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Supereulerian digraphs with given local structures

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ABSTRACT

Catlin in 1988 indicated that there exist graph families $\mathcal{F}$ such that if every edge e in a connected graph G lies in a subgraph H_e of G isomorphic to a member in $\mathcal{F}$, then G is supereulerian. In particular, if every edge of a connected graph G lies in a 3-cycle, then G is supereulerian. The purpose of this research is to investigate how Catlin's theorem can be extended to digraphs. A strong digraph D is supereulerian if D contains a spanning eulerian subdigraph. We show that there exists an infinite family of non-supereulerian strong digraphs each arc of which lies in a directed 3-cycle. We also show that there exist digraph families $\mathcal{H}$ such that a strong digraph D is supereulerian if every arc a of D lies in a subdigraph H_a isomorphic to a member of $\mathcal{H}$. A digraph D is symmetric if $(x, y) \in A(D)$ implies $(y, x) \in A(D)$; and is symmetrically connected if every pair of vertices of D are joined by a symmetric dipath. A digraph D is partially symmetric if the digraph obtained from D by contracting all symmetrically connected components is symmetrically connected. It is known that a partially symmetric digraph may not be symmetrically connected. We show that symmetrically connected digraphs and partially symmetric digraphs are such families. Sharpness of these results is discussed.

1. Introduction

We consider finite graphs and digraphs. Undefined terms and notations will follow [5] for graphs and [2] for digraphs. As in [2], (u, v) represents an arc oriented from a vertex u to a vertex v. A digraph D is **simple** if D has no loops and if for any pair of distinct vertices $u, v \in V(D)$, there is at most one arc in D oriented from u to v. For an integer $n > 0$, we use K_n^* to denote the complete digraph on n vertices. Hence for every pair of distinct vertices $u, v \in V(K_n^*)$, there is exactly one arc (u, v) in $A(K_n^*)$. A cycle on n vertices is often called an n-cycle. For a digraph D, the underlying graph of D, denoted by $G(D)$, is obtained from D by erasing the orientations of all arcs of D. A **ditrail** in D is an alternating sequence of vertices and arcs such that all the arcs are distinct. If all the vertices of a ditrail are distinct we call it a **dipath**. An arc $(u, v) \in A(D)$ is **symmetric** in D if $(u, v), (v, u) \in A(D)$. A digraph D is **symmetric** if $|V(D)| = 1$ or if $|A(D)| > 0$ and every arc of D is symmetric. In particular, a symmetric dipath P is a dipath such that every arc of P is symmetric. Following [2], if $X, Y \subseteq V(D)$, then define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X, y \in Y\}.$$

When $Y = V(D) - X$, we define

$$\partial_D^+(X) = (X, V(D) - X)_D \text{ and } \partial_D^-(X) = (V(D) - X, X)_D.$$

If $X \subseteq V(D) \cup A(D)$, then $D[X]$ denotes the subdigraph induced by X. If S is a subdigraph of a digraph D and if

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$X \subset A(S)$ and $Y \subseteq A(D) - A(S)$, we use $S - X + Y$ to denote $D[(A(S) - X) \cup Y]$. For a vertex $v \in V(D)$, the **out-degree** of v is $d_D^+(v) = |\partial_D^+(\{v\})|$ and the **in-degree** of v in D is $d_D^-(v) = |\partial_D^-(\{v\})|$. Furthermore, we define

$$\partial_D(v) = \partial_D^+(\{v\}) \cup \partial_D^-(\{v\}) \text{ and } d_D(v) = d_D^+(v) + d_D^-(v).$$

Let $N_D^+(v) = \{u \in V(D) : (v, u) \in A(D)\}$ and $N_D^-(v) = \{u \in V(D) : (u, v) \in A(D)\}$ denote the **out neighbors** and **in neighbors** of v in D, respectively. When the digraph D is understood from the context, we often omit the subscript D. Following [5] and [2], we use $\lambda(D)$ and $c(D)$ to denote the arc-strong connectivity of D and the number of components of the underlying graph $G(D)$ of D, respectively. By the definition of $\lambda(D)$ in [2], for any integer $k \geq 0$,

$$\lambda(D) \geq k \text{ if and only if for any nonempty proper subset } X \subset V(D), |\partial_D^+(X)| \geq k. \quad (1)$$

Boesch, Suffel, and Tindell [4] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning eulerian subgraphs, and they indicated that this problem would be very difficult. Pulleyblank [12] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been lots of researches on this topic. Catlin [7] in 1992 presented the first survey on supereulerian graphs. Later Chen et al. [8] gave an update in 1995, specifically on the reduction method associated with the supereulerian problem. A recent survey on supereulerian graphs is given in [11].

It is natural to consider the supereulerian problem in digraphs. A strong digraph D is **eulerian** if for each vertex $v \in V(D)$, we have $d_D^+(v) = d_D^-(v)$. A strong digraph D is **supereulerian** if D contains a spanning eulerian subdigraph. One of the central problems is to characterize supereulerian digraphs.

Theorem 1.1. *(Jaeger [10] and Catlin [6].) Every 4-edge-connected graph is supereulerian.*

Theorem 1.2. *(Catlin, Corollary 1 of [6].) there exist graph families $\mathcal{F}$ such that if every edge of a connected graph G lies in a subgraph of G isomorphic to a member in $\mathcal{F}$, then G is supereulerian. In particular, if every edge of G lies in a 3-cycle of G, then G is supereulerian.*

The purpose of this paper is to see if the above-mentioned results of Jaeger and Catlin can be extended to digraphs. Firstly, we shall show that both Theorem 1.1 and Theorem 1.2 cannot be directly extended to digraphs. To be more precise, we in the next section will show that for any integer $k > 0$, there exists infinitely many non-supereulerian digraphs D with $\lambda(D) \geq k$. To show that Theorem 1.2 cannot be extended to digraphs, we need more concepts. A digraph D is **weakly connected** if $G(D)$, its underlying graph, is connected. Let $\mathcal{H}$ be a family of digraphs. A strong digraph D is **locally** $\mathcal{H}$ if every arc $a \in A(D)$ lies in a subdigraph H_a of D, where $H_a \in \mathcal{H}$. For convenience, we also call a locally $\{H\}$ digraph D as a **locally** H digraph. Let C_3 denote a directed 3-cycle. Unlike graphs, we in Section 2 will also show that a strong and locally C_3 digraph may not be supereulerian. Thus what local structures could assure supereulerian property will be the objective of this research. We will introduce the graph families of symmetrically connected digraphs and partially symmetric digraphs, in subsequent sections below, and prove the following results.

Theorem 1.3. *Each of the following holds.*

(i) *Every symmetrically connected digraph is supereulerian.*
(ii) *Every partially symmetric digraph is supereulerian.*

Moreover, in Sections 3 and 4, we will show that every weakly connected locally symmetrically connected digraph is supereulerian and every weakly connected locally partially symmetric digraph is supereulerian. The sharpness of these results are also discussed.

2. Examples of nonsupereulerian strong digraphs

As Catlin in [6] indicated that

$$\text{every connected graph in which every edge lies in a 3-cycle is supereulerian,} \quad (2)$$

it is natural to see if every strong digraph in which every arc lies in a directed 3-cycle is supereulerian. In this section, we shall present, for any integer $k > 0$, an infinite family of $\mathcal{D}$ such that every digraph in $\mathcal{D}$ is locally $\{C_3\}$ with $\lambda(D) \geq k$ but nonsupereulerian.

We need the following necessary condition for a digraph to be supereulerian. Let D be a digraph and $U \subset V(D)$. We call a collection of ditrails $P_1, P_2, \cdots, P_t$ of the induced subdigraph $D[U]$ a **cover** of U if $\cup_{i=1}^t V(P_i) = U$ and $A(P_i) \cap A(P_j) = \emptyset$, whenever $i \neq j$. The minimum value of such t is denoted by $\tau(U)$. For any subset $A \subseteq V(D) - U$, define $B =: V(D) - U - A$. Let

$$h(U, A) =: \min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + \min\{|(U, B)_D|, |(B, U)_D|\} - \tau(U).$$

Then we have the following proposition.

Proposition 2.1. *(Hong, Lai and Liu, Proposition 2.1 of [9].) If D has a spanning eulerian subdigraph, then for any $U \subset V(D)$, and for any subset $A \subseteq V(D) - U$, we have $h(U, A) \geq 0$.*

The authors in [3] and [9] have independently presented infinite families of non-supereulerian digraphs with arbitrarily high arc-strong connectivity. In those digraphs shown in [3] and [9], there exist some arcs which are not lying in a directed 3-cycle. In this section, we will construct an infinite family of non-supereulerian digraphs with arbitrarily high arc-strong connectivity such that every arc of each of these digraphs lies in a directed 3-cycle.

Example 2.1. Let $\alpha, \beta, k > 0$ be integers with $\alpha, \beta \geq k + 1$, and let A and B be two disjoint set of vertices with $|A| =$

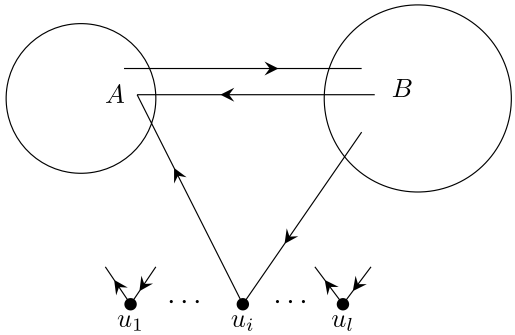

Fig. 1. The digraph $D = D(\alpha, \beta, k, \ell)$.

α and $|B| = \beta$. Let $\ell \geq \alpha\beta + 1$ be an integer, and U be a set of vertices disjoint from $A \cup B$ with $|U| = \ell$. We construct a digraph $D = D(\alpha, \beta, k, \ell)$ such that $V(D) = A \cup B \cup U$ and the arcs of D are given as required in (D1) and (D2) below. (See Fig. 1.)

(D1) $D[A \cup B] \cong K^*_{\alpha+\beta}$ is a complete digraph.
(D2) For every vertex $u \in U$, and for every $v \in A$, $(u, v) \in A(D)$ and for every $w \in B$, $(w, u) \in A(D)$. Thus for any $u \in U$, we have $N^+_D(u) = A$ and $N^-_D(u) = B$. No two vertices in U are adjacent.

A digraph D is **quasitransitive** if, for every triple of distinct vertices $x, y, z \in V(D)$, with $(x, y), (y, z) \in A(D)$, there is at least one arc between x and z. Thus the digraphs in Example 2.1 are quasitransitive. Using canonical decompositions of quasitransitive digraphs by Bang-Jensen and Huang in [1], Bang-Jensen and Maddaloni characterized supereulerian quasitransitive digraphs and further showed that there exists a polynomial algorithm to determine if a quasitransitive digraph is supereulerian in [3].

Proposition 2.2. *Let $D = D(\alpha, \beta, k, \ell)$ for some given parameters α, β, k and ℓ as defined in Example 2.1. Then each of the following holds.*

(i) $\lambda(D) > k$.
(ii) *Every arc of D lies in a directed 3-cycle.*
(iii) *D is not supereulerian.*

Proof. (i) We use (1) to show (i). Let $\emptyset \neq X \subset V(D)$ be a proper nonempty subset. Let $n_X = |X \cap (A \cup B)|$. If $0 < n_X < \alpha + \beta$, then by Example 2.1(D1), $\partial^+_D(X) \geq |(X \cap U, A - X)_D \cup (X \cap (A \cup B), (A \cup B) - X)_D| \geq n_X(\alpha + \beta - n_X) \geq \alpha + \beta - 1 \geq 2k + 1 > k$. Hence we assume that either $A \cup B \subseteq X$ or $X \cap (A \cup B) = \emptyset$. If $X \cap (A \cup B) = \emptyset$, then $X \subseteq U$, and so by Example 2.1(D2), $|\partial^+_D(X)| \geq |(X, A)_D| \geq |A| > k$. Thus we may assume that $A \cup B \subseteq X$. Then $|\partial^+_D(X)| = |(B, U - X)_D| \geq |B| > k$. Hence by (1), $\lambda(D) > k$.

(ii) Let $a = (u, v)$ be an arc in D. If $u, v \in A \cup B$, then by Example 2.1(D1), a is in a $K^*_{\alpha+\beta}$ and so a lies in a directed 3-cycle of D. Since U is an independent set of D, by Example 2.1(D2), we may assume that $u \in B$ and $v \in U$, whence for any $w \in A$, $wuvw$ is a directed 3-cycle; if $u \in U$ and $v \in A$, then for any $w' \in B$, $w'uvw'$ is a directed 3-cycle. This justifies (ii).

(iii) We apply Proposition 2.1. By (D1), $D[A \cup B] \cong K^*_{\alpha+\beta}$, and so $|\partial^+_D(A)| = \alpha\beta$. By (D2), $|(U, B)_D| = 0$ and so $\tau(U) = |U| > \alpha\beta$. Therefore we have

$$h(U, A) = |\partial^+_D(A)| + |(U, B)_D| - \tau(U) = \alpha\beta - |U| < 0.$$

It follows from Proposition 2.1 that D is not supereulerian. □

Thus Example 2.1 indicates that there exists an infinite family of non-supereulerian digraphs with arbitrarily high arc-strong connectivity such that every arc of each of these digraphs lies in a directed 3-cycle. Hence both Theorem 1.1 and Theorem 1.2 cannot be directly extended to digraphs.

3. Locally symmetrically connected supereulerian digraphs

In this section, we will introduce symmetrically connected digraphs and show that every locally symmetrically connected digraph is supereulerian. We will also show that this result is best possible in some sense.

Definition 3.1. Let D be a digraph such that either $D = K_1$ or $A(D) \neq \emptyset$. If for any $u, v \in V(D)$, D contains a symmetric dipath from u to v, then D is called a **symmetrically connected** digraph. Let $\mathcal{SC}$ be the family of all symmetrically connected digraphs.

Theorem 3.1. *Every symmetrically connected digraph is supereulerian.*

Proof. By contradiction, we assume that D is not supereulerian. By Definition 3.1 and (1), D is strong. Thus D contains a nontrivial eulerian subdigraph. Choose S to be an eulerian subdigraph of D such that

$$|V(S)| \text{ is maximized among all eulerian subdigraph of } D. \quad (3)$$

Since D is not supereulerian, we have $|V(S)| < |V(D)|$. Pick a vertex $u \in V(D) - V(S)$ and $v \in V(S)$. As $u, v \in V(D)$, and D is symmetrically connected, D contains a symmetric dipath $P = v_0 v_1 \cdots v_m$ with $v_0 = u$ and $v_m = v$. Since $u \in V(D) - V(S)$ and $v \in V(S)$, there exists a smallest integer $t > 0$ such that $v_t \in V(S)$. Since P is symmetric, the arcs $(v_1, v_0), (v_2, v_1), \cdots, (v_t, v_{t-1}) \in A(D)$. It follows that $D[A(S) \cup \{(v_0, v_1), (v_1, v_2), \cdots, (v_{t-1}, v_t), (v_1, v_0), (v_2, v_1), \cdots, (v_t, v_{t-1})\}]$ is also an eulerian subdigraph of D, contrary to (3). □

The **symmetric difference** between two digraphs D_1 and D_2, written as $D_1 \triangle D_2$, is the induced digraph on the arc set $(A(D_1) \cup A(D_2)) - (A(D_1) \cap A(D_2))$. Thus $D_1 \triangle D_2$ has no isolated vertices and contains arcs that are either in D_1 or in D_2 but not in both.

Corollary 3.1. *Every weakly connected locally $\mathcal{SC}$ digraph is supereulerian.*

Proof. Let D be a weakly connected and locally symmetrically connected digraph. We will prove that D is symmetrically connected. For any $u_1, u_k \in V(D)$, we shall follow Definition 3.1 to show there exists a symmetric dipath from u_1 to u_k. Since D is weakly connected then there exists an undirected path P in $G(D)$ lying between u_1 and u_k. If P has only one arc and since $\{(u_1, u_k), (u_k, u_1)\} \cap A(D) \neq \emptyset$, by Definition 3.1, $\{u_1, u_k\}$ are endpoints of a symmetrically connected dipath P. Then D contains a symmetric dipath from u_1 to u_k and we are done. If not, then for any continuous three vertices $\{u_{j-1}, u_j, u_{j+1}\} \in V(P)$ where $1 \leq j-1$ and $j+1 \leq k$. Since $\{(u_{j-1}, u_j), (u_j, u_{j-1})\} \cap A(D) \neq \emptyset$ and $\{(u_j, u_{j+1}), (u_{j+1}, u_j)\} \cap A(D) \neq \emptyset$, by Definition 3.1, $\{u_{j-1}, u_j\}$ are endpoints of a symmetrically connected dipath P_1 and $\{u_j, u_{j+1}\}$ are endpoints of a symmetrically connected dipath P_2. Since $u_j \in V(P_1) \cap V(P_2)$, implies that $V(P_1) \cap V(P_2) \neq \emptyset$. Then there exists a symmetrically connected dipath $P_3 \subseteq (P_1 \triangle P_2)$ with endpoints u_{j-1} and u_{j+1}. Then there exists a symmetrically connected subdigraph lies between u_{j-1} and u_{j+1}. This shows that the locally symmetrically connected digraph D is transitive for the arcs of any undirected path. Hence, by transitivity, (u_1, u_k) lies in a symmetrically connected subdigraph. This shows that D contains a symmetric dipath from u_1 to u_k implies that D is symmetrically connected. By Theorem 3.1, D is supereulerian. □

The sharpness of Corollary 3.1 will be justified in the proposition below.

Proposition 3.1. *Let H be a strong digraph with $|V(H)| = n > 1$. If there exists a vertex $v \in V(H)$ such that $d(v) \leq n-1$ and v is not incident with any symmetric arcs, then there exists an infinite family $\mathcal{F}(H)$ of strong, locally H, non-supereulerian digraphs.*

Proof. Let H be such a given digraph. Then H contains a vertex v such that v is not incident with any symmetric arcs. We have $N_H^+(v) \cap N_H^-(v) = \emptyset$.

Let $\alpha \geq |N_H^+(v)|$ and $\beta \geq |N_H^-(v)|$ be integers such that $\alpha + \beta \geq n$, and let $k = 1$, and $\ell \geq \alpha\beta + 1$ be integers. Define $D = D(\alpha, \beta, k, \ell)$ as in Example 2.1. We have the following observations, which justify the proposition.

(A) D is strong and nonsupereulerian. This observation follows from Proposition 2.2 with $k = 1$.
(B) For any arc $a \in A(D)$, there exists a subdigraph H_a of D such that H_a is isomorphic to H.

Let $a = (x, y)$ be an arc in $A(D)$. If $a \in A(D[A \cup B])$, then by (D1), $D[A \cup B] \cong K^*_{\alpha+\beta}$. It follows by the assumption that $\alpha + \beta \geq n = |V(H)|$ that $D[A \cup B]$ has a subgraph isomorphic to H which contains a. Hence by (D2), we may assume that $(x, y) \in (A \cup B, U)_D$ or $(x, y) \in (U, A \cup B)_D$. In either case, let $A' \subseteq A$ and $B' \subseteq B$ be subsets with $|A'| = |N_H^+(v)|$ and $|B'| = |N_H^-(v)|$, respectively. Then $D[A' \cup B' \cup \{x, y\}]$ contains a subdigraph isomorphic to H and contains (x, y). This proves (B) and justifies that D is locally H. □

4. Partially symmetric supereulerian digraphs

We investigate a different kind of local structural condition which warrants a digraph to be supereulerian. For any digraph D, define a relation on $V(D)$ such that $u \sim v$ if and only if $u = v$ or D has a symmetrically connected subdigraph H with $u, v \in V(H)$. If H_1 and H_2 are two subdigraphs of D, then we define $H_1 \cup H_2$ to be the subdigraph of D with vertex set $V(H_1) \cup V(H_2)$ and arc set $A(H_1) \cup A(H_2)$.

Lemma 4.1. *If H_1 and H_2 are two symmetrically connected subdigraphs of D such that $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is also a symmetrically connected subdigraph of D.*

Proof. It suffices to show that for any $u, v \in V(H_1 \cup H_2)$, $H_1 \cup H_2$ contains a symmetric dipath from u to v. If $u, v \in V(H_1)$, then since H_1 is symmetrically connected, by Definition 3.1, H_1 contains a symmetric dipath from u to v. Hence $H_1 \cup H_2$ has a symmetric dipath from u to v. Similarly, if $u, v \in V(H_2)$, then $H_1 \cup H_2$ also has a symmetric dipath from u to v.

Thus we assume that $u \in V(H_1) \backslash V(H_2)$ and $v \in V(H_2) \backslash V(H_1)$. Since $V(H_1) \cap V(H_2) \neq \emptyset$, we can take a vertex $r \in V(H_1) \cap V(H_2)$. Since H_1 and H_2 are symmetrically connected, H_1 contains a (u, r)-symmetric dipath P_1 and H_2 contains a (r, v)-symmetric dipath P_2. It follows that $D[A(P_1) \cup A(P_2)]$ contains a symmetric dipath from u to v. □

By Lemma 4.1, the relation $\sim$ is an equivalence relation on $V(D)$. Each equivalence class induces a maximal symmetrically connected subdigraph of D. We have the following observation.

Observation 4.1. *Let D be a digraph. Each of the following holds.*

(i) *D has a unique collection of maximal symmetrically connected subdigraphs.*
(ii) *If H_1 and H_2 are two maximal symmetrically connected subdigraphs, then either $H_1 = H_2$, or $V(H_1) \cap V(H_2) = \emptyset$.*

Definition 4.1. Let $c \geq 2$ be an integer and let D be a weakly connected digraph and let $\{H_1, H_2, \cdots, H_c\}$ be the set of maximal symmetrically connected subdigraphs of D.

(i) If for any proper nonempty subset $\mathcal{J} \subset \{H_1, H_2, \cdots, H_c\}$, there exist an $H_i \in \mathcal{J}$ and a vertex $v \in V(H_i)$, and an $H_j \notin \mathcal{J}$ such that

$$N_D^+(v) \cap V(H_j) \neq \emptyset \text{ and } N_D^-(v) \cap V(H_j) \neq \emptyset,$$

then D is **partially symmetric**.
(ii) Let $\mathcal{PS}$ denote the family of all partially symmetric digraphs.

For a digraph D and let $\{H_1, H_2, \cdots, H_c\}$ be the set of all symmetrically connected components of D. Define D' to be the digraph obtained from D by contracting all symmetrically connected components. By Definition 4.1,

D is partially symmetric if and only if D' is symmetrically connected. In fact, if D' is symmetrically connected, then Definition 4.1 (i) holds. Conversely, let M be a symmetrically connected component of D'. If $M \neq D'$, then $\mathcal{J} = V(M)$ is a subset of all symmetrically connected components of D, and so by Definition 4.1(i), there exists a vertex $H_i \in V(M)$ and a $H_j \in V(D') - V(M)$ such that both $(H_i, H_j), (H_j, H_i) \in A(D')$, contrary to the assumption that M is a component.

Let K_s^* and K_t^* be two complete digraphs of order $s \geq 2$ and $t \geq 2$, respectively, such that u_1, u_2 are two distinct vertices of K_s^* and v_1, v_2 are two distinct vertices of K_t^*. Let D be the digraph obtained from the disjoint union of K_s^* and K_t^* by adding the arcs (u_1, v_1) and (v_2, u_2). Then by the remark above, D is partially symmetric but not symmetrically connected.

Example 4.1. Let J_1, J_2 be digraphs with $V(J_1) = V(J_2) = \{v_1, v_2, v_3, v_4\}$ and $A(J_1) = \{(v_1, v_2), (v_2, v_1), (v_1, v_4), (v_2, v_3), (v_3, v_4), (v_4, v_3), (v_3, v_1), (v_4, v_2)\}$, and $A(J_2) = \{(v_1, v_2), (v_2, v_1), (v_1, v_3), (v_1, v_4), (v_3, v_4), (v_4, v_3), (v_3, v_2), (v_4, v_2)\}$. Then we have these observations.

(i) For $i \in \{1, 2\}$, J_i does not have a symmetric (v_2, v_3)-dipath, and so J_i is not symmetrically connected.
(ii) For $i \in \{1, 2\}$, the maximal symmetrically connected subgraph of J_i are $H_1 = D[\{v_1, v_2\}]$ and $H_2 = D[\{v_3, v_4\}]$.
(iii) By Definition 4.1(i), J_1 is partially symmetric but J_2 is not partially symmetric.
(iv) For any arc $a \in A(K_4^*) - A(J_2)$, $J_2 + a$ is symmetrically connected.

Lemma 4.2. *Let D be a partially symmetric digraph. Then each of the following holds.*

(i) *For every* $1 \leq i \leq c$, *we have that* $|V(H_i)| \geq 2$.
(ii) *D is strong.*
(iii) $|V(D)| \geq 4$.
(iv) *D does not have symmetric arcs connecting a vertex in* H_i *and a vertex in* H_j *for any distinct* $i, j \in \{1, 2, \cdots, c\}$.

Proof. Let $\{H_1, H_2, \cdots, H_c\}$ be the set of maximal symmetrically connected subgraphs of D with $c \geq 2$.

(i) By contradiction, we assume that for some maximal symmetrically connected subgraph H_i of D, $V(H_i) = \{u\}$. Let $\mathcal{J} = \{H_1, H_2, \cdots, H_c\} - \{H_i\}$. Since D is partially symmetric, there must be an $H_j \in \mathcal{J}$ and a vertex $v \in V(H_j)$ such that $(v, u), (u, v) \in A(D)$. It follows by the maximality of H_j that $u \in V(H_j)$, contrary to Observation 4.1(ii). Hence Lemma 4.2(i) must hold.

(ii) Let X be a nonempty proper subset of $V(D)$. By (1), it suffices to show that $|\partial_D^+(X)| \geq 1$. If for some i, both $X \cap V(H_i) \neq \emptyset$ and $V(H_i) - X \neq \emptyset$, then by Definition 3.1, $|\partial_D^+(X)| \geq |\partial_{H_i}^+(X \cap V(H_i))| \geq 1$. Hence we assume that no such H_i exists. Since $V(D) = \cup_{i=1}^c V(H_i)$, we may assume that for some integer m with $1 \leq m < c$, $X = \cup_{j=1}^m V(H_j)$. Since D is partially symmetric, by Definition 4.1, there exist a vertex $x \in V(H_h)$ for some $1 \leq h \leq m$ and a maximal symmetrically connected subdigraph H_r with $m+1 \leq r \leq c$ such that $N_D^+(x) \cap V(H_r) \neq \emptyset$ and $N_D^-(x) \cap V(H_r) \neq \emptyset$. This implies that $|\partial_D^+(X)| \geq 1$, and so D must be strong.

(iii) By Definition 4.1 we have that $c \geq 2$, and by (i) each maximal symmetrically connected subdigraph H_i with $|V(H_i)| \geq 2$, this shows that $|V(D)| \geq 4$.

(iv) If for some $i, j \in \{1, 2, \cdots, c\}$, D has symmetric arcs connecting a vertex in H_i and a vertex in H_j, then $H_i \cup H_j$ is symmetrically connected. By Definition 4.1, contrary to H_i is maximal symmetrically connected. Hence Lemma 4.2(iv) must hold. □

Theorem 4.1. *Every partially symmetric digraph is supereulerian.*

Proof. We argue by contradiction and assume that D is partially symmetric and

$$D \text{ is not supereulerian.} \tag{4}$$

Let $\{H_1, H_2, \cdots, H_c\}$ be the set of all maximal symmetrically connected subdigraphs of D. Since D is partially symmetric digraph, by Lemma 4.2(ii), D is strong and so D contains a nontrivial eulerian subdigraph. Choose an eulerian subdigraph S of D such that

$$|V(S)| \text{ is maximized among all eulerian subdigraphs of } D. \tag{5}$$

Since D is not supereulerian, $V(D) - V(S) \neq \emptyset$. Since D is strong, there exists an arc $(u, v) \in \partial_D^+(V(S))$.

If for some i with $1 \leq i \leq c$, $u, v \in V(H_i)$. Since H_i is symmetrically connected, by Definition 3.1, H_i has a (v, u)-dipath $P = v_1 v_2 \cdots v_k$ with $v = v_1$ and $u = v_k$ such that P is symmetric. Since $v_1 = v \notin V(S)$ and $v_k = u \in V(S)$, there exists a smallest index $i_0 > 1$ such that $v_{i_0} \in V(S)$. It follows that $D[V(S) \cup \{v_{i_0-1}, v_{i_0}\}]$ is eulerian with one more vertex than S, contrary to (5).

Therefore, there does not exist such H_i, consequently, for each maximal symmetrically connected subdigraph H_i of D, either $V(H_i) \cap V(S) = \emptyset$ or $V(H_i) \subseteq V(S)$. Without loss of generality, we assume that for some t with $1 \leq t \leq c$, $H_1, H_2, \cdots, H_t$ are contained in S and $H_{t+1} \cdots H_c$ are disjoint from $V(S)$.

Since D is partially symmetric digraphs, by Definition 4.1(i), there exist a vertex $x \in V(H_k)$ for some k with $1 \leq k \leq t$ and for some j with $t+1 \leq j \leq c$ such that $N_D^+(x) \cap V(H_j) \neq \emptyset$ and $N_D^-(x) \cap V(H_j) \neq \emptyset$. Suppose that $x' \in N_D^+(x) \cap V(H_j)$ and $x'' \in N_D^-(x) \cap V(H_j)$. Since H_j is strong, H_j has a (x', x'')-dipath $x_1 x_2 \cdots x_q$ with $x' = x_1$ and $x'' = x_q$. Since $V(H_j) \cap V(S) = \emptyset$, it follows that $C = D[A(P) \cup \{(x, x'), (x'', x)\}]$ is a dicycle of $D - A(S)$. Thus $S' = D[A(S) \cup A(C)]$ is also an eulerian subdigraph of D with $|V(S')| \geq |V(S)| + 1$, contrary to (5). □

Corollary 4.1. *Every weakly connected locally* $\mathcal{PS}$ *digraph is supereulerian.*

Proof. Let D be a weakly connected and locally partially symmetric digraph, $\{H_1, H_2, \cdots, H_c\}$ be the set of maximal symmetrically connected subdigraphs of D with $c \geq 2$.

We shall verify Definition 4.1(i) to prove that D is partially symmetric.

Let $\mathcal{J} = \{H_{i_1}, H_{i_2}, \cdots, H_{i_m}\}$ and let $\mathcal{J}' = \{H_{i_{m+1}}, H_{i_{m+2}}, \cdots, H_{i_{m+s}}\}$ with $m + s = c$. Let $X = \cup_{j=1}^{m} V(H_{i_j})$ and $Y = \cup_{k=1}^{s} V(H_{i_{m+k}})$. Since D is weakly connected then there exists an arc $a \in (X, Y)_D \cup (Y, X)_D$. By Lemma 4.2(iv), we may assume that

$$D \text{ does not have any symmetric dipath connecting a vertex in } X \text{ and a vertex in } Y. \tag{6}$$

Since D is locally partially symmetric, by Definition 4.1(ii), D contains a partially symmetric subdigraph Q of D with $a \in A(Q)$. Without loss of generality, let $a = (u, v) \in (X, Y)_D$. Since $a \in A(Q)$, we have $u \in V(Q) \cap V(X)$ and $v \in V(Q) \cap V(Y)$. Let $\{Q_1, Q_2, \cdots, Q_d\}$ be the set of maximal symmetrically connected subdigraphs of Q with $d \geq 2$. By (6) and by Definition 4.1(i), we may assume that for some index l with $1 \leq l \leq d - 1$, we have $\cup_{r=1}^{l} V(Q_{i_r}) \subseteq X$ and $\cup_{r=l+1}^{d} V(Q_{i_r}) \subseteq Y$. Since Q is partially symmetric, by Definition 4.1(i), for some h with $1 \leq h \leq l$, there exists an $x' \in V(Q_{i_h})$, and for some k with $l + 1 \leq k \leq d$, we have $N_Q^+(x') \cap V(Q_{i_k}) \neq \emptyset$ and $N_Q^-(x') \cap V(Q_{i_k}) \neq \emptyset$. Since $\{H_1, H_2, \cdots, H_c\}$ is the set of maximal symmetrically connected subdigraphs of D, there must be an s' with $1 \leq s' \leq m$ such that $x' \in V(Q_{i_h}) \subseteq V(H_{i_{s'}})$; and there must be an s'' with $m + 1 \leq s'' \leq m + s$ such that $V(Q_{i_k}) \subseteq V(H_{i_{s''}})$. Therefore, by Definition 4.1(i), D must be partially symmetric. By Theorem 4.1, D is supereulerian. $\square$

Observe that in Example 4.1 (iv), J_2 is "nearly symmetrically connected". The next example presents an infinite family of non-supereulerian digraphs, such that for each digraph D in the family, every arc of D lies in a subdigraph isomorphic to J_2. This, in some sense, indicates that Corollary 4.1 is best possible.

Example 4.2. Let $\alpha, \beta, k > 0$ be integers with $\alpha, \beta \geq k + 1$, and let A and B be two disjoint set of vertices with $|A| = \alpha$ and $|B| = \beta$. Let $\ell \geq \alpha\beta + 1$ be an integer, and U be a set of vertices disjoint from $A \cup B$ with $|U| = 2\ell$. We construct a digraph $J = J(\alpha, \beta, k, \ell)$ such that $V(J) = A \cup B \cup U$ and the arcs of J are given as required in (J1) and (J2) below.

(J1) $D[A \cup B] \cong K^*_{\alpha+\beta}$ is a complete digraph.

(J2) Write $U = \{u_1, u_1', u_2, u_2', \cdots, u_\ell, u_\ell'\}$ such that for each $1 \leq i \leq \ell$, $J[\{u_i, u_i'\}] \cong K_2^*$, $N_J^+(u_i) = N_J^+(u_i') = A$ and $N_J^-(u_i) = N_J^-(u_i') = B$.

Proposition 4.1. *Let $J = J(\alpha, \beta, k, \ell)$ for some given parameters α, β, k and ℓ as defined in Example 4.2. Then each of the following holds.*

(i) $\lambda(J) > k$.

(ii) *Every arc of J lies in a subdigraph isomorphic to J_2.*

(iii) *D is not supereulerian.*

Proof. (i) The proof (i) similar to that of Proposition 2.2(i), and will be omitted.

(ii) Let $a = (u, v)$ be an arc in D. If $u, v \in A \cup B$, then by Example 4.2(D2), a is in $K^*_{\alpha+\beta}$ and so as a lies in a subdigraph isomorphic to J_2. Thus we may assume that either $a \in A(D[\{u_i, u_i'\}])$ for some i, or $a \in (B, U)_D \cup (U, A)_D$. In any case, by Example 4.2(J2), for any $w \in A$ and $w' \in B$, and for any i with $1 \leq i \leq \ell$, $J[\{w, w', u_i, u_i'\}] \cong J_2$. Hence (ii) must hold.

(iii) We apply Proposition 2.1. By (J1), $J[A \cup B] \cong K^*_{\alpha+\beta}$, and so $|\partial_J^+(A)| = \alpha\beta$. By (J2), $|(U, B)_J| = 0$ and so $\tau(U) = |U| > \alpha\beta$. It follows that

$$h(U, A) = |\partial_J^+(A)| + |(U, B)_J| - \tau(U) = \alpha\beta - |U| < 0.$$

It follows from Proposition 2.1 that J is not supereulerian. $\square$

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