 3. *Let D be an oriented graph on n vertices and m arcs. If D has a Hamiltonian dicycle, then D has a dicycle cover $\mathscr{C}$ with $|\mathscr{C}| \le m - n + 1$.*

Proof. Let C_0 denote the directed Hamiltonian cycle of D. For each $a \in A(D) - A(C)$, let C_a denote the fundamental dicycle of a with respect to C. Then $\mathscr{C} = \{C_0\} \cup \{C_a : a \in A(D) - A(C)\}$ is a dicycle cover of D with $|\mathscr{C}| \le m - n + 1$. □

To prove that Theorem 1 is best possible, we need to construct, for each integer $n \ge 4$, a Hamiltonian oriented graph on n vertices and m arcs D such that any dicycle cover $\mathscr{C}$ of D must have at least $m - n + 1$ dicycles in $\mathscr{C}$.

Let G be a Hamiltonian simple graph. We present a construction of such an orientation $D = D(G)$. Since G is Hamiltonian, we may assume that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $C = v_1 v_2, \ldots, v_n v_1$ is a Hamiltonian cycle of G.

Definition 4. One defines an orientation $D = D(G)$ as follows.

(i) Orient the edges in the Hamiltonian cycle $C = v_1 v_2, \ldots, v_n v_1$ as follows:

$$\begin{aligned}&(v_{i+1}, v_i) \in A(D),\\ &\qquad i = 1, 2, \ldots, n-1, \ (v_1, v_n) \in A(D).\end{aligned} \tag{3}$$

(ii) For each $i = 2, 3, \ldots, n-2$, and for each $j = i+2, i+3, \ldots, n$, assign directions to edges of G not in $E(C)$ as follows:

$$\begin{aligned}&(v_i, v_j) \in A(D),\\ &\qquad \text{if } v_i v_j \in E(G) - E(C),\ i + 1 < j \le n,\\ &(v_1, v_j) \in A(D),\\ &\qquad \text{if } v_1 v_j \in E(G) - E(C),\ i + 1 < j \le n - 1.\end{aligned} \tag{4}$$

We make the following observations stated in the lemma below.

Lemma 5. *Each of the following holds for the digraph D:*

(i) *The dicycle $C_0 = v_1 v_n v_{n-1}, \ldots, v_3 v_2 v_1$ is a Hamiltonian dicycle of D.*

(ii) *The digraph $D - A(C_0)$ is acyclic.*

(iii) $N_D^+(v_n) = \{v_{n-1}\}$; $N_D^-(v_1) = \{v_2\}$; $N_D^-(v_2) = \{v_3\}$.

(iv) *The dicycle C_0 is the only dicycle of D containing the arc (v_1, v_n).*

(v) *The dicycle C_0 is the unique Hamiltonian dicycle of D.*

(vi) *If C'' is a dicycle of D, then C'' contains at most one arc in $A(D) - A(C_0)$.*

Proof. (i) follows immediately from Definition 4(i).

(ii) By Definition 4, the labels of the vertices $V(D) = \{v_1, v_2, \ldots, v_n\}$ satisfy $(v_i, v_j) \in A(D) - A(C_0)$ only if $i < j$. It follows (e.g., Section 2.1 of [2]) that $D - A(C_0)$ is acyclic, and so (ii) holds.

(iii) This follows immediately from Definition 4.

(iv) Let C' be a dicycle of D with $(v_1, v_n) \in A(C')$. Since $(v_1, v_n) \in A(C') \cap A(C_0)$, we choose the largest label $i \le n$, such that $(v_1, v_n), (v_n, v_{n-1}), \ldots, (v_{i+1}, v_i) \in A(C') \cap A(C_0)$. Since $C' \ne C_0$, we have $i \ge 3$. Since C' is a dicycle, there must be a vertex $v_j \in V(D)$ such that $(v_i, v_j) \in A(C')$. By the choice of i, we must have $(v_i, v_j) \notin A(C_0)$, and so $(v_i, v_j) \in A(D) - A(C_0)$. By Definition 4(ii), we have $i + 2 \le j \le n$, contrary to the fact that C' is a dicycle of D containing (v_1, v_n). This proves (iv).

(v) Let C' be a Hamiltonian dicycle of D. Since $V(C') = V(D)$, we have $v_n \in V(C')$. We claim that

$(v_1, v_n) \in A(C')$. If $(v_1, v_n) \notin A(C')$, then there exists $v_i \in V(C)$ $(i \in \{v_2, v_3, \ldots, v_{n-1}\})$ such that $(v_i, v_n) \in A(C')$. Hence, $(v_i, v_n), (v_n, v_{n-1}), \ldots, (v_{i+2}, v_{i+1}) \in A(C')$. By Definition 4(i) and (ii), $N^+(v_{i+1}) \subset \{v_{i+2}, v_{i+3}, \ldots, v_n\}$, contrary to the fact that C' is a Hamiltonian dicycle of D. Thus, $(v_1, v_n) \in A(C')$. It follows from Lemma 5(iv) that we must have $C' = C_0$.

(vi) By contradiction, we assume that D has a dicycle C'' which contains two arcs: $a_1, a_2 \in A(D) - A(C_0)$. Since $V(D) = \{v_1, v_2, \ldots, v_n\}$, we assume that $a_1 = (v_i, v_{i'})$ and $a_2 = (v_j, v_{j'})$. Without loss of generality and by Lemma 2, we further assume that $1 \le i < j < n$.

Let $i \ge t \ge 1$ be the smallest integer such that $v_t \in V(C'')$. Since C'' is a dicycle of D, there must be $v_s \in V(C'')$ such that $(v_s, v_t) \in A(C'')$. By Definition 4, either $(v_s, v_t) \in A(C_0)$ and $s = t + 1 < j$ or $(v_s, v_t) \in A(D) - A(C_0)$ and $1 < s + 1 < t$. By the choice of t, we can only have $s = t + 1$ and $(v_{t+1}, v_t) \in A(C'') \cap A(C_0)$. Choose the largest integer h with $t + 1 \le h < j$ such that $(v_{t+1}, v_t), (v_{t+2}, v_{t+1}), \ldots, (v_h, v_{h-1}) \in A(C'') \cap A(C_0)$. Since C'' is a dicycle, there must be v_k with $1 \le k \le n$ such that $(v_k, v_h) \in A(C'')$. By the maximality of h and by Definition 4(i), we conclude that $(v_k, v_h) \notin A(C_0)$. By Definition 4(ii), $1 \le k \le h-2$. By the minimality of t, we must have $t \le k \le h - 2$. It follows by $j > h$ that C'' cannot contain $a_2 = (v_j, v_{j'})$, contrary to the assumption. This contradiction justifies (vi). □

To complete the proof of Theorem 1, we present the next lemma.

Lemma 6. *Let G be a Hamiltonian simple graph. There exists an orientation $D = D(G)$ such that every dicycle cover of D must have at least $m - n + 1$ dicycles.*

Proof. Let G be a Hamiltonian graph and let $D = D(G)$ be the orientation of G given in Definition 4. For notational convenience, we adopt the notations in Definition 4 and denote $V(D) = \{v_1, v_2, \ldots, v_n\}$. Thus, by Lemma 5(v), $C_0 = v_1 v_n v_{n-1}, \ldots, v_2 v_1$ is the unique Hamiltonian dicycle of D.

Let $\mathscr{C}$ be a dicycle cover of D. By Lemma 5(iv), we must have $C_0 \in \mathscr{C}$. For each arc $a \in A(D) - A(C_0)$, since $\mathscr{C}$ is a dicycle cover of D, there must be a dicycle $C(a) \in \mathscr{C}$ such that $a \in A(C(a))$. By Lemma 5(vi), $A(C(a)) \cap A(D) - A(C_0) = \{a\}$. It follows that if $a, a' \in A(D) - A(C_0)$, then $a \ne a'$ implies $C(a) \ne C(a')$ in $\mathscr{C}$. Thus we have $\{C(a) \mid a \in A(D) - A(C_0)\} \subseteq \mathscr{C}$. Hence

$$\begin{aligned} |\mathscr{C}| &\ge \left|\{C(a) : a \in A(D) - A(C_0)\} \cup \{C_0\}\right| \\ &= m - n + 1. \end{aligned} \tag{5}$$

This proves the lemma. □

By Lemmas 3 and 6, Theorem 1 follows. We are about to show that Theorem 1 can be applied to obtain dicycle cover bounds for certain families of oriented graphs. Let T_n denote a tournament of order n. Then T_n is an oriented graph. Camion [13, 14] proved that every strong tournament is Hamiltonian. Hence the corollary below follows from Theorem 1.

Corollary 7. *Every strong tournament on n vertices has a dicycle cover $\mathscr{C}$ with $|\mathscr{C}| \le n(n-1)/2 - n + 1$. This bound is best possible.*

A bipartite graph $G(A, B)$ with vertex bipartition (A, B) is balanced if $|A| = |B|$. If bipartite graph $G(A, B)$ has a Hamiltonian cycle, then G is balanced. Let $K_{m,n}$ be a complete bipartite graph with vertex bipartition (A, B) and $|A| = m$, $|B| = n$; then $K_{m,n}$ has Hamiltonian cycle if and only if $m = n \ge 2$; that is, $K_{m,n}$ is balanced. Let $K_{n,n}$ denote a balanced complete bipartite graph.

Corollary 8. *Every Hamiltonian orientation of balanced complete bipartite graph $K_{n,n}$ has a dicycle cover $\mathscr{C}$ with $|\mathscr{C}| \le (n-1)^2$. This bound is best possible.*

Proof. Since an oriented balanced complete bipartite graph $K_{n,n}$ has n^2 arcs, so, by Theorem 1, we have $|\mathscr{C}| \le n^2 - 2n + 1 = (n-1)^2$.

To prove the bound is best possible, we need to construct, for each integer $n \ge 2$, a Hamiltonian oriented balanced complete bipartite graph on $2n$ vertices such that any dicycle cover $\mathscr{C}$ of $K_{n,n}$ must have at least $(n-1)^2$ dicycles in $\mathscr{C}$. We may assume that $V(K_{n,n}) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and $C = u_1 v_1 u_2 v_2, \ldots, u_n v_n u_1$ is a Hamiltonian cycle of $K_{n,n}$. We construct an orientation $D_{n,n} = D(K_{n,n})$ as the orientation of Definition 4; thus, by Lemmas 5 and 6, every dicycle cover $\mathscr{C}$ of $D_{n,n}$ must have at least $(n-1)^2$ dicycles. This proves the corollary. □

3. Dicycle Covers of 2 Sums of Digraphs

In this section, we will show that Theorem 1 can also be applied to certain non-Hamiltonian digraphs which can be built via 2 sums. We start with 2 sums of digraphs.

Definition 9. Let $D_{n_1} = (V(D_{n_1}), A(D_{n_1}))$ and $D_{n_2} = (V(D_{n_2}), A(D_{n_2}))$ be two disjoint digraphs; $a_1 = (v_{12}, v_{11}) \in A(D_{n_1})$ and $a_2 = (v_{22}, v_{21}) \in A(D_{n_2})$. The 2-sum $D_{n_1} \oplus_2 D_{n_2}$ of D_{n_1} and D_{n_2} is obtained from the union of D_{n_1} and D_{n_2} by identifying the arcs a_1 and a_2; that is, $v_{11} = v_{21}$ and $v_{12} = v_{22}$.

Definition 10. Let $D_{n_1}, D_{n_2}, \ldots, D_{n_s}$ be s disjoint digraphs with $n_1, n_2, \ldots, n_s$ vertices, respectively. Let $D_{n_1} \oplus_2 D_{n_2} \oplus_2 \cdots \oplus_2 D_{n_s}$ denote a sequence of 2 sums of $D_{n_1}, D_{n_2}, \ldots, D_{n_s}$, that is, $(((D_{n_1} \oplus_2 D_{n_2}) \oplus_2 D_{n_3}) \oplus_2 \cdots) \oplus_2 D_{n_s}$.

Theorem 11. *Let $D_{n_1}, D_{n_2}, \ldots, D_{n_s}$ be s disjoint Hamiltonian oriented graphs on $n_1, n_2, \ldots, n_s$ vertices and $m_1, m_2, \ldots, m_s$ arcs, respectively, and let $D = D_{n_1} \oplus_2 D_{n_2} \oplus_2 \cdots \oplus_2 D_{n_s}$. Then D has a dicycle cover $\mathscr{C}$ with $|\mathscr{C}| \le |A(D)| - |V(D)| + 1$. This bound is best possible.*

Proof. By Theorem 1, D_{n_i} $(i = 1, 2, \ldots, s)$ has a dicycle cover $\mathscr{C}_i$ with $|\mathscr{C}_i| \le m_i - n_i + 1$. Let $\mathscr{C} = \bigcup_{i=1}^{s} \mathscr{C}_i$. Then $|\mathscr{C}| \le (m_1 - n_1 + 1) + (m_2 - n_2 + 1) + \cdots + (m_s - n_s + 1) = (m_1 + m_2 + \cdots + m_s) - (n_1 + n_2 + \cdots + n_s) + s = (m_1 + m_2 + \cdots + m_s - (s-1)) - (n_1 + n_2 + \cdots + n_s - 2(s-1)) + 1 = |A(D)| - |V(D)| + 1$. By Definition 10, $\mathscr{C}$ is a dicycle cover of D. Thus, D has a dicycle cover $\mathscr{C}$ with $|\mathscr{C}| \le |A(D)| - |V(D)| + 1$.

Let G_{n_i} be s disjoint Hamiltonian simple graphs for $i \in \{1, 2, \ldots, s\}$. We may assume that $V(G_{n_i}) = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ and $C_i = v_{i1}v_{i2}, \ldots, v_{in_i}v_{i1}$ is a Hamiltonian cycle of G_{n_i}, and let

$$D_{n_i} = D\left(G_{n_i}\right) \text{ be the orientation of } G_{n_i} \text{ given in Definition 4.} \tag{6}$$

For notational convenience, we adopt the notations in Definition 4 and denote $V(D_{n_i}) = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$. Thus, by Lemma 5(v), $C_{i_0} = v_{i1}v_{in_i}, \ldots, v_{i2}v_{i1}$ is the unique Hamiltonian dicycle of D_{n_i}. Let $a_i = (v_{i2}, v_{i1})$ be an arc of D_{n_i}. We construct the 2-sum digraph $D_{n_1} \oplus_2 D_{n_2} \oplus_2 \cdots \oplus_2 D_{n_s}$ from the union of $D_{n_1}, D_{n_2}, \ldots, D_{n_s}$ by identifying the arcs $a_1, a_2, \ldots, a_s$ such that $v_{11} = v_{21} = \cdots = v_{s1}$ and $v_{12} = v_{22} = \cdots = v_{s2}$. We assume that $v_1 := v_{11} = v_{21} = \cdots = v_{s1}$ and $v_2 := v_{12} = v_{22} = \cdots = v_{s2}$ (the case when $s = 2$ is depicted in Figure 1).

Claim 1. There does not exist a dicycle whose arcs intersect arcs in two or more D_{n_i}'s ($i = 1, 2, \ldots, s$).

By Definition 9, we have $V(D_{n_i}) \cap V(D_{n_j}) = \{v_1, v_2\}$ ($i \neq j$). Without loss of generality, we consider oriented graphs D_{n_1} and D_{n_2}; suppose that there exists a dicycle C_0 such that

$$\begin{aligned} \{A(C_0) - (v_2, v_1)\} \cap A\left(D_{n_1}\right) &\neq \emptyset, \\ \{A(C_0) - (v_2, v_1)\} \cap A\left(D_{n_2}\right) &\neq \emptyset. \end{aligned} \tag{7}$$

Thus, there must exist four different arcs

$$\left\{(v_{1i'}, v_1), (v_1, v_{2i''}), (v_{2j''}, v_2), (v_2, v_{1j'})\right\} \in A(C_0) \tag{8}$$

with $(v_{1i'}, v_1), (v_2, v_{1j'}) \in A(D_{n_1})$ and $(v_1, v_{2i''}), (v_{2j''}, v_2) \in A(D_{n_2})$, as shown in Figure 2, or four different arcs

$$\{(v_{1s'}, v_2), (v_2, v_{2s''}), (v_{2k''}, v_1), (v_1, v_{1k'})\} \in A(C_0) \tag{9}$$

with $(v_{1s'}, v_2), (v_1, v_{1k'}) \in A(D_{n_1})$ and $(v_2, v_{2s''}), (v_{2k''}, v_1) \in A(D_{n_2})$, as shown in Figure 3.

By Definition 9, Lemma 5(iii), and (6), we have $N_D^-(v_1) = \{v_2\}$, and so $v_{1i'} = v_2$ or $v_{2k''} = v_2$, contrary to the assumption that C_0 is a dicycle. This proves Claim 1.

By Claim 1, for every dicycle C in D, all arcs in C (except for the arc (v_2, v_1)) belong to exactly one of oriented graphs D_{n_i} ($i = 1, 2, \ldots, n$). By Definition 4 and Lemma 6, every dicycle cover of oriented graph D_{n_i} ($i = 1, 2, \ldots, n$) must have at least $m_i - n_i + 1$ dicycles. This completes the proof. □

By Corollary 7 and Theorem 11, we have the following corollary.

Corollary 12. *Let $D_{n_1}, D_{n_2}, \ldots, D_{n_s}$ be s disjoint strong tournaments with $n_1, n_2, \ldots, n_s$ vertices, respectively. Then $D_{n_1} \oplus_2 D_{n_2} \oplus_2 \cdots \oplus_2 D_{n_s}$ has a dicycle cover $\mathscr{C}$ with $|\mathscr{C}| \leq (n_1(n_1 - 1)/2 + n_2(n_2 - 1)/2 + \cdots + n_s(n_s - 1)/2) - (n_1 + n_2 + \cdots + n_s) + s$. This bound is best possible.*

Let G_n be a Hamiltonian graph with n vertices and m arcs; let D_n^i (i is an integer) denote a Hamiltonian orientation of G_n. For a positive integer s, let $H(G_n, s)$ denote the family of all 2-sum generated digraphs $D_n^1 \oplus_2 D_n^2 \oplus_2 \cdots \oplus_2 D_n^s$, as well as a member in the family (for notational convenience). By the definition of $H(G_n, s)$, we have $H(G_n, 1) = D_n^1$ and $H(G_n, s) = H(G_n, s-1) \oplus_2 D_n^s$. The conclusions of the next corollaries follow from Theorem 1. The sharpness of these corollaries can be demonstrated using similar constructions displayed in Lemma 6 and Corollary 8.

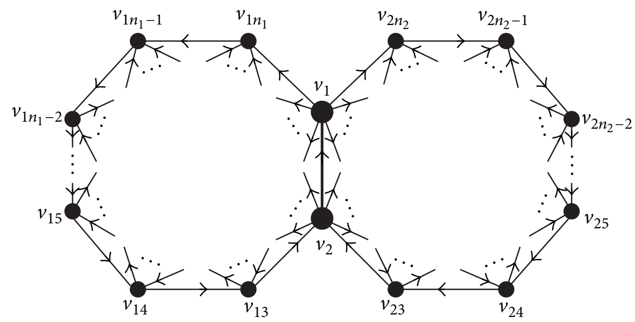

Figure 1: The 2-sum digraph for D_{n_1} and D_{n_2}.

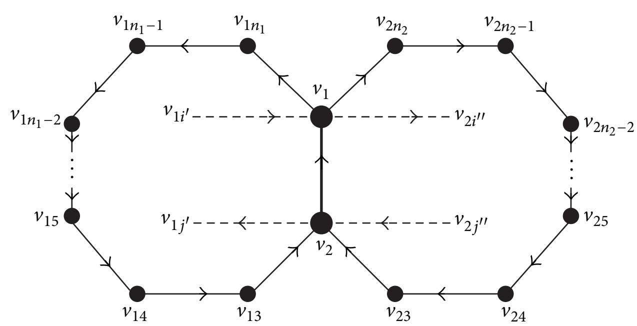

Figure 2

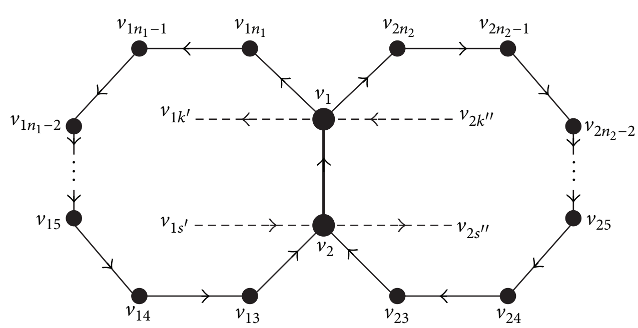

Figure 3

Corollary 13. *Let $m, n \geq 3$ be integer, let G_n be a Hamiltonian graph with n vertices and m edges, and let K_n be a complete graph on $n \geq 3$ vertices:*

(i) *Any member in $H(G_n, s)$ has a dicycle cover $\mathscr{C}$ with $|\mathscr{C}| \leq s(m - n + 1)$. This bound is best possible.*

(ii) *In particular, any $H(K_n, s)$ has a dicycle cover $\mathscr{C}$ with $|\mathscr{C}| \leq s(n(n-1)/2 - n + 1)$. This bound is best possible.*

Corollary 14. *Let $m, n \geq 3$ be integer, let B_n be a Hamiltonian bipartite graph with $2n$ vertices and m edges, and let $K_{n,n}$ be a complete bipartite graph:*

(i) *Any $H(B_n, s)$ has a dicycle cover $\mathscr{C}$ with $|\mathscr{C}| \leq s(m - 2n + 1)$. This bound is best possible.*

(ii) *In particular, any $H(K_{n,n}, s)$ has a dicycle cover $\mathscr{C}$ with $|\mathscr{C}| \leq s(n-1)^2$. This bound is best possible.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, vol. 244 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2008.

[2] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer, 2nd edition, 2010.

[3] J. A. Bondy, "Small cycle double covers of graphs," in *Cycles and Rays*, pp. 21–40, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[4] Y. X. Luo and R. S. Chen, "Cycle covers of 2-connected 3-regular graphs," *Mathematics and Applications*, vol. 9, supplement, pp. 23–25, 1996 (Chinese).

[5] H.-J. Lai, "Cycle covers of planar graphs," *Congressus Numerantium*, vol. 122, pp. 33–46, 1996.

[6] H.-J. Lai and H. Y. Lai, "Cycle covering of plane triangulations," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 10, pp. 3–21, 1991.

[7] H.-J. Lai and H. Y. Lai, "Cycle covers in graphs without subdivisions of K4," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 9, pp. 201–207, 1991.

[8] H.-J. Lai and H. Y. Lai, "Small cycle covers of planar graphs," *Congressus Numerantium*, vol. 85, pp. 203–209, 1991.

[9] D. W. Barnette, "Cycle covers of planar 3-connected graphs," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 20, pp. 245–253, 1996.

[10] G. Fan, "Subgraph coverings and edge switchings," *Journal of Combinatorial Theory Series B*, vol. 84, no. 1, pp. 54–83, 2002.

[11] H.-J. Lai and X. Li, "Small cycle cover of 2-connected cubic graphs," *Discrete Mathematics*, vol. 269, no. 1–3, pp. 295–302, 2003.

[12] F. Yang and X. Li, "Small cycle covers of 3-connected cubic graphs," *Discrete Mathematics*, vol. 311, no. 2-3, pp. 186–196, 2011.

[13] P. Camion, "Chemins et circuits hamiltoniens des graphes complets," *Comptes Rendus de l'Académie des Sciences*, vol. 249, pp. 2151–2152, 1959.

[14] J. W. Moon, "On subtournaments of a tournament," *Canadian Mathematical Bulletin*, vol. 9, pp. 297–301, 1966.