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On dense strongly $\mathbb{Z}_{2s+1}$-connected graphs

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ABSTRACT

Let G be a graph and $s > 0$ be an integer. If, for any function $b : V(G) \to \mathbb{Z}_{2s+1}$ satisfying $\sum_{v \in V(G)} b(v) \equiv 0 \pmod{2s+1}$, G always has an orientation D such that the net outdegree at every vertex v is congruent to $b(v)$ mod $2s+1$, then G is strongly $\mathbb{Z}_{2s+1}$-connected. For a graph G, denote by $\alpha(G)$ the cardinality of a maximum independent set of G. In this paper, we prove that for any integers $s, t > 0$ and real numbers a, b with $0 < a < 1$, there exist an integer $N(a, b, s)$ and a finite family $\mathcal{Y}(a, b, s, t)$ of non-strongly $\mathbb{Z}_{2s+1}$-connected graphs such that for any connected simple graph G with order $n \geq N(a, b, s)$ and $\alpha(G) \leq t$, if G satisfies one of the following conditions:

(i) for any edge $uv \in E(G)$, $\max\{d_G(u), d_G(v)\} \geq an + b$, or
(ii) for any $u, v \in V(G)$ with $dist_G(u, v) = 2$, $\max\{d_G(u), d_G(v)\} \geq an + b$,

then G is strongly $\mathbb{Z}_{2s+1}$-connected if and only if G is not contractible to a member in the finite family $\mathcal{Y}(a, b, s, t)$.

1. Introduction

We consider finite graphs without loops, but multiple edges are allowed, and we follow [2] for undefined terms and notations. In particular, for a graph G, $\kappa'(G)$, $\delta(G)$ and $\alpha(G)$ denote the edge-connectivity, the minimum degree and the cardinality of a maximum independent set of G, respectively. Throughout this paper, $s > 0$ denotes an integer and $\mathbb{Z}$ denotes the set of all integers. For an $m \in \mathbb{Z}$, let $\mathbb{Z}_m$ be the set of integers modulo m, as well as the (additive) cyclic group on m elements.

If $X \subseteq E(G)$, the **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If H is a subgraph of G, we write G/H for $G/E(H)$. If H is a connected subgraph of G and v_H is the vertex in G/H onto which H is contracted, then H is the **preimage** of v_H and is denoted by $PI_G(v_H)$.

Let D denote an orientation of G. Following [2], for each $v \in V(G)$, we use $d_D^+(v)$ and $d_D^-(v)$ to denote the out-degree and the in-degree of v under the orientation D, respectively. For an integer $m > 1$, if a graph G has an orientation D such that at every vertex $v \in V(G)$, $d_D^+(v) - d_D^-(v) \equiv 0 \pmod{m}$, then we say that G admits a **mod** m-**orientation**. The set of all graphs which have mod m-orientations is denoted by M_m. If $m = 2s$ is an even integer, then a connected graph G is in M_{2s} if and only if G is Eulerian. Hence we are only interested in the case when $m = 2s + 1$ is an odd integer.

Let A be an (additive) abelian group, and let G be a graph with an orientation $D = D(G)$. For any vertex $v \in V(G)$, let $E_D^+(v)$ denote the set of all edges directed out from v, and let $E_D^-(v)$ denote the set of all edges directed into v. For a function

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$f : E(G) \to A$, define $\partial f : V(G) \to A$, called the **boundary** of f, as follows:

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) \quad \text{for any vertex } v \in V(G).$$

A function $b : V(G) \to A$ is a **zero-sum function** on A if $\sum_{v \in V(G)} b(v) \equiv 0$, where 0 denotes the additive identity of A. The set of all zero-sum functions on A of G is denoted by $Z(G, A)$. Let A' be a subset of A. We define $F(G, A') = \{f : E(G) \to A'\}$. For any zero-sum function b on A of G, a function $f \in F(G, A')$ satisfying $\partial f = b$ is referred to as an (A', b)-flow. When $b = 0$, an $(A - \{0\}, 0)$-flow is known as a **nowhere-zero** A-**flow** in the literature (see [9,10], among others). Following [10], if for any zero-sum function b on A of G, G always has an $(A - \{0\}, b)$-flow, then G is A-**connected**.

A graph G is **strongly** $\mathbb{Z}_m$-**connected** if, under a given orientation D, for any zero-sum function b on $\mathbb{Z}_m$ of G, there exists a function $f \in F(G, \{\pm 1\})$ such that $\partial f = b$. Again, for a given $b \in Z(G, \mathbb{Z}_m)$ and an $f \in F(G, \{\pm 1\})$ with $\partial f = b$, one can keep the orientation of each edge with $f(e) = 1$ and reverse the orientation of each edge with $f(e) = -1$ to obtain a new orientation D' of G such that for any vertex $v \in V(G)$, $d^+_{D'}(v) - d^-_{D'}(v) = b(v) = \partial f(v)$. This orientation D' will be referred to as a $(\mathbb{Z}_m, b)$-orientation of G. Thus a graph G is strongly $\mathbb{Z}_m$-connected if and only if for any $b \in Z(G, \mathbb{Z}_m)$, G always has a $(\mathbb{Z}_m, b)$-orientation. By definition, a graph G is $\mathbb{Z}_3$-connected if and only if G is strongly $\mathbb{Z}_3$-connected. But for an odd number $m \geq 5$, while every strongly $\mathbb{Z}_m$-connected graph is $\mathbb{Z}_m$-connected, not every $\mathbb{Z}_m$-connected graph is strongly $\mathbb{Z}_m$-connected. In Lemma 2.7 of [14](ii), it is known that K_4 is not strongly $\mathbb{Z}_{2s+1}$-connected for any $s \geq 1$. However, as every edge of K_4 lies in a 3-cycle, it is known (see for example, [10], or Lemma 2.1 of [12]) that K_4 is $\mathbb{Z}_m$-connected for any $m \geq 4$. It has been proved [14,15] that strongly $\mathbb{Z}_{2s+1}$-connected graphs must be $2s$-edge-connected and are precisely the graphs H such that for any graph G containing H as a subgraph, G is in M_{2s+1} if and only if G/H is in M_{2s+1}. Therefore, following the notation of Catlin [3] and Catlin, Hobbs and Lai [4], the family of strongly $\mathbb{Z}_{2s+1}$-connected graphs is denoted by M^o_{2s+1}.

Tutte and Jaeger proposed the following conjectures concerning mod $(2s + 1)$-orientations. A conjecture on strongly $\mathbb{Z}_{2s+1}$-connected graphs has also been proposed recently.

Conjecture 1.1. *Let $s > 0$ denote an integer.*

(i) *(Tutte [21]) Every 4-edge-connected graph has a mod 3-orientation.*
(ii) *(Jaeger [8,10]) Every $4s$-edge-connected graph has a mod $(2s + 1)$-orientation.*
(iii) *(Jaeger [8,10]) Every 5-edge-connected graph is $\mathbb{Z}_3$-connected.*
(iv) [13,15] *Every $(4s + 1)$-edge-connected graph is strongly $\mathbb{Z}_{2s+1}$-connected.*

Conjecture 1.1(i) is well known as Tutte's 3-flow conjecture. Conjecture 1.1(ii) is an extension of Tutte's 3-flow conjecture, which includes Conjecture 1.1(i) as the special case of $s = 1$. In [11], Kochol showed that to prove Conjecture 1.1(i), it suffices to prove that every 5-edge-connected graph has a mod 3-orientation. Consequently, Conjecture 1.1(iii) implies Conjecture 1.1(i). To the best of our knowledge, all these conjectures remain open. The best known results so far have been recently obtained by Thomassen [20], Wu [22] and Lovász et al. [18].

Theorem 1.1 (*Thomassen [20]*)**.** *Every 8-edge-connected graph is $\mathbb{Z}_3$-connected.*

Theorem 1.2 (*Lovász et al. [18], Wu [22]*)**.** *Let $s > 0$ be an integer. Every $6s$-edge-connected graph is strongly $\mathbb{Z}_{2s+1}$-connected.*

Barat and Thomassen presented the first degree condition to ensure a simple graph to be $\mathbb{Z}_3$-connected. This was later improved by Fan and Zhou [5] and Luo et al. [19].

Theorem 1.3 (*Barat and Thomassen, Theorem 5.2 of [1]*)**.** *There exists a positive integer N such that every simple graph G on $n \geq N$ vertices with $\delta(G) \geq n/2$ is $\mathbb{Z}_3$-connected.*

Theorem 1.4 (*Fan and Zhou [5]*)**.** *Let G be a simple graph on $n \geq 3$ vertices such that $d_G(u) + d_G(v) \geq n$, for every pair of nonadjacent vertices u and v in G. With six exceptional graphs, G has a nowhere-zero 3-flow.*

Theorem 1.5 (*Luo et al., Theorem 1.8 of [19]*)**.** *Let G be a simple graph on $n \geq 3$ vertices such that $d_G(u) + d_G(v) \geq n$, for every pair of nonadjacent vertices u and v in G. With 12 exceptional graphs, G is $\mathbb{Z}_3$-connected.*

The results in Theorems 1.3–1.5 have the format that if a simple graph satisfies the Dirac condition or the Ore condition, then the graph has a nowhere-zero 3-flow or is $\mathbb{Z}_3$-connected, with finitely many exceptional cases. This motivates the proofs of the following results.

Theorem 1.6 (*Li and Lai, Theorem 1.6 of [16]*)**.** *Let G be a simple graph on n vertices. For any integer $s > 0$ and for any real numbers a and b with $0 < a < 1$, there exist an integer $N = N(a, b, s)$ and a finite family $\mathcal{J}(a, s)$ of non-strongly $\mathbb{Z}_{2s+1}$-connected graphs such that if $n \geq N$ and if $d_G(u) + d_G(v) \geq an + b$ for every pair of nonadjacent vertices u and v in G, then G is strongly $\mathbb{Z}_{2s+1}$-connected if and only if G cannot be contracted to a member in $\mathcal{J}(a, s)$.*

For positive integers n and s with $n \geq 2s$, let $K^+_{s,n-s}$ denote the simple graph obtained from the complete bipartite graph $K_{s,n-s}$ by adding one new edge joining two vertices of maximum degree.

Theorem 1.7 (*Fan and Zhou, Theorem 1.7 of [6]*). *Let G be a 2-edge-connected simple graph on n vertices such that $d_G(u) + d_G(v) \geq n$, for every pair of adjacent vertices u and v in G. Then G has a nowhere-zero 3-flow if and only if G is not isomorphic to a $K^+_{3,n-3}$ or to one of the 5 other exceptional graphs.*

Theorem 1.8 (*Zhang et al., Theorem 1.3 of [24]*). *Let G be a simple graph on $n \geq 3$ vertices such that $d_G(u) + d_G(v) \geq n$, for every pair of adjacent vertices u and v in G. Then G is $\mathbb{Z}_3$-connected if and only if G is not isomorphic to a member of $\{K_{2,n-2}, K^+_{2,n-2}, K_{3,n-3}, K^+_{3,n-3}\}$ or to one of the 15 other exceptional graphs.*

Theorem 1.9 (*Li et al., Theorem 1.4 of [17]*). *Let G be a simple 2-edge-connected graph on $n \geq 3$ vertices. If for every $uv \notin E(G)$, $\max\{d_G(u), d_G(v)\} \geq n/2$, then G is $\mathbb{Z}_3$-connected if and only if G is not contractible to one of 22 exceptional graphs.*

Note that if $d_G(u) + d_G(v) \geq 2(an + b)$, then $\max\{d_G(u), d_G(v)\} \geq an + b$. Hence Theorem 1.6 implies the following theorem, which, in some sense, generalizes Theorem 1.9.

Theorem 1.10. *For any integer $s > 0$ and real numbers a, b with $0 < a < 1$, there exist an integer $N = N(a, b, s)$ and a finite family $\mathcal{J}_0(a, s)$ of non-strongly $\mathbb{Z}_{2s+1}$-connected graphs such that for any connected simple graph G with order $n \geq N$, if*

$$\text{for any } uv \notin E(G),\ \ \max\{d_G(u), d_G(v)\} \geq an + b,$$

then G is strongly $\mathbb{Z}_{2s+1}$-connected if and only if G cannot be contracted to a member in $\mathcal{J}_0(a, s)$.

For vertices u, v in G, let $dist_G(u, v)$ denote the length of a shortest (u, v)-path in G.

Theorem 1.11 (*Yan, Theorem 5.4 of [23]*). *Let G be a 2-edge-connected simple graph on n vertices. If for every pair $u, v \in V(G)$ with $dist_G(u, v) = 2$, $\max\{d_G(u), d_G(v)\} \geq n/2$, then G is $\mathbb{Z}_3$-connected if and only if G is not contractible to a member of a well defined graph family and G is not isomorphic to one of 26 exceptional graphs.*

Let $B, s > 0$ be integers. We shall define an M^o_{2s+1}-reduced graph in Section 2. Let $\mathcal{F}(B, s)$ be the family of graphs such that a graph H is in $\mathcal{F}(B, s)$ if and only if $H \notin M^o_{2s+1}$, $|V(H)| \leq B$ and H is M^o_{2s+1}-reduced. As implied by Lemma 2.2, for any given positive integers B and s, $\mathcal{F}(B, s)$ is a finite family. The main results of this paper, motivated by the results above, are the following.

Theorem 1.12. *For any integers $s, t > 0$ and real numbers a, b with $0 < a < 1$, there exist integers $N = \max\{\lceil \frac{12s-b-1}{a} \rceil, \lceil \frac{24s-2b-6}{a} \rceil\}$ and $B_1 = (12s-1)(\lfloor \frac{2}{a} \rfloor + t - 1) + 1$ such that for any connected simple graph G with order $n \geq N$ and $\alpha(G) \leq t$, if*

$$\text{for every edge } uv \in E(G),\ \max\{d_G(u), d_G(v)\} \geq an + b, \tag{1.1}$$

then either G is strongly $\mathbb{Z}_{2s+1}$-connected or G is contractible to a member in $\mathcal{F}(B_1, s)$.

Theorem 1.13. *For any integers $s, t > 0$ and real numbers a, b with $0 < a < 1$, there exist integers $N = \max\{\lceil \frac{12s-b-1}{a} \rceil, \lceil \frac{24s-2b-6}{a} \rceil\}$ and $B_2 = (12s-1)(\lfloor \frac{2}{a} \rfloor + 4st - 1) + 1$ such that for any connected simple graph G with order $n \geq N$ and $\alpha(G) \leq t$, if*

$$\text{for every pair } u, v \in V(G) \text{ with } dist_G(u, v) = 2,\ \max\{d_G(u), d_G(v)\} \geq an + b, \tag{1.2}$$

then either G is strongly $\mathbb{Z}_{2s+1}$-connected or G is contractible to a member in $\mathcal{F}(B_2, s)$.

In the next section, we shall present some useful facts on strongly $\mathbb{Z}_{2s+1}$-connected graphs. In the last section, we first prove a useful lemma, which facilitates the proofs of Theorems 1.12 and 1.13 later in Section 3.

2. Some useful facts

Let $B, s > 0$ be integers. By definition, every graph in $\mathcal{F}(B, s)$ is not in M^o_{2s+1}. Lemma 2.1(i) indicates that any graph G contractible to a member in $\mathcal{F}(B, s)$ cannot be in M^o_{2s+1}.

Lemma 2.1 (*[13]*). *For any integer $s > 0$, each of the following holds.*

(i) *If G is strongly $\mathbb{Z}_{2s+1}$-connected and e is an edge of G, then G/e is strongly $\mathbb{Z}_{2s+1}$-connected.*
(ii) *If H is a subgraph of G and both H and G/H are strongly $\mathbb{Z}_{2s+1}$-connected, then so is G.*

Let $K_2^{(m)}$ denote the loopless graph with two vertices and m parallel edges. Some examples of strongly $\mathbb{Z}_{2s+1}$-connected graphs have been found in [15].

Lemma 2.2 (*Lemmas 2.2 and 2.7 of [14], and Liang [15]*). *Let G be a graph and let $m, s > 0$ be integers. Each of the following holds:*

(i) *$K_2^{(m)}$ is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $m \geq 2s$;*
(ii) *K_n is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $n = 1$ or $n \geq 4s + 1$.*

For any graph G, every vertex lies in a maximal strongly $\mathbb{Z}_{2s+1}$-connected subgraph. Let $H_1, H_2, \ldots, H_c$ denote the collection of all maximal subgraphs that are in M^o_{2s+1}. Then $G' = G/(\cup_{i=1}^{c} E(H_i))$ is the M^o_{2s+1}-**reduction** of G. If G is strongly $\mathbb{Z}_{2s+1}$-connected, then its M^o_{2s+1}-reduction is K_1, a singleton. A graph which does not have any nontrivial subgraph in M^o_{2s+1} is M^o_{2s+1}-**reduced**. Thus by definition, the M^o_{2s+1}-reduction of a graph is always M^o_{2s+1}-reduced. Even when G is a simple graph, its M^o_{2s+1}-reduction may have multiple edges. Lemma 2.2 implies that there are only finitely many nontrivial M^o_{2s+1}-reduced graphs with a given order.

Define $\overline{\kappa}'(G) = \max\{\kappa'(H) : H \subseteq G \text{ with } |E(H)| > 0\}$. The following is a corollary of Theorem 1.2.

Lemma 2.3. *Let G' be the M^o_{2s+1}-reduction of a connected graph G such that $G' \neq K_1$. Then $\overline{\kappa}'(G') \leq 6s - 1$.*

For multigraphs with bounded values of $\overline{\kappa}'$, the number of edges will also be bounded.

Lemma 2.4 (Gu et al. [7]). *Let G be a graph with order n and let $k > 0$ be an integer. If $\overline{\kappa}'(G) \leq k$, then $|E(G)| \leq (n-1)k$.*

3. Proof of the main results

For notational convenience, throughout this section, we follow the notation in [3,4] to denote the family of strongly $\mathbb{Z}_{2s+1}$-connected graphs by M^o_{2s+1}, and let G be a connected simple graph on n vertices. Let G' be the M^o_{2s+1}-reduction of G and $n' = |V(G')|$. Define

$$\begin{aligned} X_c &= \{v \in V(G') : d_{G'}(v) \leq c\}, & X'_c &= \{v \in X_c : PI_G(v) \neq K_1\}, \\ Y &= \{v \in V(G) : d_G(v) \geq an + b\}, & W &= \{v \in X'_c : PI_G(v) \cap Y \neq \emptyset\}. \end{aligned} \tag{3.3}$$

Lemma 3.1. *If $G' \neq K_1$, each of the following holds.*
(i) *If $n \geq \max\{\lceil \frac{c-b}{a} \rceil, \lceil \frac{2c-2b-2}{a} \rceil\}$, then $|W| \leq \lfloor \frac{2}{a} \rfloor$.*
(ii) *If $c = 12s - 2$, then $n' - |X_c| \leq c|X_c| - 2(6s - 1)$.*

Proof. (i) If $W = \emptyset$, $|W| \leq \lfloor \frac{2}{a} \rfloor$ holds immediately. If $W \neq \emptyset$, for any vertex $v \in W$, by the definition of W, there exists a vertex $u \in PI_G(v) \cap Y$ such that

$$|V(PI_G(v))| - 1 + c \geq d_G(u) \geq an + b.$$

This implies that $|V(PI_G(v))| \geq an + b - c + 1$. Then

$$|W|(an + b - c + 1) \leq \sum_{v \in W} |V(PI_G(v))| \leq n. \tag{3.4}$$

As $n \geq \max\{\lceil \frac{c-b}{a} \rceil, \lceil \frac{2c-2b-2}{a} \rceil\}$, by (3.4), we have

$$|W| \leq \frac{n}{an + b - c + 1} \leq \left\lfloor \frac{2}{a} \right\rfloor.$$

(ii) We first suppose that $V(G' - X_c) \neq \emptyset$ and $G' - X_c \neq K_1$. Since G' is M^o_{2s+1}-reduced, $G' - X_c$ is also M^o_{2s+1}-reduced. By counting the edges in $G' - X_c$ and by Lemmas 2.3 and 2.4, we have

$$(c + 1)(n' - |X_c|) - c|X_c| \leq 2|E(G' - X_c)| \leq 2(6s - 1)(n' - |X_c| - 1). \tag{3.5}$$

Set $c = 12s - 2$. By (3.5), we have

$$n' - |X_c| = (c - 12s + 3)(n' - |X_c|) \leq c|X_c| - 2(6s - 1). \tag{3.6}$$

We now show that if $V(G' - X_c) = \emptyset$ or if $G' - X_c = K_1$, then $|X_c| \geq 2$, and so (3.6) holds as well. Since $G' \neq K_1$, $n' > 1$. If $V(G' - X_c) = \emptyset$, then $|X_c| = |V(G')| \geq 2$. Suppose that $G' - X_c = K_1$ but $|X_c| = 1$. Then $G' = K_2^{(m)}$. It contradicts the fact that G' has a unique vertex with degree no more than c and a unique vertex with degree no less than $c + 1$. Hence we must have $|X_c| \geq 2$ in this case. This completes the proof of Lemma 3.1. ■

Slightly stronger versions of Theorems 1.12 and 1.13 will be proved, which are presented as Theorems 3.1 and 3.2, respectively. By Lemma 2.1, if G' is the M^o_{2s+1}-reduction of G, then $G \in M^o_{2s+1}$ if and only if $G' \in M^o_{2s+1}$. Therefore, the necessity of each of Theorems 3.1 and 3.2 is implied by Lemma 2.1. To prove the sufficiency of each of Theorems 3.1 and 3.2, we shall show that if G is a graph satisfying the hypothesis and if G is not in M^o_{2s+1}, then the M^o_{2s+1}-reduction G' of G must be in the specified finite family.

Recall that for integers $B, s > 0$, we define the family $\mathcal{F}(B, s)$ such that a graph H is in $\mathcal{F}(B, s)$ if and only if $H \notin M^o_{2s+1}$, $|V(H)| \leq B$ and H is M^o_{2s+1}-reduced.

Theorem 3.1. *For any integers $s, t > 0$ and real numbers a, b with $0 < a < 1$, there exist an integer $N = \max\{\lceil \frac{12s-b-1}{a} \rceil, \lceil \frac{24s-2b-6}{a} \rceil\}$ and a finite family $\mathcal{Y}_1(a, b, s, t)$ of non-strongly $\mathbb{Z}_{2s+1}$-connected graphs such that for any connected simple graph G with order $n \geq N$ and $\alpha(G) \leq t$, if (1.1) holds, then G is strongly $\mathbb{Z}_{2s+1}$-connected if and only if the M^o_{2s+1}-reduction of G is not a member in $\mathcal{Y}_1(a, b, s, t)$.*

Proof. Suppose that the real numbers a, b with $0 < a < 1$ and the integers $s, t > 0$ are given. Set

$$N = \max\left\{\left\lceil \frac{12s-b-1}{a} \right\rceil, \left\lceil \frac{24s-2b-6}{a} \right\rceil\right\},$$

and let $\mathcal{G}_1(a, b, s, t)$ denote the family of all connected simple graphs of order $n \geq N$ and satisfying (1.1) with $\alpha(G) \leq t$. Define

$$\mathcal{Y}_1(a, b, s, t) = \{G' \mid G' \text{ is the } M^o_{2s+1}\text{-reduction of a graph } G \in \mathcal{G}_1(a, b, s, t)\} \setminus \{K_1\}.$$

Let $G \in \mathcal{G}_1(a, b, s, t)$ be a graph, let G' be the M^o_{2s+1}-reduction of G and $n' = |V(G')|$. Since $K_1 \notin \mathcal{Y}_1(a, b, s, t)$, if $G \in M^o_{2s+1}$, then the M^o_{2s+1}-reduction of G is not in $\mathcal{Y}_1(a, b, s, t)$. Now we assume that $G \notin M^o_{2s+1}$. By Lemma 2.1(ii), the M^o_{2s+1}-reduction of G is in $\mathcal{Y}_1(a, b, s, t)$. It remains to show that $\mathcal{Y}_1(a, b, s, t)$ is a finite family.

Since $G \notin M^o_{2s+1}$, we have $n' > 1$. We first show that there exists an integer $B_1 = B_1(a, s, t)$ such that $n' \leq B_1$. Define X_c, X'_c, Y, W as in (3.3) and set $c = 12s - 2$. Since $n \geq \lceil \frac{12s-b-1}{a} \rceil$, we have $c < an + b$. Hence for any vertex $v \in X_c - X'_c$, $d_G(v) \leq c < an + b$. As G satisfies (1.1), $X_c - X'_c$ is an independent set of G and $X'_c = W$. Since $n \geq \max\{\lceil \frac{c-b+1}{a} \rceil, \lceil \frac{2c-2b-2}{a} \rceil\}$, it follows by Lemma 3.1(i) and $\alpha(G) \leq t$ that

$$|X_c| = |X'_c| + |X_c - X'_c| \leq \left\lfloor \frac{2}{a} \right\rfloor + \alpha(G) \leq \left\lfloor \frac{2}{a} \right\rfloor + t. \tag{3.7}$$

By Lemma 3.1(ii) and (3.7),

$$\begin{aligned} n' &= |X_c| + (n' - |X_c|) \\ &\leq |X_c| + c|X_c| - 2(6s-1) \\ &\leq (c+1)\left(\left\lfloor \frac{2}{a} \right\rfloor + t\right) - 2(6s-1) \\ &= (12s-1)\left(\left\lfloor \frac{2}{a} \right\rfloor + t - 1\right) + 1. \end{aligned}$$

Let $B_1 = (12s-1)(\lfloor \frac{2}{a} \rfloor + t - 1) + 1$. Then $n' \leq B_1$.

Since G' is M^o_{2s+1}-reduced and $1 < n' \leq B_1$, it follows by the definition of $\mathcal{F}(B, s)$ that $G' \in \mathcal{F}(B_1, s)$, and so $\mathcal{Y}_1(a, b, s, t) \subseteq \mathcal{F}(B_1, s)$. By Lemma 2.2, $\mathcal{F}(B_1, s)$ is a finite family, and so $\mathcal{Y}_1(a, b, s, t)$ is a finite family. This completes the proof of Theorem 3.1. ■

Theorem 3.2. *For any integers $s, t > 0$ and real numbers a, b with $0 < a < 1$, there exist an integer $N = \max\{\lceil \frac{12s-b-1}{a} \rceil, \lceil \frac{24s-2b-6}{a} \rceil\}$ and a finite family $\mathcal{Y}_2(a, b, s, t)$ of non-strongly $\mathbb{Z}_{2s+1}$-connected graphs such that for any connected simple graph G with order $n \geq N$ and $\alpha(G) \leq t$, if (1.2) holds, then G is strongly $\mathbb{Z}_{2s+1}$-connected if and only if the M^o_{2s+1}-reduction of G is not a member in $\mathcal{Y}_2(a, b, s, t)$.*

Proof. Suppose that the real numbers a, b with $0 < a < 1$ and the integers $s, t > 0$ are given. Set

$$N = \max\left\{\left\lceil \frac{12s-b-1}{a} \right\rceil, \left\lceil \frac{24s-2b-6}{a} \right\rceil\right\},$$

and let $\mathcal{G}_2(a, b, s, t)$ denote the family of all connected simple graphs of order $n \geq N$ and satisfying (1.2) with $\alpha(G) \leq t$. Define

$$\mathcal{Y}_2(a, b, s, t) = \{G' \mid G' \text{ is the } M^o_{2s+1}\text{-reduction of a graph } G \in \mathcal{G}_2(a, b, s, t)\} \setminus \{K_1\}.$$

Let $G \in \mathcal{G}_2(a, b, s, t)$ be a graph, let G' be the M^o_{2s+1}-reduction of G and $n' = |V(G')|$. Since $K_1 \notin \mathcal{Y}_2(a, b, s, t)$, if $G \in M^o_{2s+1}$, then the M^o_{2s+1}-reduction of G is not in $\mathcal{Y}_2(a, b, s, t)$. Now we assume that $G \notin M^o_{2s+1}$. By Lemma 2.1(ii), the M^o_{2s+1}-reduction of G is in $\mathcal{Y}_2(a, b, s, t)$. It remains to show that $\mathcal{Y}_2(a, b, s, t)$ is a finite family.

Since $G \notin M^o_{2s+1}$, we have $n' > 1$. We first show that there exists an integer $B_2 = B_2(a, s, t)$ such that $n' \leq B_2$. Define X_c, X'_c, Y, W as in (3.3) and set $c = 12s - 2$. Then the following claim holds.

Claim A. $|X_c - X'_c| + |X'_c - W| \leq 4st$.

Define $T = (X_c - X'_c) \cup (\bigcup_{v \in X'_c - W} PI_G(v))$. Since $n \geq \lceil \frac{12s-b-1}{a} \rceil$, we have $c < an + b$, and so for any vertex $u \in X_c - X'_c$, $d_G(u) \leq c < an + b$. Hence $T \subseteq V(G) - Y$. As G satisfies (1.2), every connected component of $G[T]$ (respectively, $G[X_c - X'_c]$, $G[\bigcup_{v \in X'_c - W} PI_G(v)]$) is a complete graph. Furthermore, a connected component of $G[X_c - X'_c]$ (respectively, $G[\bigcup_{v \in X'_c - W} PI_G(v)]$) is also a connected component of $G[T]$. Assume that $G[X_c - X'_c]$ has t_1 connected components and $G[\bigcup_{v \in X'_c - W} PI_G(v)]$ has t_2 connected components, i.e., $|X'_c - W| = t_2$. Then $G[T]$ has $t_1 + t_2$ connected components and so $t_1 + t_2 \leq \alpha(G) \leq t$. Since each component of $G[X_c - X'_c]$ is not strongly $\mathbb{Z}_{2s+1}$-connected, by Lemma 2.2(ii), its order is no more than $4s$.

Hence

$$|X_c - X'_c| + |X'_c - W| \leq t_1 \cdot 4s + t_2 \leq 4s(t_1 + t_2) \leq 4s \cdot \alpha(G) \leq 4st.$$

This justifies Claim A.

Since $n \geq \max\{\lceil \frac{c-b+1}{a} \rceil, \lceil \frac{2c-2b-2}{a} \rceil\}$, it follows by Claim A and Lemma 3.1(i) that

$$|X_c| = |W| + |X_c - X'_c| + |X'_c - W| \leq \left\lfloor \frac{2}{a} \right\rfloor + 4st. \tag{3.8}$$

By Lemma 3.1(ii) and (3.8),

$$\begin{aligned} n' &= |X_c| + (n' - |X_c|) \\ &\leq |X_c| + c|X_c| - 2(6s - 1) \\ &\leq (c + 1)\left(\left\lfloor \frac{2}{a} \right\rfloor + 4st\right) - 2(6s - 1) \\ &= (12s - 1)\left(\left\lfloor \frac{2}{a} \right\rfloor + 4st - 1\right) + 1. \end{aligned}$$

Let $B_2 = (12s - 1)(\lfloor \frac{2}{a} \rfloor + 4st - 1) + 1$. Then $n' \leq B_2$.

Since G' is M^o_{2s+1}-reduced and $1 < n' \leq B_2$, it follows by the definition of $\mathcal{F}(B, s)$ that $G' \in \mathcal{F}(B_2, s)$, and so $\mathcal{Y}_2(a, b, s, t) \subseteq \mathcal{F}(B_2, s)$. By Lemma 2.2, $\mathcal{F}(B_2, s)$ is a finite family, and so $\mathcal{Y}_2(a, b, s, t)$ is a finite family. This completes the proof of Theorem 3.2. ■

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