Discrete Applied Mathematics 200 (2016) 79–94

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Supereulerian graphs with width s and s-collapsible graphs

Ping Li [a], Hao Li [b], Ye Chen [c], Herbert Fleischner [d], Hong-Jian Lai [e,*]

[a] Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China
[b] Department of Mathematics, Renmin University of China, Beijing 100872, China
[c] Bioinformatics and Systems Biology Core, Systems Biology Center, National Heart, Lung, and Blood Institute, National Institutes of Health, Bethesda, MD 20892, United States
[d] Faculty of Informatics, DBAI, TU Wien, Austria
[e] Department of Mathematics, West Virginia University, Morgantown, WV 26506, United States

ARTICLE INFO

Article history:
Received 31 July 2013
Received in revised form 29 June 2015
Accepted 13 July 2015
Available online 6 August 2015

Keywords:
Supereulerian graphs
Collapsible graphs
Edge-connectivity
Edge-disjoint trails
Supereulerian graphs with width s
The supereulerian width of a graph
s-collapsible graphs
Eulerian-connected graphs

ABSTRACT

For an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$-trail-system of G is a subgraph H consisting of s edge-disjoint (u, v)-trails. A graph is **supereulerian with width** s if for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(s; u, v)$-trail-system. The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G is supereulerian with width k for every integer k with $0 \leq k \leq s$. Thus a graph G with $\mu'(G) \geq 2$ has a spanning Eulerian subgraph. Catlin (1988) introduced collapsible graphs to study graphs with spanning Eulerian subgraphs, and showed that every collapsible graph G satisfies $\mu'(G) \geq 2$ (Catlin, 1988; Lai et al., 2009). Graphs G with $\mu'(G) \geq 2$ have also been investigated by Luo et al. (2006) as Eulerian-connected graphs. In this paper, we extend collapsible graphs to s-collapsible graphs and develop a new related reduction method to study $\mu'(G)$ for a graph G. In particular, we prove that $K_{3,3}$ is the smallest 3-edge-connected graph with $\mu' < 3$. These results and the reduction method will be applied to determine a best possible degree condition for graphs with supereulerian width at least 3, which extends former results in Catlin (1988) and Lai (1988).

1. Introduction

Graphs in this paper are finite and may have multiple edges but no loops. Terminology and notation not defined here can be found in [3]. In particular, for a graph G, $\delta(G)$, $\Delta(G)$, $\kappa(G)$ and $\kappa'(G)$ represent the minimum degree, the maximum degree, the connectivity and the edge connectivity of a graph G, respectively. For subgraphs H_1, H_2 of G, $H_1 \bigcup H_2$ and $H_1 \bigcap H_2$ denote the union and intersection of H_1 and H_2, respectively, as defined in [3]. For vertices $u, v \in V(G)$, a trail with end vertices being u and v will be called a (u, v)-trail. We use $O(G)$ to denote the set of all odd degree vertices in G. A graph G is **Eulerian** if $O(G) = \emptyset$ and G is connected, and is **supereulerian** if G has a spanning Eulerian subgraph.

Let G be a graph, and $s > 0$ be an integer. For any distinct $u, v \in V(G)$, an $(s; u, v)$-**trail-system** of G is a subgraph H consisting of s edge-disjoint (u, v)-trails. A graph is **supereulerian with width** s if for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(s; u, v)$-trail-system. The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G is supereulerian with width k for any integer k with $1 \leq k \leq s$. Luo et al. in [19] defined graphs with $\mu'(G) \geq 2$ as **Eulerian-connected**

* Corresponding author.
E-mail addresses: pingli@bjtu.edu.cn (P. Li), lihao1982@gmail.com (H. Li), yechent@gmail.com (Y. Chen), fleisch@dbai.tuwien.ac.at (H. Fleischner), hongjianlai@gmail.com (H.-J. Lai).

http://dx.doi.org/10.1016/j.dam.2015.07.013

graphs and investigated, for a given integer $r > 0$, the minimum value $\psi(r)$ such that if G is a $\psi(r)$-edge-connected graph, then for any $X \subseteq E(G)$ with $|X| \le r$, $\mu'(G-X) \ge 2$. Note that if for some vertices u and v, G does not have a spanning (u, v)-trail, then $\mu'(G) = 0$. The vertex counter-part of $\mu'(G)$, called the spanning connectivity of a graph, has been intensively studied, as can be seen in Chapters 14 and 15 of [11].

Following [3], if $V' \subseteq V(G)$ is a vertex subset, then $G[V']$ is the subgraph of G induced by V'; if $X \subseteq E(G)$ is an edge subset, then $G[X]$ is the subgraph of G induced by X. If $v \in V(G)$, then $N_G(v)$ denotes the vertices of G adjacent to v in G. If H is a graph and Z is a set of edges such that the end vertices of each edge in Z are in $V(H)$, then $H + Z$ denotes the graph with vertex set $V(H)$ and edge set $E(H) \bigcup Z$.

In [2], Boesch et al. first raised the problem of characterizing supereulerian graphs. They remarked that such a problem would be difficult. In [20], Pulleyblank confirmed the remark by showing that the problem to determine if a graph is supereulerian, even within planar graphs, is NP-complete. Jaeger [12] first proved that every 4-edge-connected graph is supereulerian. In [4], Catlin introduced collapsible graphs as a tool to study supereulerian graphs. Catlin [4] and Lai et al. [16] showed that if G is collapsible, then $\mu'(G) \ge 2$. (See also Chapter 3 of [21] and [26].) Most of the studies on supereulerian graphs with width at most 2 can be found in Catlin's survey [5] and its updates [9,15]. By definition, we have the obvious inequality

$$\mu'(G) \le \kappa'(G), \quad \text{for any connected graph } G. \tag{1}$$

Determining when equality holds in (1) is one of the most natural questions. One purpose of this paper is to investigate graphs G such that for a given integer k, $\mu'(G) \ge k$ if and only if $\kappa'(G) \ge k$. Motivated by Catlin's work in [4], in Section 2 we extend the concept of collapsible graphs to s-collapsible graphs, and use it to develop a new reduction method using s-collapsible graphs. In Section 3, we study the s-collapsibility of complete graphs and some other dense graphs, and prove that $K_{3,3}$ is the smallest among all 3-edge-connected graphs G such that $\mu'(G) < \kappa'(G)$. In the last section, we apply the reduction method associated with s-collapsible graphs to study the structure of reduced graphs under a degree condition. These allow us to obtain a best possible degree condition for supereulerian graphs with width at least 3, extending former results in [4] and [13].

2. Reductions with s-collapsible graphs

Throughout this paper, we adopt the convention that any graph is 0-edge-connected, and so $\kappa'(G) \ge 0$ holds for any graph G, and let $s \ge 1$ denote an integer. For sets X and Y, the **symmetric difference** of X and Y is

$$X \Delta Y = \left(X \bigcup Y\right) - \left(X \bigcap Y\right).$$

Definition 2.1. A graph G is s-**collapsible** if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning subgraph Γ_R such that

(i) both $O(\Gamma_R) = R$ and $\kappa'(\Gamma_R) \ge s - 1$, and
(ii) $G - E(\Gamma_R)$ is connected.

A spanning subgraph Γ_R of G with both properties in Definition 2.1 is an (s, R)-**subgraph** of G. Let $\mathcal{C}_s$ denote the collection of s-collapsible graphs. Then $\mathcal{C}_1$ is the collection of all collapsible graphs, defined in [4]. By definition, any $(s+1, R)$-subgraph of G is also an (s, R)-subgraph of G. This implies that

$$\mathcal{C}_{s+1} \subseteq \mathcal{C}_s, \quad \text{for any positive integer } s. \tag{2}$$

Proposition 2.2. *Let G be a graph, and let $s \ge 1$ be an integer. Then the following are equivalent.*

(i) $G \in \mathcal{C}_s$.
(ii) *For any $X \subseteq V(G)$ with $|X| \equiv 0 \pmod 2$, G has a spanning connected subgraph L_X such that $O(L_X) = X$ and such that $\kappa'(G - E(L_X)) \ge s - 1$.*

Proof. (i) $\Longrightarrow$ (ii). Given $X \subseteq V(G)$ with $|X| \equiv 0 \pmod 2$, let $R = O(G)\Delta X$. By the definition of R, it follows that $|R| \equiv 0 \pmod 2$. Since $G \in \mathcal{C}_s$, G has a spanning subgraph Γ_R such that $O(\Gamma_R) = R$, $\kappa'(\Gamma_R) \ge s - 1$, and $G - E(\Gamma_R)$ is connected. Let $L_X = G - E(\Gamma_R)$. Then L_X is a spanning connected subgraph such that $O(L_X) = R\Delta O(G) = X\Delta O(G)\Delta O(G) = X$. Moreover $\kappa'(G - E(L_X)) = \kappa'(\Gamma_R) \ge s - 1$.

(ii) $\Longrightarrow$ (i). Given $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, let $X = R\Delta O(G)$. By the definition of X, it follows that $|X| \equiv 0 \pmod 2$. By (ii), G has a spanning connected subgraph L_X such that $O(L_X) = X$ and such that $\kappa'(G-E(L_X)) \ge s-1$. Let $\Gamma_R = G-E(L_X)$. Then both $\kappa'(\Gamma_R) \ge s - 1$ and $O(\Gamma_R) = O(G)\Delta X = R$. As $G - E(\Gamma_R) = L_X$ is connected, $G \in \mathcal{C}_s$. □

For a graph G, and for $X \subseteq E(G)$, the **contraction** G/X is obtained from G by identifying the two ends of each edge in X and then by deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$. When H is connected, we use v_H to denote the vertex in G/H onto which H is contracted.

Lemma 2.3. *Suppose that H is a connected subgraph of G, and $R \subseteq V(G)$ is a subset with $|R| \equiv 0 \pmod 2$. Define*

$$R' = \begin{cases} R - V(H) & \text{if } |R \bigcap V(H)| \equiv 0 \pmod 2 \\ (R - V(H)) \bigcup \{v_H\} & \text{if } |R \bigcap V(H)| \equiv 1 \pmod 2. \end{cases}$$

If G/H has an (s, R')-subgraph $\Gamma_{R'}$, and if $H \in \mathcal{C}_s$, then G has an (s, R)-subgraph Γ_R.

Proof. Let $\Gamma_{R'}$ be an (s, R')-subgraph of G/H. Define $R^* = V(H) \bigcap O(G[E(\Gamma_{R'})])$. Thus R^* consists of the vertices in H that are incident with an odd number of edges in $E(\Gamma_{R'})$. By the definition of R', $|R^*| \equiv d_{\Gamma_R}(v_H) \equiv |R \bigcap V(H)| \pmod 2$. Define $R'' = R^* \Delta (R \bigcap V(H))$. By definition, $|R''| \equiv |R^*| + |R \bigcap V(H)| \equiv 0 \pmod 2$ and $R'' \subseteq V(H)$. Since $H \in \mathcal{C}_s$, H has an (s, R'')-subgraph $\Gamma_{R''}$. Define

$$\Gamma_R = G\left[E(\Gamma_{R'}) \bigcup E(\Gamma_{R''})\right].$$

Since $\kappa'(\Gamma_{R'}) \geq s - 1$ and $\kappa'(\Gamma_{R''}) \geq s - 1$, and as $\Gamma_R/\Gamma_{R''} = \Gamma_{R'}$ when $s \geq 2$, we conclude that $\kappa'(\Gamma_R) \geq s - 1$. By the definition of R' and R'', we observe that $O(\Gamma_R) - V(H) = R - V(H)$; since $R \bigcap V(H) \subseteq V(H)$ and $R^* \subseteq V(H)$, we have $(R^* \Delta (R \bigcap V(H))) \bigcap V(H) = R^* \Delta (R \bigcap V(H))$, and so $O(\Gamma_R) \bigcap V(H) = (O(G[E(\Gamma_{R'})]) \bigcap V(H)) \; \Delta ((R^* \Delta (R \bigcap V(H))) \bigcap V(H)) = R^* \Delta ((R^* \Delta (R \bigcap V(H))) \bigcap V(H)) = R^* \Delta R^* \Delta (R \bigcap V(H)) = (R \bigcap V(H))$. Thus

$$O(\Gamma_R) = O(G[E(\Gamma_{R'})]) \Delta O(\Gamma_{R''}) = (R - V(H)) \bigcup \left(R \bigcap V(H)\right) = R.$$

Moreover, $G - E(\Gamma_R) = G[E(G/H - E(\Gamma_{R'})) \bigcup E(H - E(\Gamma_{R''}))]$. Since $\Gamma_{R'}$ is an (s, R')-subgraph of G/H, and since $\Gamma_{R''}$ is an (s, R'')-subgraph of H, $G/H - E(\Gamma_{R'})$ contains a spanning tree of G/H and $H - E(\Gamma_{R''})$ contains a spanning tree of H. It follows that $G - E(\Gamma_R)$ contains a spanning tree of G, and so by definition, Γ_R is an (s, R)-subgraph of G. □

Corollary 2.4. *Let $s \geq 1$ be an integer. Then $\mathcal{C}_s$ satisfies the following.*
(C1) *$K_1 \in \mathcal{C}_s$.*
(C2) *If $G \in \mathcal{C}_s$ and if $e \in E(G)$, then $G/e \in \mathcal{C}_s$.*
(C3) *If H is a subgraph of G and if $H, G/H \in \mathcal{C}_s$, then $G \in \mathcal{C}_s$.*

Proof. (C1) and (C2) follow immediately from definitions, and (C3) follows from Lemma 2.3. □

Corollary 2.5. *Let $s \geq 1$ be an integer. If a graph $G \in \mathcal{C}_s$, then $\mu'(G) \geq s + 1$.*

Proof. Let u and v be two distinct vertices of G. Let $X = \emptyset$. Since $G \in \mathcal{C}_s$, by Proposition 2.2, G has a spanning connected subgraph L_X with $O(L_X) = \emptyset$ and $\kappa'(G - E(L_X)) \geq s - 1$. Since L_X is Eulerian, L_X can be partitioned into two edge-disjoint (u, v)-trails T_1, T_2. By the edge version of Menger's Theorem, $G - E(L_X)$ has $s - 1$ edge-disjoint (u, v)-paths, $T_3, T_4, \ldots, T_{s+1}$. Since $T_1 \bigcup T_2 = L_X$ is spanning, $\{T_1, T_2, \ldots, T_{s+1}\}$ is spanning $(s + 1; u.v)$-trail-system. □

A subgraph H of G is $\mathcal{C}_s$-**maximal** if $H \in \mathcal{C}_s$ and if G has no subgraph in $\mathcal{C}_s$ that properly contains H.

Lemma 2.6. *Let G be a graph and let $s > 0$ be an integer. Each of the following holds.*
(i) *Let L_1, L_2 be vertex induced subgraphs of G. If $V(L_1) \bigcap V(L_2) \neq \emptyset$ and if $L_1, L_2 \in \mathcal{C}_s$, then $L_1 \bigcup L_2 \in \mathcal{C}_s$.*
(ii) *The graph G has a unique set of vertex disjoint $\mathcal{C}_s$-maximal subgraphs $H_1, H_2, \ldots, H_c$ such that $V(G) = \bigcup_{i=1}^{c} V(H_i)$, and if $G' = G/(\bigcup_{i=1}^{c} E(H_i))$, then G' contains no nontrivial subgraph in $\mathcal{C}_s$.*

Proof. (i) Let $J_1, J_2, \ldots, J_t$ be the connected components of $L_1 \bigcap L_2$. Since $L_1 \in \mathcal{C}_s$, by Corollary 2.4(C2), $L_1/(L_1 \bigcap L_2) \in \mathcal{C}_s$. Let v_{J_j} be the vertex in $L_1/(L_1 \bigcap L_2)$ onto which J_j is contracted, $(1 \leq j \leq t)$, and let X be a set of $t - 1$ additional edges, (i.e. $X \bigcap E(G) = \emptyset$), such that the graph with vertices $\{v_{J_1}, \ldots, v_{J_t}\}$ and edge set X is a tree. Since $L_1/(L_1 \bigcap L_2) \in \mathcal{C}_s$, it follows by definition of s-collapsible graphs that $L_1/(L_1 \bigcap L_2) + X \in \mathcal{C}_s$, and so by Corollary 2.4(C2), $(L_1/(L_1 \bigcap L_2) + X)/X \in \mathcal{C}_s$. By definition of contraction and since L_1, L_2 are vertex induced connected subgraphs of G, we have

$$\left(L_1 \bigcup L_2\right)/L_2 = \left(L_1/\left(L_1 \bigcap L_2\right) + X\right)/X \in \mathcal{C}_s.$$

It follows from $L_2 \in \mathcal{C}_s$ and by Corollary 2.4(C3) that $L_1 \bigcup L_2 \in \mathcal{C}_s$.
(ii) The existence and the uniqueness of this set of $\mathcal{C}_s$-maximal subgraphs $H_1, H_2, \ldots, H_c$ follow from Corollary 2.4(C1) and from (i). Let $V(G') = \{u_1, u_2, \ldots, u_c\}$, where u_i is the vertex onto which the subgraph H_i is contracted, $(1 \leq i \leq c)$. Suppose that G' has a nontrivial subgraph $H' \in \mathcal{C}_s$. We may assume that $V(H') = \{u_1, u_2, \ldots, u_t\}$ with $t \geq 2$. Then by repeated applications of Corollary 2.4(C3),

$$H = G\left[E(H') \bigcup \left(\bigcup_{i=1}^{t} E(H_i)\right)\right] \in \mathcal{C}_s,$$

contrary to the assumption that these H_i's are $\mathcal{C}_s$-maximal. □

A graph is $\mathcal{C}_s$-**reduced** if it contains no nontrivial subgraph in $\mathcal{C}_s$. By Lemma 2.6, the graph $G' = G/(\bigcup_{i=1}^{c} E(H_i))$ is $\mathcal{C}_s$-reduced; call it the $\mathcal{C}_s$-**reduction** of G.

Corollary 2.7. *Let $s \geq 1$ be an integer. Let T be a spanning tree of a graph G. If for any $e \in E(T)$, e lies in a subgraph $H_e \in \mathcal{C}_s$, then $G \in \mathcal{C}_s$.*

Proof. The hypothesis implies that G has a nontrivial subgraph in $\mathcal{C}_s$. Let H be a subgraph of G such that $H \in \mathcal{C}_s$ with $|V(H)|$ being maximized. If $G = H$, then we are done. Assume that $|V(H)| < |V(G)|$. Since T is a spanning tree, there must be an edge $e \in E(T) - E(H)$ such that e is incident with a vertex in H. By assumption, G has a subgraph $H_e \in \mathcal{C}_s$ such that $e \in E(H_e)$. Since $V(H) \bigcap V(H_e) \neq \emptyset$, by Lemma 2.6(i), $H \bigcup H_e \in \mathcal{C}_s$, contrary to the maximality of H. Hence we must have $G = H$ in $\mathcal{C}_s$. □

Lemma 2.8. *Let $s \geq 1$ be an integer. Suppose that H is a connected subgraph of a given graph G, and let v_H denote the vertex in G/H onto which H is contracted. For any $x \in V(G)$, define $x' = x$ if $x \in V(G) - V(H)$ and $x' = v_H$ if $x \in V(H)$. If $H \in \mathcal{C}_s$, then for any $u, v \in V(G)$ with $u \neq v$, the following are equivalent.*

(i) *G has a spanning $(s+1; u, v)$-trail-system.*

(ii) *If $u' \neq v'$, then G/H has a spanning $(s+1; u', v')$-trail-system; and if $u' = v' = v_H$, then G/H is supereulerian.*

Proof. (i) $\Longrightarrow$ (ii). Let $T_1, T_2, \dots T_{s+1}$ be edge-disjoint (u, v)-trails in G such that $\bigcup_{i=1}^{s+1} T_i$ is spanning in G. For $i \in \{1, 2, \dots, s+1\}$, define T_i' to be the graph obtained from $(T_i \bigcup H)/H$ by deleting the possible isolated vertex v_H. Then in G/H, if $u' \neq v'$, $T_1', T_2', \dots, T_{s+1}'$ are edge-disjoint (u', v')-trails. Since $\bigcup_{i=1}^{s+1} T_i$ is spanning in G, $\{T_1', T_2', \dots, T_{s+1}'\}$ is a spanning $(s+1; u', v')$-trail-system of G/H. If $u' = v'$, then since $u \neq v$ in G, we must have $u' = v' = v_H$, and so $T_1', T_2', \dots T_{s+1}'$ are edge-disjoint closed trails in G/H. Since $\bigcup_{i=1}^{s+1} T_i$ is spanning in G, $\bigcup_{i=1}^{s+1} T_i'$ is a spanning closed trail in G/H, and so G/H is supereulerian.

(ii) $\Longrightarrow$ (i). Suppose first that $u' = v' = v_H$, and G/H is supereulerian. Let T' denote a spanning closed trail in G/H and let $X' = O(G[E(T')])$. Since T' is an Eulerian subgraph of G/H, we conclude that $X' \subseteq V(H)$ and $|X'| \equiv 0 \pmod 2$. Since $H \in \mathcal{C}_s$, by Proposition 2.2, H has a spanning connected subgraph $L_{X'}$ with $O(L_{X'}) = X'$ such that $\kappa'(H - E(L_{X'})) \geq s - 1$. Thus $H - E(L_{X'})$ has $s - 1$ edge-disjoint (u, v)-paths $T_1, T_2, \dots, T_{s-1}$. Let $\Gamma = G[E(T') \bigcup E(L_{X'})]$. Since T' is spanning and connected in G/H, and since $L_{X'}$ is spanning and connected in H, Γ is a spanning connected subgraph of G with $O(\Gamma) = O(G[E(T')]) \Delta O(L_{X'}) = X' \Delta X' = \emptyset$. Thus Γ is a spanning Eulerian subgraph of G, and so Γ can be partitioned into two edge-disjoint (u, v)-trails T_s and T_{s+1}, such that $T_s \bigcup T_{s+1} = \Gamma$ is spanning in G. Note that Γ is edge-disjoint from $H - E(L_{X'})$ and from $T_1, T_2, \dots, T_{s-1}$. It follows that $\{T_1, T_2, \dots, T_{s+1}\}$ is a spanning $(s+1; u, v)$-trail-system.

Therefore we may assume that $u' \neq v'$ and $u' \neq v_H$. Choose a spanning $(s+1; u', v')$-trail-system $\{T_1', T_2', \dots, T_{s+1}'\}$ of G/H such that $d_{T_1'}(v_H) \geq d_{T_2'}(v_H) \geq \cdots \geq d_{T_{s+1}'}(v_H)$ and such that $d_{T_1'}(v_H)$ is maximized. Since the T_i''s are trails, the maximality of $d_{T_1'}(v_H)$ implies that we must have $d_{T_i'}(v_H) \leq 2$ for each i with $2 \leq i \leq s+1$. Since for each i, T_i' is a (u', v')-trail in G/H,

$$O(G[E(T_i')]) \subseteq V(H) \bigcup \{u, v\}, \quad 1 \leq i \leq s+1. \tag{3}$$

Define $Y_i = O(G[E(T_i')]) \bigcap V(H)$, $(1 \leq i \leq s+1)$. Without loss of generality, we assume that t is an integer such that $Y_i \neq \emptyset$ when $1 \leq i \leq t$, and $Y_i = \emptyset$, for all $i > t$. (If $v_H \in \{u', v'\}$, then $\{u, v\} \bigcap V(H) \neq \emptyset$ and so $t = s+1$.) For each i with $1 \leq i \leq t$, T_i' is an (u', v')-trail containing v_H, and so there must be $u_i, v_i \in Y_i$ such that $G[E(T_i')]$ contains an (u, u_i)-trail J_i and a (v_i, v)-trail J_i' such that J_i and J_i' are edge-disjoint. (If $v' = v_H$, we choose $v_i = v$ and in this case, J_i' consists of only one vertex.)

Since T_1' and T_2' are edge disjoint, the maximality of $d_{T_1'}(v_H)$ implies that $J' = T_1' \bigcup T_2'$ is an Eulerian subgraph of G/H containing $\{u', v', v_H\}$. Let $X = O(G[E(J')])$. As J' is an Eulerian subgraph of G/H, we have $X \subseteq V(H)$ and $|X| \equiv 0 \pmod 2$. Since $H \in \mathcal{C}_s$, and since $X \subseteq V(H)$ with $|X| \equiv 0 \pmod 2$, by Proposition 2.2, H has a spanning connected subgraph L_X with $O(L_X) = X$, such that $\kappa'(H - E(L_X)) \geq s - 1$.

Let $J = G[E(J') \bigcup E(L_X)]$. Then J is an Eulerian subgraph of G containing $V(H) \bigcup \{u, v\}$. Hence J can be partitioned into two edge disjoint (u, v)-trails T_1, T_2.

Since $\kappa'(H - E(L_X)) \geq s - 1$, for some permutation π on $\{3, 4, \dots, t\}$, $H - E(L_X)$ has edge-disjoint $(u_i, v_{\pi(i)})$-trails J_i'', $(3 \leq i \leq t)$. Define edge induced subgraphs as follows:

$$T_i = \begin{cases} G\left[E(J_i) \bigcup E(J_{\pi(i)'}) \bigcup E(J_i'')\right] & \text{if } 3 \leq i \leq t \\ G[E(T_i')] & \text{if } t+1 \leq i \leq s+1. \end{cases}$$

Recall that $\{T_1', T_2', \dots, T_{s+1}'\}$ is a spanning $(s+1; u', v')$-trail-system of G/H, that J_i and J_i' are subgraphs of T_i', and that the (u_i, v_i)-trails J_i'' $(3 \leq i \leq t)$ in $H - E(L_X)$ are edge-disjoint subgraphs. By the definition of the T_i's, for all $1 \leq i \leq s+1$, these T_i's are edge-disjoint (u, v)-trails. Since $V(G/H) = \bigcup_{i=1}^{s+1} V(T_i')$ and since $V(H) \subseteq V(T_1) \bigcup V(T_2)$, it follows that $\bigcup_{i=1}^{s+1} V(T_i) = V(G)$ and so $\{T_1, T_2, \dots, T_{s+1}\}$ is a spanning $(s+1; u, v)$-trail-system of G. □

Corollary 2.9. *Let G be a graph and H be a subgraph of G with $H \in \mathcal{C}_s$. Each of the following holds.*
(i) *$G \in \mathcal{C}_s$ if and only if $G/H \in \mathcal{C}_s$.*
(ii) *If $\mu'(G) \geq s+1$, then for any $e \in E(G)$, $\mu'(G/e) \geq s+1$.*
(iii) *$\mu'(G) \geq s+1$ if and only if $\mu'(G/H) \geq s+1$.*

Proof. (i) follows from Corollary 2.4. To prove (ii), we assume that $e = xy$ and use v_e to denote the vertex in G/e onto which e is contracted. Let $u, v' \in V(G/e)$ such that $u \neq v'$. We may assume that $u \neq v_e$ and so $u \in V(G)$. Define $v = v'$ if $v' \neq v_e$ and $v = x$ if $v' = v_e$. Since $\mu'(G) \geq s+1$, for any integer k with $1 \geq k \geq s+1$, G has a spanning $(k; u, v)$-trail system consisting of k edge-disjoint (u, v)-trails $L_1, L_2, \ldots, L_k$. For each $1 \leq i \leq k$, define $L_i' = (L_i + e)/\{e\}$ if $x, y \in V(L_i)$ or $L_i' = L_i$ if $|\{x, y\} \bigcap V(L_i)| \leq 1$. By definition of the L_i''s, $L_1', L_2', \ldots, L_{s+1}'$ form a spanning $(k; u, v')$-trail system in G/e. Thus (ii) must hold.

By (ii), if $\mu'(G) \geq s+1$, then $\mu'(G/H) \geq s+1$. Thus to prove (iii), we only need to assume that $\mu'(G/H) \geq s+1$ to prove $\mu'(G) \geq s+1$. Let k be an integer with $1 \leq k \leq s+1$, and let v_H denote the vertex in G/H onto which H is contracted. For any $x \in V(G)$, define $x' = x$ if $x \notin V(H)$ and $x' = v_H$ if $x \in V(H)$. For any $u, v \in V(G)$, if $u' \neq v'$, then since $\mu'(G/H) \geq s+1$, G/H has a spanning $(k; u'v')$-trail system. If $u' = v'$, then as $\mu'(G/H) \geq s+1 \geq 2$, by the definition of μ', G/H is supereulerian. It follows by Lemma 2.8 that G has a spanning $(k; u, v)$-trail system, and so as u, v are arbitrary vertices of G, $\mu'(G) \geq s+1$. □

For a graph G, let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees of G. By the well known spanning tree packing theorem of Nash-Williams [22] and Tutte [24], every $2k$-edge-connected graph must have k edge-disjoint spanning trees. (For a direct proof of this fact, see [10], or Theorems 1.1 and 1.3 of [7]). Following Catlin's notation, let $F(G, s)$ denote the minimum number of additional edges that must be added to G to result in a graph G' (possibly having multiple edges) with $\tau(G') \geq s$. The value of $F(G, s)$ has been studied and determined in [18], whose matroidal versions are proved in [14] and [17]. Catlin proved the following when $s = 2$.

Theorem 2.10 (*Catlin, Theorem 7 of [4]*). *If $F(G, 2) \leq 1$, then $G \in \mathcal{C}_1$ if and only if $\kappa'(G) \geq 2$.*

Further studies on $F(G, 2)$ can be found in [6]. We extend this theorem to all other values of s.

Theorem 2.11. *Let $s \geq 1$ be an integer. If $F(G, s+1) \leq 1$, then $G \in \mathcal{C}_s$ if and only if $\kappa'(G) \geq s+1$.*

Proof. Suppose first that $G \in \mathcal{C}_s$. By Corollary 2.5 and by (1), we have $\kappa'(G) \geq \mu'(G) \geq s+1$.

Conversely, we assume that $\kappa'(G) \geq s+1$ to prove that $G \in \mathcal{C}_s$. By Theorem 2.10, we may assume that $s > 1$. Let $n = |V(G)|$.

Since $F(G, s+1) \leq 1$, G has spanning trees $T_1, T_2, \ldots, T_s$ such that $J = G - \bigcup_{i=1}^{s} E(T_i)$ is a spanning subgraph of G with at most two components. Let $X \subseteq V(G)$ be a subset with $|X| \equiv 0 \pmod 2$. By Proposition 2.2, it suffices to show that G has a spanning connected subgraph L_X with $O(L_X) = X$ and with $\kappa'(G - E(L_X)) \geq s-1$.

Claim 1. *If for some i with $1 \leq i \leq s$, $T_i \bigcup J \in \mathcal{C}_1$, then $G \in \mathcal{C}_s$.*

Suppose that $H = T_1 \bigcup J \in \mathcal{C}_1$. Then $V(H) = V(T_1) = V(G)$. By Proposition 2.2, as $H \in \mathcal{C}_1$, H has a spanning connected subgraph L_X with $O(L_X) = X$. Note that $V(L_X) = V(H) = V(G)$. Since $G - E(L_X)$ contains spanning trees $T_2, \ldots, T_s$, we have $\kappa'(G - E(L_X)) \geq s-1$. By Proposition 2.2 again, $G \in \mathcal{C}_s$. This proves Claim 1.

By Theorem 2.10 and by Claim 1, if J is connected, then $G \in \mathcal{C}_s$ and we are done. Hence J has two components J' and J''. For each i with $1 \leq i \leq s$, let $H_i = T_i \bigcup J$. By Claim 1, we may assume that for each i, $H_i \notin \mathcal{C}_1$. By definition, $F(H_i, 2) = 1$, for $1 \leq i \leq s$, and so by Theorem 2.10, we may assume that for all i, $\kappa'(H_i) = 1$. Thus for each i with $1 \leq i \leq s$, there must be an edge $e_i \in E(T_i)$ which is a cut edge of H_i, such that if T_i', T_i'' are the components of $T_i - e_i$, then $V(J') = V(T_i')$ and $V(J'') = V(T_i'')$. It follows that $\{e_1, e_2, \ldots, e_s\}$ is an edge cut of G separating $V(J')$ and $V(J'')$, contrary to the assumption that $\kappa'(G) \geq s+1$. Hence we may assume that $\kappa'(H_1) \geq 2$. By Theorem 2.10, $H_1 \in \mathcal{C}_1$. By Claim 1, we conclude that $G \in \mathcal{C}_s$. □

We need a theorem of Nash-Williams to derive a corollary of Theorem 2.11. For an explicit proof of this theorem, see Theorem 2.4 of [25].

Theorem 2.12 (*Nash-Williams [23]*). *Let G be a graph. If $\frac{|E(G)|}{|V(G)|-1} \geq s+1$, then G has a nontrivial subgraph L with $\tau(L) \geq s+1$.*

Corollary 2.13. *Let G be a connected nontrivial graph, and $s \geq 1$ be an integer.*
(i) *If $\tau(G) \geq s+1$, then $G \in \mathcal{C}_s$.*
(ii) *If G is $\mathcal{C}_s$-reduced, then for any nontrivial subgraph H of G, $\frac{|E(H)|}{|V(H)|-1} < s+1$.*
(iii) *If $\kappa'(G) \geq s+1$ and G is $\mathcal{C}_s$-reduced, then*

$$F(G, s+1) = (s+1)(|V(G)|-1) - |E(G)| \geq 2.$$

Proof. (i) If $\tau(G) \geq s+1$, then $F(G, s+1) = 0$ and $\kappa'(G) \geq \tau(G) \geq s+1$. By Theorem 2.11, $G \in \mathcal{C}_s$.

(ii) Assume that G is $\mathcal{C}_s$-reduced and for some connected subgraph H of G, $\frac{|E(H)|}{|V(H)|-1} \geq s+1$. Then by Theorem 2.12, H (and so G) has a nontrivial subgraph L with $\tau(L) \geq s+1$. It follows from Corollary 2.13(i) that $L \in \mathcal{C}_s$, contrary to the assumption that G is $\mathcal{C}_s$-reduced.

(iii) The formula $F(G, s+1) = (s+1)(|V(G)|-1) - |E(G)|$ follows from Lemma 3.1 of [14] (or indirectly, from Theorem 3.4 of [18]). Since G is nontrivial and $\mathcal{C}_s$-reduced, $G \notin \mathcal{C}_s$. Now the inequality follows from Theorem 2.11. □

The following theorem of Chen is useful when dealing with graphs with small order.

Theorem 2.14 (*Chen [8]*). *If G satisfies $\kappa'(G) \geq 3$ and $|V(G)| \leq 11$, then $G \in \mathcal{C}_1$ if and only if G cannot be contracted to the Petersen graph.*

3. Complete graphs and other examples

In this section, we shall study the $\mathcal{C}_s$ membership and the μ' values of certain graphs, which will be useful in our arguments in later sections. For a graph G, if $X, Y \subseteq V(G)$ are disjoint vertex subsets, then $[X, Y]_G$ denotes the set of edges in G with one end in X and the other end in Y. We start with a simple example. For an integer $\ell > 1$, and a graph H, lH denotes the graph obtained from H by replacing each edge of H by a set of ℓ parallel edges joining the same pair of vertices. For example, ℓK_2 is the loopless connected graph with two vertices and ℓ edges. By Corollaries 2.5 and 2.13 and as $\mu'(G) \leq \kappa'(G)$ for any graph G, we have

Corollary 3.1. *Let $\ell \geq 2, s \geq 1$ be integers. Then $\ell K_2 \in \mathcal{C}_s$ if and only if $\ell \geq s+1$.*

Next we consider the problem to determine the values of n such that $K_n \in \mathcal{C}_s$, for a given integer $s \geq 1$.

Lemma 3.2. *Let $n \geq 2, s \geq 2$ be integers.*
(i) *If both n and s are odd and if $sn > n^2 - 3n + 3$, then $K_n \notin \mathcal{C}_s$.*
(ii) *If at least one of n and s is even, and if $sn > n^2 - 3n + 2$, then $K_n \notin \mathcal{C}_s$.*

Proof. In the proofs below, for each n satisfying the inequalities, we will choose a particular $R \subseteq V(K_n)$, and assume that if K_n has an (s, R)-subgraph Γ, then a contradiction will be obtained.

(i) Take $R \subset V(G)$ with $|R| = n-1 \equiv 0 \pmod 2$. Since Γ is an (s, R)-subgraph, by Definition 2.1, we have $\kappa'(\Gamma) \geq s-1$, $s-1 \equiv 0 \pmod 2$ and $O(\Gamma) = R$. Thus for any $v \in R$, we must have $d_\Gamma(v) \geq s$. It follows that $2|E(\Gamma)| = \sum_{v \in V(\Gamma)} d_\Gamma(v) \geq s(n-1) + (s-1) = sn-1$. As $sn > n^2 - 3n + 3$, we have

$$|E(K_n) - E(\Gamma)| \leq \frac{n(n-1)}{2} - \frac{sn-1}{2} < \frac{(n^2-n) - (n^2-3n+3-1)}{2} = n-1.$$

Hence $K_n - E(\Gamma)$ cannot be connected, contrary to the assumption that Γ is an (s, R)-subgraph of K_n.

(ii) Set $R = V(K_n)$ if $s \equiv 1 \pmod 2$, and $R = \emptyset$ if $s \equiv 0 \pmod 2$. Then since $\kappa'(\Gamma) \geq s-1$, we have $\delta(\Gamma) \geq s$, and so $2|E(\Gamma)| \geq sn$. Since $sn > n^2 - 3n + 2$, we have

$$|E(K_n) - E(\Gamma)| \leq \frac{n(n-1)}{2} - \frac{sn}{2} < \frac{(n^2-n) - (n^2-3n+2)}{2} = n-1.$$

Hence $K_n - E(\Gamma)$ cannot be connected, contrary to the assumption that Γ is an (s, R)-subgraph of G. □

Theorem 3.3. *Let $s \geq 2$ and $n \geq 2$ be integers. Then $K_n \in \mathcal{C}_s$ if and only if $n \geq s+3$.*

Proof. By Corollary 2.5 and (1), if $K_n \in \mathcal{C}_s$, then $\kappa'(K_n) \geq s+1$. Thus if $n < s+1$, then $K_n \notin \mathcal{C}_s$. Since $s \geq 2$, if $s+1 \leq n \leq s+2$, then by simple elementary computation in the respective two cases, we obtain $sn > n^2 - 3n + 3$, and so by Lemma 3.2, $K_{s+1}, K_{s+2} \notin \mathcal{C}_s$. This completes the proof of necessity.

To prove sufficiency, we first consider $n > s+3$. Note that K_n/K_{s+3} contains a spanning tree isomorphic to $K_{1,n-(s+3)}$ with the contraction image of K_{s+3} being a vertex of degree $n-(s+3)$, such that every edge e of this spanning tree lies in a subgraph $H_e \cong (s+3)K_2$. By Corollaries 3.1 and 2.7, $K_n/K_{s+3} \in \mathcal{C}_s$. Thus if we can show $K_{s+3} \in \mathcal{C}_s$, then it follows from Corollary 2.4(C3) that $K_n \in \mathcal{C}_s$.

Let $n = s+3$ and denote $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Then as $s \geq 2$, $n = s+3 \geq 5$. Let $R \subseteq V(K_n)$ be a subset with $|R| \equiv 0 \pmod 2$. We shall show that for any possible values of $|R|$, K_n always has an (s, R)-subgraph Γ_R.

In the arguments below, we will utilize the fact that if $n-3 > \frac{n}{2}$, then the quadratic function $x(n-x) - 3x$ has minimum value $n-4$ when $1 \leq x \leq \frac{n}{2}$. As for integer value n, we have $n-3 > \frac{n}{2}$ if and only if $n \geq 7$, we first consider the cases when $n \geq 7$.

Case 1. $n = 2k + 1$, for some integer $k \geq 3$.

For each even subset $R \subset V(G)$ with $|R| = 2\ell \geq 0$ with $0 \leq \ell \leq k$, we will find an (s, R)-subgraph Γ_R below. By symmetry and since $n \geq 7$ is odd, we may assume that $v_1 \notin R$, and when $\ell > 0$, $R = \{v_i, v_{2k-i+3} : i = 2, 3, 4, \ldots, \ell + 1\}$. Let $C_n = v_1 v_2 \ldots v_n v_1$ be a hamiltonian cycle of K_n. Since $s = n - 3$, $K_n - E(C_n)$ is an s-edge-connected, s-regular graph. If $|R| = 0$, then define $\Gamma_R = K_n - E(C_n)$; if $\ell > 0$, then define $M_{(\ell)} = \{v_i v_{2k-i+2} : \text{with } i = 2, 3, \ldots, \ell\} \bigcup \{v_{\ell+1} v_{2k}\}$. Note that $M_{(k)}$ is a perfect matching of $K_n - E(C_n) - v_1$, and observe that $M_{(\ell)} \bigcap E(C_n) = \emptyset$. Let $\Gamma_R = K_n - E(C_n) - M_{(\ell)}$. We claim that

$$\kappa'(\Gamma_R) \geq n - 4 = s - 1. \tag{4}$$

Let X, Y be a vertex partition of $V(K_n) = V(\Gamma_R)$ with $|X| = x$ and $|Y| = n - x$ such that $1 \leq x \leq n - x$. Then in $[X, Y]_{K_n}$, there are at most $2x$ edges in C_n and at most x edges in $M_{(\ell)}$. It follows that $|[X, Y]_{\Gamma_R}| \geq x(n - x) - 3x \geq n - 4$, where $1 \leq x \leq n/2$, and so (4) must hold.

By the definition of R, we have $O(\Gamma_R) = R$; as $G - E(\Gamma_R)$ contains the hamiltonian cycle C_n, it is connected. These, together with (4), imply that $K_n \in \mathcal{C}_s$.

Case 2. $n = 2k$, for some integer $k \geq 4$.

By symmetry and since n is even, we may assume that if $|R| = 2l > 0$, then $R = \{v_1, v_{k+1}, \ldots, v_\ell, v_{k+l}\}$. Let $M_1 = \{v_i v_{k+i} : i = 1, 2, \ldots, k\}$, $M_2 = \{v_i v_{k+i+1} : i = 1, 2, \ldots, k - 1\} \bigcup \{v_k v_{k+1}\}$, and $M_3 = \{v_i v_{k+i+2} : i = 1, 2, \ldots, k - 2\} \bigcup \{v_{k-1} v_{k+1}, v_k v_{k+2}\}$. Then M_1, M_2, M_3 are mutually edge disjoint perfect matchings of K_n. Let $L = G[M_1 \bigcup M_2 \bigcup M_3]$, and define

$$\Gamma_R = \begin{cases} K_n - E(L) & \text{if } |R| = 0, \\ K_n - E(L - \{v_i v_{k+i} : 1 \leq i \leq \ell\}) & \text{if } |R| = 2\ell \text{ for some } 0 < \ell \leq k. \end{cases}$$

We claim that

$$\kappa'(\Gamma_R) \geq \kappa'(\Gamma_R) \geq n - 4 = s - 1. \tag{5}$$

Let X, Y be a vertex partition of $V(K_n) = V(\Gamma_R)$ with $|X| = x$ and $|Y| = n - x$ such that $1 \leq x \leq n - x$. Then in $[X, Y]_{K_n}$, there are at most x edges in each M_i. It follows that $|[X, Y]_{\Gamma_R}| \geq x(n - x) - 3x \geq n - 4$, and so (5) must hold.

By the definition of Γ_R, we have $O(\Gamma_R) = R$; as $G - E(\Gamma_R)$ contains a hamiltonian cycle $v_1 v_{k+2} v_k v_{k+1} v_{k-1}\ v_{2k} v_{k-2}\ v_{2k-1} \cdots v_2 v_{k+3} v_1$, whose edge set is $M_2 \bigcup M_3$, it is connected. These, together with (5), imply that $K_n \in \mathcal{C}_s$.

Case 3. $n \in \{5, 6\}$.

For $n = 5$, we have $s = 2$; let $C_5 = v_1 v_3 v_5 v_2 v_4 v_1$. Define

$$\Gamma_R = \begin{cases} C_5 & \text{if } R = \emptyset, \\ C_5 \bigcup \{v_3 v_4\} & \text{if } R = \{v_3, v_4\}, \\ \left(C_5 \bigcup \{v_3 v_4\}\right) - v_2 v_5 & \text{if } R = \{v_2, v_3, v_4, v_5\}. \end{cases}$$

In any case, $O(\Gamma_R) = R$ and both Γ_R and $G - E(\Gamma_R)$ are connected. By symmetry and by definition, $K_5 \in \mathcal{C}_2$.

Suppose that $n = 6$ and so $s = 3$. Let $C_6 = v_1 v_2 v_3 v_4 v_5 v_6 v_1$, and $H = C_6 + v_2 v_5$. Define

$$\Gamma_R = \begin{cases} C_6 & \text{if } R = \emptyset, \\ H & \text{if } R = \{v_2, v_5\}, \\ H \bigcup \{v_4 v_6\} & \text{if } R = \{v_2, v_4, v_5, v_6\}, \\ H \bigcup \{v_1 v_3, v_4 v_6\} & \text{if } R = V(K_6). \end{cases}$$

In any case, we have $O(\Gamma_R) = R$ with $\kappa'(\Gamma_R) \geq 2$ such that $G - E(\Gamma_R)$ is connected. By symmetry and by definition, $K_6 \in \mathcal{C}_3$. □

Example 3.1. We present some examples G with $\kappa'(G) = \mu'(G) = 3$. Let $C_n = v_1 v_2 \cdots v_n v_1$ denote a cycle on n vertices and let $v_0 \notin \{v_1, v_2, \ldots, v_n\}$ be a vertex. The **wheel** on $n + 1$ vertices, denoted by W_n, is obtained from C_n and v_0 by adding n new edges $v_0 v_i$, $(1 \leq i \leq n)$. These new edges $v_0 v_i$, $(1 \leq i \leq n)$, are referred to as spokes of W_n. The graph W_n' is obtained from W_n by contracting a spoke. Isomorphically, we can write $W_n' = W_n/\{v_0 v_n\}$. The following can be routinely verified (hint: apply Corollary 2.9(ii) for Example 3.1(ii)).

(i) $\mu'(K_n) = \kappa'(K_n) = n - 1$.
(ii) if $G \in \{W_n, W_n'\}$ for $n \geq 3$, then $\mu'(G) = \kappa'(G) = 3$.

4. $K_{3,3}$ is the smallest graph G with $\mu'(G) < \kappa'(G) = 3$

The main result of this section will determine the smallest graph G with $\mu'(G) < \kappa'(G) = 3$. For a vertex $v \in V(G)$, define

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

We start by quoting a conditional reduction lemma; its proof is straightforward.

Table 1
$\mu'(K^+_{2,3}) \geq 3$.

u	v	Spanning $(3; u, v)$-trail system	Similar cases by symmetry
v_1	v_2	$\{e_1\}, \{e'_1, e_2, e'_2\}, \{e_3, v_4v_5, v_5v_2\}$	$v \in \{v_3, v_4\}, u = v_1$
v_1	v_5	$\{e_1, v_2v_5\}, \{e_2, v_3v_5\}, \{e_3, v_4v_5\}$	
v_2	v_3	$\{e_1, e_2\}, \{e'_1, e_3, e'_3, e'_2\}, \{v_2v_5, v_5v_3\}$	$(u, v) \in \{(v_2, v_4), (v_3, v_4)\}$
v_2	v_5	$\{v_2v_5\}, \{e_1, e_2, v_3v_5\}, \{e'_1, e_3, v_4v_5\}$	$(u, v) \in \{(v_3, v_5), (v_4, v_5)\}$

Lemma 4.1 (*Lemma 5.4.1 of [17]*). *Let G be a graph and let $H = 2K_2$ be a subgraph of G. Denote $V(H) = \{z_1, z_2\}$ and $E(H) = \{e_1, e_2\}$. Suppose that*

$$|E_G(z_i) - E(H)| \leq 2, \quad \text{for each } i = 1, 2. \tag{6}$$

Let v_H denote the vertex in G/H onto which H is contracted. For each vertex $v \in V(G)$, define $v' = v$ if $v \in V(G) - V(H)$ and $v' = v_H$ if $v \in V(H)$. Each of the following holds for any $u, v \in V(G)$.

(i) If $\{u', v'\} - \{v_H\} \neq \emptyset$, and if G/H has a spanning $(3; u', v')$-trail-system, then G has a spanning $(3; u, v)$-trail-system.

(ii) If $\{u, v\} = \{z_1, z_2\}$ and if $G - E(H)$ has a spanning (u, v)-trail, then G has a spanning $(3; u, v)$-trail-system.

A subgraph $2K_2$ of G is a **contractible** $2K_2$ of G if it satisfies (6) and Lemma 4.1(ii).

Example 4.1. Let C_n be a cycle on $n \geq 3$ vertices. Then $\forall e \in E(2C_n)$, repeat the application of Lemma 4.1 to digons not containing e to result in a $4K_2$. This shows that $\mu'(2C_n - e) = 3$.

Lemma 4.2. *Let $K_{3,3}$, $K^+_{2,3}$, $K'_{2,4}$, $K''_{2,4}$, $K'''_{2,4}$, and $S(2, 1)$ be the graphs depicted in Fig. 1A. Each of the following holds.*

(i) $\mu'(K_{3,3}) = 2$.

(ii) For each $G \in \{K^+_{2,3}, K'_{2,4}, K''_{2,4}, K'''_{2,4}\}$, $\mu'(G) = 3$.

(iii) If G is a non-hamiltonian graph spanned by a $S(2, 1)$, and if $\kappa'(G) \geq 3$, then $\mu'(G) = 3$.

Proof. We shall use the notations in Fig. 1A in the proofs.

(i) By Theorem 2.10, $K_{3,3} \in \mathcal{C}_1$, and so by Corollary 2.5, $\mu'(K_{3,3}) \geq 2$. It suffices to show that for some $u, v \in V(K_{3,3})$, $K_{3,3}$ does not have a spanning $(3; u, v)$-trail-system.

Suppose that $K_{3,3}$ has a spanning $(3; v_1, v_3)$-trail-system $\{P_1, P_2, P_3\}$. Let $e_1 = v_1v_2$, $e_2 = v_1v_4$, and $e_3 = v_1v_6$; and $f_1 = v_3v_2$, $f_2 = v_3v_4$ and $f_3 = v_3v_6$. Since P_1, P_2, P_3 are edge-disjoint, we must have

$$|\{e_1, e_2, e_3\} \bigcap E(P_i)| = 1 = |\{f_1, f_2, f_3\} \bigcap E(P_i)|, \quad \forall i \in \{1, 2, 3\}. \tag{7}$$

By (7), we may assume that $e_i \in E(P_i)$, $(1 \leq i \leq 3)$. If $f_1 \notin E(P_1)$, then since $K_{3,3}$ is 3-regular, P_1 must use v_2v_5, which will force f_1 lying in no P_i's, contrary to (7). Therefore, we must have $f_1 \in E(P_1)$. Similarly, we must have $f_2 \in E(P_2)$ and $f_3 \in E(P_3)$. Since $v_5 \notin V(P_i)$, $(1 \leq i \leq 3)$, it follows that $K_{3,3}$ does not have a spanning $(3; v_1, v_3)$-trail-system, and so $\mu'(K_{3,3}) = 2$. This proves (i).

(ii) To show that $\mu'(K^+_{2,3}) = 3$, by (1), it suffices to show that for any distinct $u, v \in V(K^+_{2,3})$ and any integer $1 \leq s \leq 3$, there will always be a spanning $(s; u, v)$-trail system. Since $\tau(K^+_{2,3}) = 2$, it follows by Corollaries 2.13 and 2.5 that $\mu'(K^+_{2,3}) \geq 2$. Table 1 shows that we can always find spanning $(3; u, v)$-trail systems for any $u, v \in V(K^+_{2,3})$. This proves that $\mu'(K^+_{2,3}) = 3$. The proofs for the cases when $G \in \{K'_{2,4}, K''_{2,4}, K'''_{2,4}\}$ are similar but somewhat more elaborate, and will thus be omitted. This proves (ii).

(iii) Let G be a minimally 3-edge-connected non-hamiltonian graph spanned by an $S(2, 1)$, and let $\tilde{G}$ be the underlying simple graph of G. We adopt the labels of $S(2, 1)$ in Fig. 1A, and denote $e_1 = v_1v_2$, $e_2 = v_1v_3$, $e_3 = v_3v_5$, $e_4 = v_1v_4$, $e_5 = v_5v_6$. If e_i has a duplicated edges, then we assume that e_i, e'_i are parallel edges in the discussions below. Since G is not hamiltonian,

$$v_2v_3 \notin E(G), \quad \text{and for any } i \in \{2, 3\} \text{ and for any } j \in \{4, 6\}, v_iv_j \notin E(G). \tag{8}$$

Since G is minimally 3-edge-connected, and by (8),

$$\text{for every } i \in \{2, 3\}, \text{ there exists exactly one } j \in \{1, 5\} \text{ such that } v_iv_j \text{ is a parallel edge in } G. \tag{9}$$

By (9) and by symmetry, we assume that v_1, v_2 are joined by parallel edges e_1 and e'_1.

Case 1. $\tilde{G} = S(2, 1)$ and v_1, v_3 are joined by parallel edges e_2, e'_2.

If v_1, v_4 are also joined by parallel edges, then by $\kappa'(G) \geq 3$, either $G[\{v_4, v_6\}]$ or $G[\{v_5, v_6\}]$ is a contractible $2K_2$; and contracting this $2K_2$ results in a graph isomorphic to $K^+_{2,3}$. By Lemma 4.2(ii), and by Lemma 4.1, $\mu'(G) = 3$. Hence we assume that $G[\{v_1, v_4\}] \cong K_2$. Then by $\kappa'(G) \geq 3$, we have $G[\{v_4, v_6\}] \cong G[\{v_5, v_6\}] \cong 2K_2$, and both are contractible $2K_2$. Contracting these $2K_2$ results in a graph $J(4)$, depicted in Fig. 1B, with

$$V(J(4)) = \{v_1, v_2, v_3, v_4\} \quad \text{and} \quad E(J(4)) = \{e_1, e'_1, e_2, e'_2, v_1v_4, v_2v_4, v_3v_4\}. \tag{10}$$

It is routine to verify that $\mu'(J(4)) = 3$, and so by Lemma 4.1, $\mu'(G) = 3$. This proves Case 1.

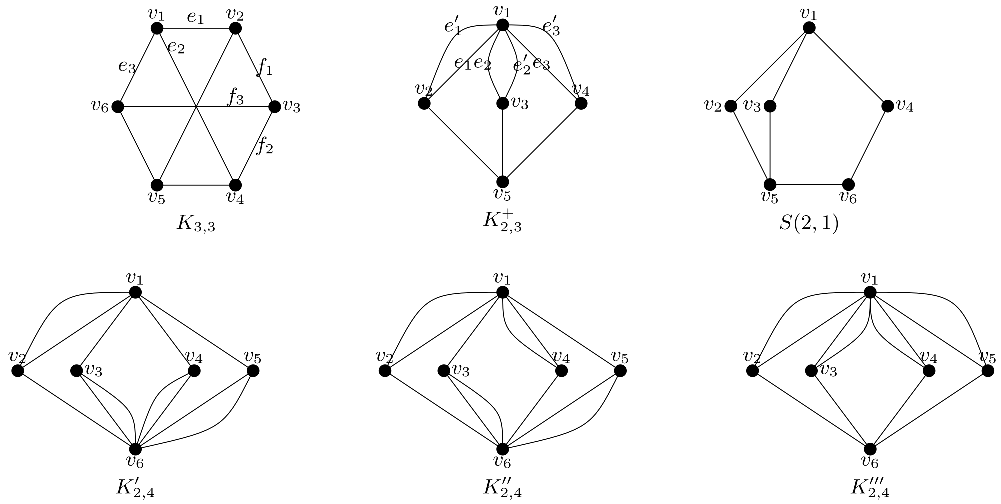

Fig. 1A. Graphs $K_{3,3}$, $K_{2,3}^{+}$, $S(2, 1)$, $K'_{2,4}$, $K''_{2,4}$ and $K'''_{2,4}$.

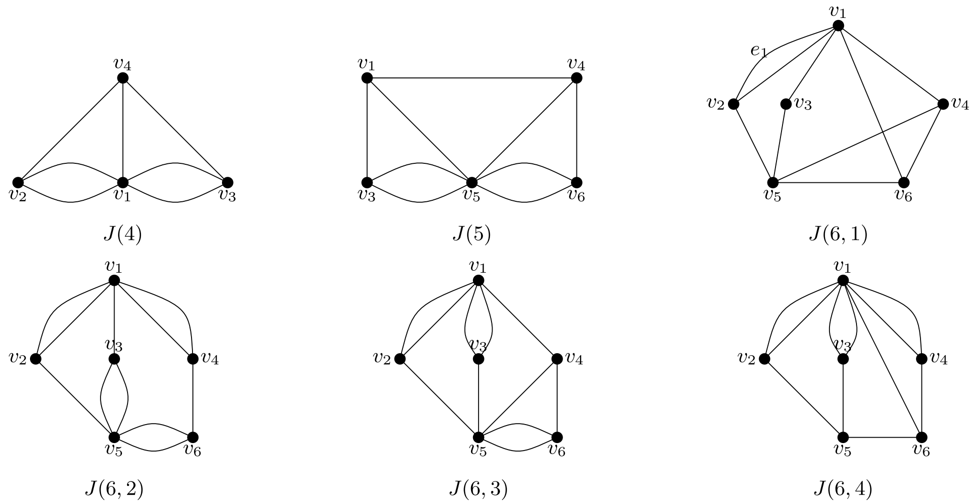

Fig. 1B. Graphs $J(4)$, $J(5)$ and $J(6, i)$, $1 \le i \le 4$.

Case 2. $\tilde{G} = S(2, 1)$ and v_1, v_3 are not joined by parallel edges.

By (9), v_3, v_5 are joined by parallel edges e_3, e'_3. If $G[\{v_4, v_6\}] \cong 2K_2$, then as G is minimally 3-edge-connected, either $G[\{v_1, v_4\}] \cong 2K_2$ or $G[\{v_5, v_6\}] \cong 2K_2$. In the first case, $G[\{v_3, v_5\}]$ and $G[\{v_4, v_6\}]$ are contractible $2K_2$'s; in the second case, $G[\{v_1, v_2\}]$ and $G[\{v_4, v_6\}]$ are contractible $2K_2$'s. As contracting the corresponding $2K_2$'s results in a graph isomorphic to $J(4)$ defined in (10), and as $\mu'(J(4)) = 3$, it follows by Lemma 4.1 that $\mu'(G) = 3$. Hence we may assume that $G[\{v_4, v_6\}] \cong K_2$, and so by $\kappa'(G) \ge 3$, we have both $G[\{v_1, v_4\}] \cong 2K_2$ and $G[\{v_5, v_6\}] \cong 2K_2$. In order for $G[\{v_1, v_4\}]$ not to be a contractible $2K_2$, we must have $G[\{v_1, v_2\}] \cong 2K_2$. Thus $G \cong J(6, 2)$ depicted in Fig. 1B. Now it is routine to verify that $\mu'(G) = 3$. This proves Case 2.

In the cases of Cases 3, 4, and 5, $\tilde{G}$ differs from $S(2, 1)$ but contains $S(2, 1)$ as a spanning subgraph.

Case 3. $\tilde{G} \ne S(2, 1)$ and $v_1v_6, v_4v_5 \in E(\tilde{G})$.

Then either e_2, e'_2 are parallel edges joining v_1, v_3 or e_3, e'_3 are parallel edges joining v_3, v_5 in G. Define $J(6, 1)$, depicted in Fig. 1B, as follows:

$$V(J(6, 1)) = V(S(2, 1)), \quad \text{and} \quad E(J(6, 1)) = E(S(2, 1)) \bigcup \{e'_1, v_1v_6, v_4v_5\}, \tag{11}$$

and define $G'_2 = J(6, 1) + e'_2$ and $G''_2 = J(6, 1) + e'_3$. By the assumption of Case 3, and since G is minimally 3-edge-connected, we have $G \in \{G'_2, G''_2\}$. It is routine to verify that $\mu'(G) = 3$. This proves Case 3.

Case 4. $\tilde{G} \ne S(2, 1)$ and $v_4v_5 \in E(\tilde{G})$ and $v_1v_6 \notin E(\tilde{G})$.

If $G[\{v_4, v_6\}] \cong 2K_2$, then $G[\{v_4, v_6\}]$ is always a contractible $2K_2$. It follows that either $G[\{v_1, v_3\}] \cong 2K_2$, whence $\{v_4v_5, v_5v_6\}$ induces another contractible $2K_2$ in $G/G[\{v_4, v_6\}]$; or $G[\{v_1, v_3\}] \cong K_2$, whence $G[\{v_3, v_5\}] \cong 2K_2$ and

$G[\{v_1, v_2\}]$ is a contractible $2K_2$ in G. After contracting these contractible $2K_2$'s, we obtain a graph isomorphic to $J(4)$ defined in (10). As we already know that $\mu'(J(4)) = 3$, by Lemma 4.1, $\mu'(G) = 3$.

Hence we assume that $G[\{v_4, v_6\}] \cong K_2$. Then by $\kappa'(G) \geq 3$, $G[\{v_5, v_6\}] \cong 2K_2$. Thus either $G[\{v_3, v_5\}] \cong 2K_2$, or $G[\{v_1, v_3\}] \cong 2K_2$. If $G[\{v_3, v_5\}] \cong 2K_2$, then $G[\{v_1, v_2\}]$ is a contractible $2K_2$, and $G/G[\{v_1, v_2\}] \cong J(5)$, depicted in Fig. 1B, with

$$V(J(5)) = \{v_1, v_3, v_4, v_5, v_6\} \quad \text{and} \quad E(J(5)) = \{v_1v_3, e_3, e'_3, e_5, e'_5, v_1v_4, v_1v_5, v_4v_5\}. \tag{12}$$

If $G[\{v_1, v_3\}] \cong 2K_2$, then $G = S(2, 1) + \{e'_1, e'_2, e'_5, v_4v_5\}$, which is the graph $J(6, 3)$ depicted in Fig. 2B. It is routine to verify that $\mu'(G) = 3$.

Case 5. $\tilde{G} \neq S(2, 1)$ and $v_4v_5 \notin E(\tilde{G})$ and $v_1v_6 \in E(\tilde{G})$.

If $G[\{v_4, v_6\}] \cong 2K_2$, then $G[\{v_4, v_6\}]$ is a contractible $2K_2$. By $\kappa'(G) \geq 3$, either $G[\{v_3, v_5\}] = 2K_2$ or $G[\{v_1, v_3\}] = 2K_2$. If $G[\{v_3, v_5\}] = 2K_2$, then all the $2K_2$'s in G are contractible, and contracting all these contractible $2K_2$'s results in a $J(4)$. Thus by $\mu'(J(4)) \geq 3$ and Lemma 4.1, $\mu'(G) = 3$ in this case. If $G[\{v_1, v_3\}] = 2K_2$, then $G/G[\{v_4, v_6\}] \cong K^+_{2,3}$. By Lemma 4.2(ii), $\mu'(K^+_{2,3}) = 3$, and so by Lemma 4.1, $\mu'(G) = 3$.

Therefore, we assume that $G[\{v_4, v_6\}] \cong K_2$. Then by $\kappa'(G) \geq 3$, $G[\{v_1, v_4\}] \cong 2K_2$. By $\kappa'(G) \geq 3$, either $G[\{v_3, v_5\}] = 2K_2$ or $G[\{v_1, v_3\}] = 2K_2$. If $G[\{v_3, v_5\}] = 2K_2$, then $G[\{v_3, v_5\}]$ is contractible, and $G/G[\{v_3, v_5\}] \cong J(5)$ defined in (12). As we already know that $\mu'(J(5)) = 3$, by Lemma 4.1, $\mu'(G) = 3$. If $G[\{v_1, v_3\}] = 2K_2$, then $G \cong (J(6, 1) + \{e'_2, e'_4\}) - v_4v_5$, where $J(6, 1)$ is defined in (11). We denote $J(6, 4) = (J(6, 1) + \{e'_2, e'_4\}) - v_4v_5$, as depicted in Fig. 1B. It is routine to verify that $\mu'(J(6, 4)) = 3$, and so by Lemma 4.1, $\mu'(G) = 3$.

By (8) and (9), these cases cover all the possibilities and so the proof of (iii) is complete. □

Lemma 4.3. *If $e \notin E(K_{3,3})$ is an edge whose ends are in $V(K_{3,3})$, and if $G = K_{3,3} + e$, then $\mu'(G) \geq 3$.*

Proof. We use the notation of Fig. 1A for $K_{3,3}$ and let $G = K_{3,3} + e$. By symmetry, we may assume that $e = v_1v_i$. If $G[v_1, v_i]$ is a contractible $2K_2$ of G, then $i \in \{2, 4, 6\}$ and $G/G[v_1, v_i]$ is isomorphic to W_4, the wheel on 5 vertices. By Example 3.1, $\mu'(W_4) = 3$ and so by Lemma 4.1, $\mu'(G) \geq 3$. Now assume that $i \in \{3, 5\}$. It is routine to show that $\mu'(G) \geq 3$. (Detailed verification can be found in Chapter 5 of [17].) □

Before proving the next theorem, we observe that, for every integer $k \geq 1$,

$$\mu'(G) \geq k \quad \text{if and only if} \quad \text{very block } H \text{ of } G \text{ satisfies } \mu'(H) \geq k. \tag{13}$$

Theorem 4.4. *Let G be a graph on n vertices.*

(i) *(Lemma 5 of [4]) If $n \leq 4$, and if $\kappa'(G) \geq 2$, then $\mu'(G) \geq 2$ if and only if $G \neq K_{2,2}$.*
(ii) *If $n \leq 6$, and if $\kappa'(G) \geq 3$, then $\mu'(G) \geq 3$ if and only if $G \neq K_{3,3}$.*

Proof of (ii). By Lemma 4.2(i), $\mu'(K_{3,3}) < 3$. It suffices to show that if $G \neq K_{3,3}$, then $\mu'(G) \geq 3$. We argue by contradiction and assume that

$$G \text{ is a counterexample with } |E(G)| + |V(G)| \text{ minimized.} \tag{14}$$

If $n \leq 3$, then $\kappa'(G) \geq 3$ implies that $F(G, 3) \leq 1$, and so in (ii), it follows from Theorem 2.11 for $s = 2$ and from Corollary 2.5 that $n \geq 4$. We claim that

$$4 \leq n \leq 6, \kappa(G) \geq 2, \quad G \text{ is } \mathcal{C}_2\text{-reduced and minimally 3-edge-connected.} \tag{15}$$

As $n \geq 4$, by assumption, $n \leq 6$, hence $4 \leq n \leq 6$. By (13) and by (14), we conclude that $\kappa(G) \geq 2$. If G has a nontrivial subgraph H with $H \in \mathcal{C}_2$, then G/H satisfies both $|V(G/H)| < 6$ and $\kappa'(G/H) \geq 3$. It follows from $|V(G/H)| \leq 5$ that $G/H \neq K_{3,3}$ and so by (14), we have $\mu'(G/H) \geq 3$. By Corollary 2.9(iii) with $s = 2$, and by $H \in \mathcal{C}_2$, we conclude that $\mu'(G) \geq 3$, contrary to (14). Thus G must be $\mathcal{C}_2$-reduced. If there exists an edge $e \in E(G)$ such that $\kappa'(G-e) \geq 3$, then by (14), we have $\mu'(G-e) \geq 3$. But $\mu'(G) \geq \mu'(G-e) \geq 3$, contrary to (14). Therefore, G must be minimally 3-edge-connected. This justifies (15).

If G has a subgraph H which is a contractible $2K_2$, then as $\kappa'(G/H) \geq \kappa'(G) \geq 3$, by (14), $\mu'(G/H) \geq 3$. By Lemma 4.1, $\mu'(G) \geq 3$, contrary to (14). Thus

$$G \text{ has no contractible } 2K_2. \tag{16}$$

By (15) and (16), we make the following observations.

Observation 1. *Let $\tilde{G}$ denote the underlying simple graph of G, and suppose that $\tilde{G}$ has a hamiltonian cycle C.*

(i) *If $\tilde{G}$ has at most one vertex of degree at least 4, then the vertices of degree 2 in $\tilde{G}$ must be an independent set of $\tilde{G}$.*
(ii) *Every edge of $\tilde{G}$ not lying in a 2-edge-cut of $\tilde{G}$ is not a parallel edge in G. For every edge cut X of size 2 in $\tilde{G}$, exactly one edge in X is a parallel edge in G.*
(iii) *Every chord of C in $\tilde{G}$ cannot have parallel edges in G.*
(iv) *Every edge of G must be in a 3-edge-cut of G.*

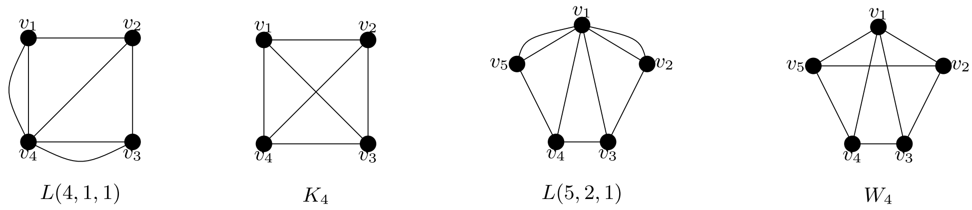

Fig. 2. Graphs in Claim 2.

In fact, if $\tilde{G}$ has two adjacent vertices (say v_1, v_2) of degree 2 in $\tilde{G}$, then since $\tilde{G}$ has at most one vertex of degree at least 4, we may assume that v_1 is not incident with a vertex of degree at least 4 in $\tilde{G}$. Since $\kappa'(G) \geq 3$, at least one edge incident with v_1 must be a parallel edge, and so by definition, G has a contractible $2K_2$, violating (16). This justifies Observation 1(i). Observation 1(ii) and (iv) follow from the assumption that G is minimally 3-edge-connected, stated in (15). Since any chord of C is not lying in a 2-edge-cut of $\tilde{G}$, Observation 1(iii) follows from Observation 1(ii).

Note that by Theorem 2.14, every such graph has a spanning Eulerian subgraph. By (15) and by $n \leq 6$, we further claim that

$$\text{every such graph } G \text{ has a Hamilton cycle } C = v_1 v_2 \cdots v_n v_1. \quad (17)$$

To justify (17), we observe that every 2-connected graph on 4 vertices must be hamiltonian, and so we assume that $n \in \{5, 6\}$. Now we proceed by contradiction. Let c be the length of a longest cycle of G. Since $\kappa(G) \geq 2$ and $n \geq 5$, we have $n > c \geq 4$.

Assume first that $c = 4$. Hence G has a $K_{2,2}$. Let $K \cong K_{2,t}$ be a subgraph of G with t maximized. For any $v \in V(G) - V(K)$, by $\kappa(G) \geq 2$, v must have two internally disjoint paths from v to K. As $c = 4$, v must be adjacent to the two vertices of degree t in $K \cong K_{2,t}$, violating the maximality of K. Hence G is spanned by a $K_{2,3}$ or a $K_{2,4}$. Since $c = 4$, G must be obtained from a $K_{2,3}$ or a $K_{2,4}$ by duplicating some edges in the $K_{2,3}$ or $K_{2,4}$, as otherwise G has a cycle longer than 4.

If G is spanned by a $K_{2,3}$, then by (16) and (15), we conclude that $G \cong K^+_{2,3}$, and so by Lemma 4.2, $\mu'(G) = 3$, contrary to (14). Now assume that G is spanned by a $K_{2,4}$. By $\kappa'(G) \geq 3$ and $c = 4$, one of the two edges incident with a vertex of degree 2 in this $K_{2,4}$ must be a parallel edge. It follows from (16) and (15) that $G \in \{K'_{2,4}, K''_{2,4}, K'''_{2,4}\}$. By Lemma 4.2(ii), we have $\mu'(G) = 3$, contrary to (14). This finishes the case when $c = 4$.

Next, we assume that $c = 5$; $n = 6$ follows from necessity. By $\kappa(G) \geq 2$, and by $c = 5$, we conclude that G is a non-hamiltonian graph spanned by an $S(2, 1)$ with $\kappa'(G) \geq 3$, and so by Lemma 4.2(iii), $\mu'(G) = 3$, contrary to (14). This justifies (17).

Recall that $\tilde{G}$ denotes the underlying simple graph of G. Let C be a hamiltonian cycle of $\tilde{G}$. Let $f(G, C) = |E(\tilde{G})| - n$ denote the number of chords of C in $\tilde{G}$. If $f(G, C) = 0$, then $G = 2C_n - e$ by (15), and so by Example 4.1, $\mu'(G) = 3$, contrary to (14). Hence $f(G, C) \geq 1$. If $n \geq 5$ and $f(G, C) = 1$, then by $\kappa'(G) \geq 3$ and by (15), it is straightforward to verify that G must have a contractible $2K_2$, violating (16). Therefore, we have

Claim 1. *When $n \geq 5$, $f(G, C) \geq 2$.*

Claim 2. *Theorem 4.4(ii) holds if $4 \leq n \leq 5$.*

We shall use the notations in Fig. 2 in our arguments below. By (16), G cannot have a contractible $2K_2$. Therefore, if $n = 4$, G must be either K_4 or $L(4, 1, 1)$ as depicted in Fig. 2. In fact, as $n = 4$, $1 \leq F(G, C) \leq 2$, where $F(G, C) = 2$ if and only if $G = K_4$. By Example 3.1, $\mu'(K_4) = 3$. We assume that $F(G, C) = 1$, and without lose of generality, that $v_2v_4 \in E(G)$ and $v_1v_3 \notin E(G)$ (see Fig. 2). By $\kappa'(G) \geq 3$, one of the two edges incident with v_1 or v_3 must have parallel edges. By (16) and (15), these parallel edges must be all incident with v_2 or all incident with v_4, and so $G \cong L(4, 1, 1)$. It is straightforward to verify that $\mu'(L(4, 1, 1)) = 3$, and so we assume $n = 5$.

By Claim 1 and (15), $2 \leq f(G, C) \leq 4$. If $f(G, C) = 4$, then one of the chords of C may be removed and the resulting graph is still 3-edge-connected, contrary to (15). Next we assume $f(G, C) = 3$. As G is spanned by a 5-cycle, $\tilde{G}$ has a vertex of degree 4. We assume that v_1 has degree 4 in $\tilde{G}$, and so $v_1v_3, v_1v_4 \in E(\tilde{G})$. By symmetry, we assume that the third chord of C in $\tilde{G}$ is v_2v_5, resulting in a wheel W_4. As W_4 is already 3-edge-connected, we conclude that if $f(G, C) = 3$, then $G = W_4$, (see Fig. 2). By Example 3.1, $\mu'(W_4) = 3$. Finally we assume that $f(G, C) = 2$. If these two chords of C are not incident with the same vertex in C, then $\Delta(\tilde{G}) = 3$. By $\kappa'(G) \geq 3$, any vertex of degree 2 in $\tilde{G}$ must be incident with parallel edges in G. As $\Delta(\tilde{G}) = 3$, G must have a contractible $2K2$, contrary to (16). Hence we may assume that v_1 has degree 4 in $\tilde{G}$ and $v_1v_3, v_1v_4 \in E(\tilde{G})$. As v_1 is the only vertex of $\tilde{G}$ with degree 4, any parallel edge not incident with v_1 must be a contractible $2K_2$. By (15) and (16), G must be isomorphic to a $L(5, 2, 1)$, (see Fig. 2). It is routine to verify that $\mu'(L(5, 2, 1)) = 3$. (Detailed verifications can be found in Chapter 5 of [17].) This completes the proof for Claim 2.

We are now ready to complete the proof of Theorem 4.4(ii). By Claim 2 and Lemma 4.3, we may assume that $n = 6$ and G is not spanned by a $K_{3,3}$. If $f(G, C) \leq 1$, then $\Delta(\tilde{G}) = 3$ with 4 vertices of degree 2, which cannot be independent, contrary

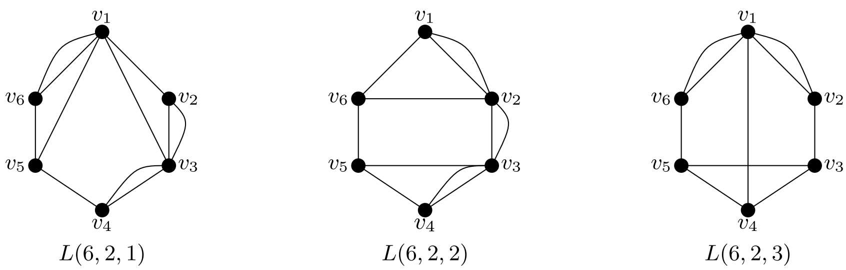

Fig. 3A. The graphs $L(6, 2, j)$ with $1 \le j \le 3$ in Case 1.

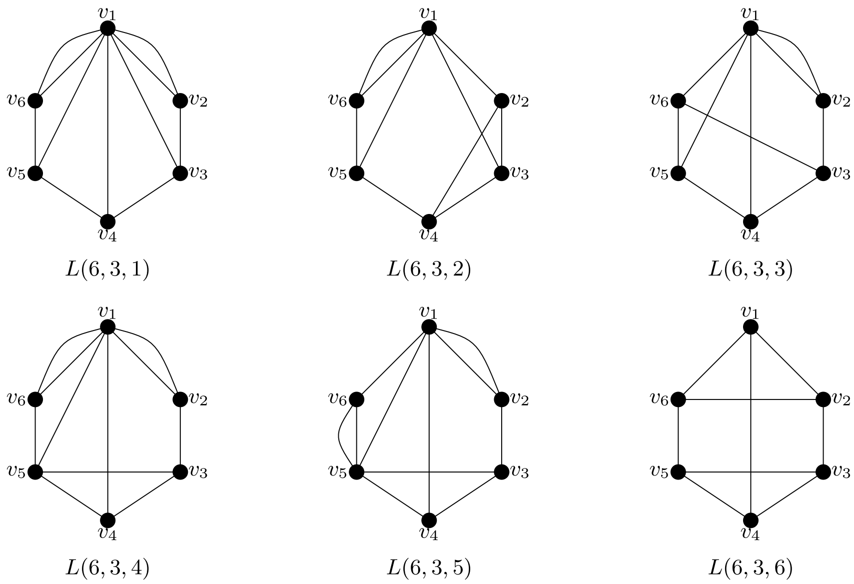

Fig. 3B. G has 6 vertices with 3 chords of C in Case 2.

to Observation 1(i). If $f(G, C) \ge 5$, then $\tilde{G}$ is not minimally 3-edge-connected, violating (15). Hence $2 \le f(G, C) \le 4$. Let $d = \Delta(\tilde{G})$.

Case 1. $f(G, C) = 2$. Then $3 \le d \le 4$.

If $d = 4$, we may assume that v_1 has degree 4. By Observation 1(i), we must have $v_1v_3, v_1v_5 \in E(\tilde{G})$. By $\kappa'(G) \ge 3$, we may assume that $G[\{v_3, v_4\}] \cong 2K_2$. By (16), we must have $G[\{v_2, v_3\}] \cong 2K_2$. By $\kappa'(G) \ge 3$, either that $G[\{v_5, v_6\}] \cong 2K_2$, which is a contractible $2K_2$ of G; or $G[\{v_1, v_6\}] \cong 2K_2$, and so $G = L(6, 2, 1)$, (see Fig. 3A).

If $d = 3$, then by symmetry and by Observation 1(i), we may assume either $v_1v_4, v_2v_5 \in E(\tilde{G})$, or $v_2v_6, v_3v_5 \in E(\tilde{G})$ or $v_1v_4, v_3v_5 \in E(\tilde{G})$. If $v_1v_4, v_2v_5 \in E(\tilde{G})$, then by Observation 1(ii), both v_1v_2 and v_4v_5 are not parallel edges in G. It follows that G will always have a contractible $2K_2$, contrary to (16). Next we assume that $v_2v_6, v_3v_5 \in E(\tilde{G})$. By $\kappa'(G) \ge 3$ and by symmetry, we may assume that $G[\{v_1, v_2\}] \cong 2K_2$. As $G[\{v_1, v_2\}]$ cannot be a contractible $2K_2$, we must have $G[\{v_2, v_3\}] \cong 2K_2$. By (15) and (16), either both $G[\{v_4, v_5\}] \cong 2K_2$ and $G[\{v_5, v_6\}] \cong 2K_2$, whence $\kappa'(G - v_3v_5) \ge 3$, contrary to (15); or $G[\{v_3, v_4\}] \cong 2K_2$, whence $G = L(6, 2, 2)$, (see Fig. 3A).

Finally we assume that $d = 3$ and $v_1v_4, v_3v_5 \in E(\tilde{G})$. It is straightforward to verify that if $G[\{v_2, v_3\}] \cong 2K_2$, then it will be a contractible $2K_2$. Thus we must have $G[\{v_1, v_2\}] \cong 2K_2$. By symmetry and (16), we also have $G[\{v_1, v_6\}] \cong 2K_2$. Hence $G = L(6, 2, 3)$, (see Fig. 3A).

Therefore, if $f(G, C) = 2$, then $G \in \{L(6, 2, 1), L(6, 2, 2), L(6, 2, 3)\}$. It is routine to verify that in any of these cases, $\mu'(G) \ge 3$. This proves Case 1.

Case 2. $f(G, C) = 3$. Then $3 \le d \le 5$.

If $d = 5$, then we may assume that $v_1v_3, v_1v_4, v_1v_5 \in E(\tilde{G})$. As before, it is routine to verify that if $G[\{v_2, v_3\}] \cong 2K_2$, then $G[\{v_2, v_3\}]$ is a contractible $2K_2$. Hence by Observation 1(ii), $G[\{v_1, v_2\}] \cong 2K_2$. By symmetry, $G[\{v_1, v_6\}] \cong 2K_2$, and so $G = L(6, 3, 1)$ (depicted in Fig. 3B).

If $d = 3$, then C has 3 independent chords in $\tilde{G}$, forcing $G \in \{K_{3,3}, L(6, 3, 6)\}$. However, $G \ne K_{3,3}$ by hypothesis, and so $G = L(6, 3, 6)$, (see Fig. 3B).

Next we suppose that $d = 4$ and v_1 has degree 4 in $\tilde{G}$. Assume first that v_1 is adjacent to v_2, v_3, v_5, v_6. If $v_3v_5 \in E(\tilde{G})$, then v_3v_5 is not in any 3-edge-cut of G; if $v_3v_6 \in E(\tilde{G})$, then v_1v_3 is not in any 3-edge-cut of G. By Observation 1(iv), neither

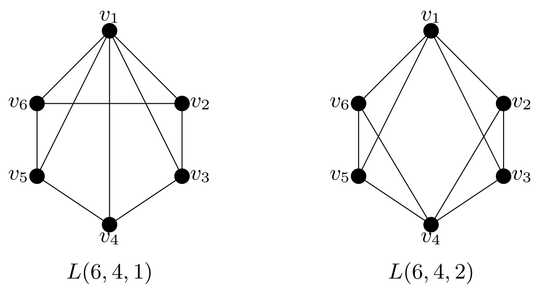

Fig. 4. *G* has at least 4 chords of *C* in Case 3.

possibility holds. By symmetry, we must have $v_2v_4 \in E(\tilde{G})$. By Observation 1(ii) and by (16), we must have $G[\{v_1, v_6\}] \cong 2K_2$, and so $G = L(6, 3, 2)$ (depicted in Fig. 3B).

Therefore, by symmetry, we may assume that v_1 is adjacent to v_2, v_4, v_5, v_6. To avoid a contractible $2K_2$, v_3 must have degree 3 in $\tilde{G}$. Hence either $v_3v_6 \in E(\tilde{G})$ or $v_3v_5 \in E(\tilde{G})$. If $v_3v_6 \in E(\tilde{G})$, then by (15) and (16), $G[\{v_1, v_2\}] \cong 2K_2$, and so $G = L(6, 3, 3)$ (depicted in Fig. 3B).

Suppose that $v_3v_5 \in E(\tilde{G})$. By (15) and (16), we must have $G[\{v_1, v_2\}] \cong 2K_2$, and either $G[\{v_1, v_6\}] \cong 2K_2$ or $G[\{v_5, v_6\}] \cong 2K_2$. It follows that $G \in \{L(6, 3, 4), L(6, 3, 5)\}$ (depicted in Fig. 3B). However, v_1v_5 is not in any 3-edge-cut of G if $G \in \{L(6, 3, 4), L(6, 3, 5)\}$, contrary to Observation 1(iv).

Therefore, if $f(G, C) = 3$, then $G \in \{L(6, 3, j) : j = 1, 2, 3, 6\}$. It is routine to verify that in any of these cases, $\mu'(G) \geq 3$. (Detailed verifications can be found in Chapter 5 of [17].)

Case 3. $f(G, C) = 4$. Then as $n = 6$ and C has at least 4 chords, $4 \leq d \leq 5$.

If $\tilde{G}$ has a vertex v of degree 2, then at least 4 edges in $E(\tilde{G}) - E(C)$ will be joining the vertices of $V(C) - \{v\}$, and so G must have at least one edge e, both of whose ends are of degree at least 4 in $\tilde{G}$, such that $\kappa'(G - e) \geq 3$. Thus G is not minimally 3-edge-connected, contrary to (15). This, together with Lemma 4.3, implies that

$$\delta(\tilde{G}) \geq 3, \text{ and } G \text{ is not spanned by a } K_{3,3} \text{ or any } L(6, 3, j) \text{ with } 1 \leq j \leq 6. \tag{18}$$

If $d = 5$, then we assume that v_1 is adjacent to all other 5 vertices of $\tilde{G}$. By (18), $\delta(\tilde{G}) \geq 3$, and so $v_2v_6 \in E(\tilde{G})$. Thus $G = L(6, 4, 1)$ (depicted in Fig. 4). Assume that $d = 4$ and that v_1 is a vertex of degree 4 in $\tilde{G}$.

Case 3.1. v_1 is adjacent to all but v_4.

By (18), $\delta(\tilde{G}) \geq 3$, and so by symmetry, we may assume that $v_2v_4 \in E(\tilde{G})$, and either v_2v_6 or $v_4v_6 \in E(\tilde{G})$. If $v_2v_6 \in E(\tilde{G})$, then $\kappa'(G - v_1v_2) \geq 3$, violating (15). Hence we have $v_4v_6 \in E(\tilde{G})$ and so $G = L(6, 4, 2)$ (depicted in Fig. 4).

Case 3.2. v_1 is adjacent to v_2, v_i, v_4, v_6, where $i \in \{3, 5\}$.

By symmetry, we may assume that $i = 3$. By (18), $\delta(\tilde{G}) \geq 3$. By Observation 1(iv), $v_2v_4 \notin E(\tilde{G})$; but also $v_3v_5, v_3v_6, v_4v_6 \notin E(\tilde{G})$, whence $v_2v_5, v_2v_6 \in E(\tilde{G})$, contrary to Observation 1(iv).

Thus in Case 3, when $f(G, C) = 4$, we must have $G \in \{L(6, 4, 1), L(6, 4, 2)\}$. It is routine to show that $\mu'(L(6, 4, 1)) = \mu'(L(6, 4, 2)) = 3$. Detailed verifications can be found in Chapter 5 of [17].

This completes the proof of the theorem. □

5. Degree condition for supereulerian graphs with larger width

Settling three open problems of Bauer in [1], Catlin and Lai proved the following.

Theorem 5.1. *Let G be a 2-edge-connected simple graph G on n vertices.*

(i) *(Catlin, Theorem 9 of [4]) If $\delta(G) > \frac{n}{5} - 1$, then for sufficiently large n, G is supereulerian.*

(ii) *(Lai, Theorem 5 of [13]) If G is bipartite, or G is triangle free, and if $\delta(G) > \frac{n}{10}$, then for sufficiently large n, G is supereulerian.*

Both bounds in Theorem 5.1 are best possible in the sense that there exist an infinite family of non-supereulerian 2-edge-connected graphs G on n vertices with $\delta(G) = \frac{n}{5} - 1$ (for Theorem 5.1(i)) and an infinite family of non-supereulerian bipartite graphs on n vertices with $\delta(G) = \frac{n}{10}$ (for Theorem 5.1(ii)). The main purpose of this section is to extend the theorem above, by using a more general argument than in the proofs in both [4] and [13]. We start with some additional notations and a preparatory lemma before presenting our main arguments. If G is a graph and G' is the $\mathcal{C}_s$-reduction of G, then for any vertex $u \in V(G')$, G has a maximal $\mathcal{C}_s$-subgraph H_u such that u is the vertex onto which H_u is contracted. The subgraph H_u is called the **preimage** of u in G. It is possible that H_u consists of a single vertex, in which case u is a **trivial vertex** of the contraction. If H is a subgraph of G, then define

$$A_G(H) = \{v \in V(H) : N_G(v) - V(H) \neq \emptyset\}.$$

Lemma 5.2. *Let n, p, c be positive integers, and $f(n,p)$ be a function of n and p such that for every fixed $p > 0$, $\lim_{n\to\infty} f(n,p) = \infty$. Suppose that G is a simple graph on n vertices such that one of the following holds:*

(i) $\delta(G) \geq f(n,p) - 1$;

(ii) *G is triangle free and* $\delta(G) \geq \frac{f(n,p)}{2}$.

Then for sufficiently large n (such that $f(n,p) \geq 2c+2$, say), any vertex u in the $\mathcal{C}_s$-reduction of G whose degree is at most c has as its preimage the maximal $\mathcal{C}_s$-subgraph H_u with

$$|V(H_u)| \geq f(n,p). \tag{19}$$

Proof. Let G' be the $\mathcal{C}_s$-reduction of G. Define $W = \{u \in V(G') : d_{G'}(u) \leq c\}$ and for each $u \in W$, choose $v \in V(H_u)$. Then $V(H_u)$ contains all vertices in $N_G(v)$ except at most c vertices in $A_G(H) \bigcup (V(G) - V(H_u))$. Hence

$$\left|\left(V(H_u) \bigcap N_G(v)\right) - A_G(H)\right| \geq d_G(v) - c. \tag{20}$$

By assumption, there exists an N such that for any $n \geq N$, $f(n,p) \geq 2c+2$. We assume that $n \geq N$ in the rest of the proof.

Suppose first that (i) holds. By (20), $|(V(H_u) \bigcap N_G(v)) - A_G(H)| \geq d_G(v) - c \geq f(n,p) - 1 - c \geq (2c+2) - 1 - c = c+1$. It follows that there exists a vertex $z \in V(H_u) \bigcap N_G(v) - A_G(H)$ such that $N_G(z) \subseteq V(H_u)$. By (i), we have $|V(H_u)| \geq |N_G(z) \bigcup \{z\}| \geq d_G(z) + 1 \geq f(n,p)$.

Now suppose that (ii) holds and so G is triangle free and $\delta(G) \geq \frac{f(n,p)}{2}$. Again by (20), $|(V(H_u) \bigcap N_G(v)) - A_G(H)| \geq d_G(v) - c \geq \frac{f(n,p)}{2} - c \geq \frac{2c+2}{2} - c > 0$. It follows that there exists a vertex $z' \in V(H_u) \bigcap N_G(v) - A_G(H)$ such that $N_G(z') \subseteq V(H_u)$. By (20) again with v replaced by z', we have $|N_G(z') - A_G(H_u)| \geq d_G(z') - c > 0$. This implies that there exists a $z'' \in N_G(z') - A_G(H_u) \subseteq V(H_u) - A_G(H_u)$. By the choices of z' and z'', we have $N_G(z') \bigcup N_G(z'') \subseteq V(H_u)$. Since G is triangle free and since $z'z'' \in E(G)$, we have $N_G(z') \bigcap N_G(z'') = \emptyset$. It follows that $|V(H_u)| \geq |N_G(z') \bigcup N_G(z'')| \geq d_G(z') + d_G(z'') \geq 2\delta(G) \geq f(n,p)$. This completes the proof of the lemma. □

Theorem 5.3. *Let n, p, s be positive integers such that $p \geq 2$. Suppose that G is a simple graph on n vertices.*

(i) *If n is sufficiently large (say $n \geq 2p((2s+2)p-2)$) and if*

$$\delta(G) \geq \frac{n}{p} - 1, \tag{21}$$

then the $\mathcal{C}_s$-reduction of G has at most p vertices.

(ii) *If G is triangle free, n is sufficiently large (say $n \geq 2p((2s+2)p-2)$), and if*

$$\delta(G) \geq \frac{n}{2p}, \tag{22}$$

then the $\mathcal{C}_s$-reduction of G has at most p vertices.

Proof. As the arguments to prove both conclusions are similar, we shall prove them simultaneously.

For given $p > 0$ and $s > 0$, choose an integer $c = (2s+2)p - 3$. Let G' be the $\mathcal{C}_s$-reduction of G, and assume that $n' = |V(G')| > 1$. Define

$$W = \{u \in V(G') : d_{G'}(u) \leq c\}.$$

Choose $f(n,p) = \frac{n}{p}$. Then as $c = (2s+2)p - 3$ and as $n \geq 2p((2s+2)p-2) = 2p(c+1)$, we have $f(n,p) \geq 2c+2$. Choose any $u \in W$ and any $z \in V(H_u)$. By Lemma 5.2, (19) must hold, and so,

$$n \geq \sum_{u\in W} |V(H_u)| \geq |W| \cdot f(n,p) = \frac{n|W|}{p}.$$

This implies that

$$|W| \leq p. \tag{23}$$

Since G' is $\mathcal{C}_s$-reduced, by Corollary 2.13(iii), we have

$$|E(G')| \leq (s+1)n' - (s+3). \tag{24}$$

By the definition of W, we have

$$2|E(G')| = \sum_{v\in V(G')} d_{G'}(v) = \sum_{v\in V(G')-W} d_{G'}(v) + \sum_{v\in W} d_{G'}(v) \geq \sum_{v\in V(G')-W} d_{G'}(v) \geq c|V(G') - W|.$$

This, together with (23) and (24), implies that $cn' - cp \leq c|V(G') - W| \leq 2|E(G')| \leq 2(s+1)n' - 2(s+3)$. Hence

$$n' \leq \frac{cp - 2(s+3)}{c - 2(s+1)}. \tag{25}$$

As $c > p(2s+2) - 4 = 2p(s+1) - 2(s+3) + 2(s+1)$, it follows that $c(p+1) > cp - 2(s+3) + 2(p+1)(s+1)$, and so algebraic manipulations lead to $(c - 2(s+1))(p+1) > cp - 2(s+3)$. This, together with (25), implies

$$n' \le \frac{cp - 2(s+3)}{c - 2(s+1)} < p + 1.$$

Hence $n' \le p$, and so the theorem follows. □

The theorem above can be applied to study the supereulerian width of some dense graphs, as shown in Corollary 5.4. By definition of $\mu'(G)$, $\mu'(G) \ge 2$ implies that G is supereulerian. It follows that when $s = 1$ and $p = 5$, Corollary 5.4 yields the results as stated in Theorem 5.1.

Corollary 5.4. *Let n, s be positive integers such that $1 \le s \le 2$. Suppose that G is a simple graph on n vertices with $\kappa'(G) \ge s+1$. Let $p(s) = 2s + 3$. Each of the following holds for sufficiently large n.*

(i) *If*

$$\delta(G) \ge \frac{n}{p(s)} - 1, \tag{26}$$

then $\mu'(G) \ge s + 1$ if and only if the $\mathcal{C}_s$-reduction of G is not a $K_{s+1,s+1}$.

(ii) *If G is triangle free, and if*

$$\delta(G) \ge \frac{n}{2p(s)}, \tag{27}$$

then $\mu'(G) \ge s + 1$ if and only if the $\mathcal{C}_s$-reduction of G is not a $K_{s+1,s+1}$.

Proof. Let $p = p(s)$. Let G be a simple graph G satisfying (26) or a triangle free graph satisfying (27). Let G' denote the $\mathcal{C}_s$-reduction of G.

If $|V(G')| = 1$, then $G' = K_1 \in \mathcal{C}_s$. By Corollary 2.4, $G \in \mathcal{C}_s$. By Corollary 2.5, $\mu'(G) \ge s + 1$. Hence we may assume that $|V(G')| > 1$.

By Theorem 5.3, there exists an integer $N_1(s)$ such that if $n \ge N_1(s)$, $|V(G')| \le p$. We shall further show that $|V(G')| \le p - 1$, for all sufficiently large n. Assume by contradiction that we always have $|V(G')| = p$. By Lemma 5.2 with $c = p$ and $f(n,p) = \frac{n+1}{p}$, we conclude that there exists an integer $N = N_2(s) \ge N_1$ such that when $n \ge N$, every vertex v in G' has a nontrivial preimage H_v with at least $\lceil f(n,p) \rceil$ vertices. It follows that

$$n = \sum_{v \in V(G')} |V(H_v)| \ge pf(n,p) = n + 1.$$

This contradiction shows that, when $n \ge N$, we must have $1 < |V(G')| \le p - 1$.

Since $p(1) = 5$ and $p(2) = 7$, by Theorem 4.4, the conclusions of Corollary 5.4(i) and (ii) must hold. □

Final Remark: There exist natural bounds of $\mu'(G)$: if $\kappa'(G) \ge 2k \ge 4$, then $\kappa'(G) \ge \mu'(G) \ge k$. It is not known to which extent this inequality can be improved. In particular, we do not know when $\kappa'(G)$ equals $\mu'(G)$.

Acknowledgments

The research of Ping Li is partially supported by National Natural Science Foundation of China (11301023) and the Fundamental Research Funds for the Central Universities (2015JBM105). The research of Hao Li is supported in part by the National Natural Science Foundation of China (No. 11301538). The research of Herbert Fleischner is supported in part by FWF Project P27615-N25.

References

[1] D. Bauer, A note on degree conditions for hamiltonian cycles in line graphs, Congres. Numer. 49 (1985) 11–18.
[2] F.T. Boesch, C. Suffel, R. Tindell, The spanning subgraphs of eulerian graphs, J. Graph Theory 1 (1977) 79–84.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[4] P.A. Catlin, A reduction method to find spanning eulerian subgraphs, J. Graph Theory 12 (1988) 29–45.
[5] P.A. Catlin, Super-Eulerian graphs, a survey, J. Graph Theory 16 (1992) 177–196.
[6] P.A. Catlin, Z. Han, H.-J. Lai, Graphs without spanning eulerian subgraphs, Discrete Math. 160 (1996) 81–91.
[7] P.A. Catlin, H.-J. Lai, Y.H. Shao, Edge-connectivity and edge-disjoint spanning trees, Discrete Math. 309 (2009) 1033–1040.
[8] Z.H. Chen, Supereulerian graphs and the Petersen graph, J. Combinat. Math. Comb. Comput. 9 (1991) 79–89.
[9] Z.H. Chen, H.-J. Lai, Reduction techniques for super-Eulerian graphs and related topics (a survey), in: Combinatorics and Graph Theory 95, vol. 1 (Hefei), World Sci. Publishing, River Edge, NJ, 1995, pp. 53–69.
[10] D. Gusfield, Connectivity and edge-disjoint spanning trees, Inform. Process. Lett. 16 (1983) 87–89.
[11] L.-H. Hsu, C.-K. Lin, Graph Theory and Interconnection Networks, CRC Press, Boca Raton, London and New York, 2009.
[12] F. Jaeger, A note on subeulerian graphs, J. Graph Theory 3 (1979) 91–93.
[13] H.-J. Lai, Contractions and hamiltonian line graphs, J. Graph Theory 12 (1988) 11–15.

[14] H.-J. Lai, P. Li, Y. Liang, J. Xu, Reinforcing a matroid to have k disjoint bases, Appl. Math. 1 (2010) 244–249.
[15] H.-J. Lai, Y. Shao, H. Yan, An update on supereulerian graphs, WSEAS Transactions on Mathematics 12 (2013) 926–940.
[16] H.-J. Lai, Y. Shao, G. Yu, M. Zhan, Hamiltonian connectedness in 3-connected line graphs, Discrete Appl. Math. 157 (2009) 982–990.
[17] P. Li, Cycles and bases in graphs and matroids (Ph.D. Dissertation), West Virginia University, 2012.
[18] D. Liu, H.-J. Lai, Z.-H. Chen, Reinforcing the number of disjoint spanning trees, Ars Combin. 93 (2009) 113–127.
[19] W. Luo, Z.-H. Chen, W.-G. Chen, Spanning trails containing given edges, Discrete Math. 306 (2006) 87–98.
[20] W.R. Pulleyblank, A note on graphs spanned by eulerian graphs, J. Graph Theory 3 (1979) 309–310.
[21] Y. Shao, Claw-free graphs and line graphs (Ph.D. Dissertation), West Virginia University, 2005.
[22] C. St, J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. Lond. Math. Soc. 36 (1961) 445–450.
[23] C. St, J.A. Nash-Williams, Decomposition of finite graphs into forest, J. Lond. Math. Soc. 39 (1964) 12.
[24] W.T. Tutte, On the problem of decomposing a graph into n connected factors, J. Lond. Math. Soc. 36 (1961) 221–230.
[25] X. Yao, X. Li, H.-J. Lai, Degree conditions for group connectivity, Discrete Math. 310 (2010) 1050–1058.
[26] S.M. Zhan, On Hamiltonian line graphs and connectivity, Discrete Math. 89 (1991) 89–95.