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On r-hued coloring of planar graphs with girth at least 6

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ABSTRACT

For integers $k, r > 0$, a (k, r)-coloring of a graph G is a proper k-coloring c such that for any vertex v with degree $d(v)$, v is adjacent to at least $\min\{d(v), r\}$ different colors. Such coloring is also called as an r-hued coloring. The *r-hued chromatic number* of G, $\chi_r(G)$, is the least integer k such that a (k, r)-coloring of G exists. In this paper, we proved that if G is a planar graph with girth at least 6, then $\chi_r(G) \le r + 5$. This extends a former result in Bu and Zhu (2012). It also implies that a conjecture on r-hued coloring of planar graphs is true for planar graphs with girth at least 6.

1. Introduction

Graphs in this paper are simple and finite. Undefined terminologies and notations are referred to [1]. Thus $\Delta(G)$, $\delta(G)$, $g(G)$ and $\chi(G)$ denote the maximum degree, the minimum degree, the girth and the chromatic number of a graph G, respectively. When no confusion on G arises, we often use Δ for $\Delta(G)$. For $v \in V(G)$, let $N_G(v)$ be the set of vertices adjacent to v in G, $N_G[v] = N_G(v) \cup \{v\}$, and $d_G(v) = |N_G(v)|$. When G is understood from the context, the subscript G is often omitted in these notations.

Let k, r be integers with $k > 0$ and $r > 0$, and let $[k] = \{1, 2, \ldots, k\}$. If $c : V(G) \mapsto [k]$ is a mapping, and if $V' \subseteq V(G)$, then define $c(V') = \{c(v) | v \in V'\}$. A (k, r)*-coloring* of a graph G is a mapping $c : V(G) \mapsto [k]$ satisfying both the following.

(C1) $c(u) \ne c(v)$ for every edge $uv \in E(G)$;
(C2) $|c(N_G(v))| \ge \min\{d_G(v), r\}$ for any $v \in V(G)$.

The condition (C2) is often referred to as the r-hued condition. Such coloring is also called as an r-hued coloring. For a fixed integer $r > 0$, the *r-hued chromatic number* of G, denoted by $\chi_r(G)$, is the smallest integer k such that G has a (k, r)-coloring. The concept was first introduced in [10] and [6], where $\chi_2(G)$ was called *the dynamic chromatic number* of G. The study of r-hued-colorings can be traced a bit earlier, as the square coloring of a graph is the special case when $r = \Delta$.

By the definition of $\chi_r(G)$, it follows immediately that $\chi(G) = \chi_1(G)$, and $\chi_\Delta(G) = \chi(G^2)$, where G^2 is the square graph of G. Thus r-hued coloring is a generalization of the classical vertex coloring. For any integer $i > j > 0$, any (k, i)-coloring of G is also a (k, j)-coloring of G, and so

$$\chi(G) \le \chi_2(G) \le \cdots \le \chi_r(G) \le \cdots \le \chi_\Delta(G) = \chi_{\Delta+1}(G) = \cdots = \chi(G^2).$$

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In [9], it was shown that (3, 2)-colorability remains NP-complete even when restricted to planar bipartite graphs with maximum degree at most 3 and with arbitrarily high girth. This differs considerably from the well-known result that the classical 3-colorability is polynomially solvable for graphs with maximum degree at most 3.

The r-hued chromatic numbers of some classes of graphs are known. For example, the r-hued chromatic numbers of complete graphs, cycles, trees and complete bipartite graphs have been determined in [5]. In [6], an analogue of Brooks Theorem for χ_2 was proved. It was shown in [3] that $\chi_2(G) \leq 5$ holds for any planar graph G. A *Moore graph* is a regular graph with diameter d and girth $2d+1$. Ding et al. [4] proved that $\chi_r(G) \leq \Delta^2 + 1$, where equality holds if and only if G is a Moore graph, which was improved to $r\Delta + 1$ in [8]. Wegner [12] conjectured that if G is a planar graph, then

$$\chi_\Delta(G) = \begin{cases} \Delta(G) + 5, & \text{if } 4 \leq \Delta(G) \leq 7; \\ \lfloor 3\Delta(G)/2 \rfloor + 1, & \text{if } \Delta(G) \geq 8. \end{cases}$$

A graph G has a graph H as a *minor* if H can be obtained from a subgraph of G by edge contraction, and G is called H-*minor free* if G does not have H as a minor.

Define

$$K(r) = \begin{cases} r + 3, & \text{if } 2 \leq r \leq 3; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \geq 4. \end{cases}$$

Lih et al. proved the following towards Wegner's conjecture.

Theorem 1.1 (*Lih et al. [7]*). *Let G be a K_4-minor free graph. Then*

$$\chi_\Delta(G) \leq K(\Delta(G)).$$

Song et al. extended this result by proving the following theorem. Theorem 1.1 is the special case when $r = \Delta$ of Theorem 1.2.

Theorem 1.2 (*Song et al. [11]*). *Let G be a K_4-minor free graph. Then $\chi_r(G) \leq K(r)$.*

A conjecture similar to the above-mentioned Wegner's conjecture is proposed in [11].

Conjecture 1.3. *Let G be a planar graph. Then*

$$\chi_r(G) \leq \begin{cases} r + 3, & \text{if } 1 \leq r \leq 2 \\ r + 5, & \text{if } 3 \leq r \leq 7; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \geq 8. \end{cases}$$

In this paper, we prove the following theorem.

Theorem 1.4. *If $r \geq 3$ and G is a planar graph with $g(G) \geq 6$, then $\chi_r(G) \leq r + 5$.*

When $r \geq 8$, we have $r + 5 \leq \lfloor 3r/2 \rfloor + 1$. Thus Theorem 1.4, together with Theorem 1.1 of [3] with $1 \leq r \leq 2$, justifies Conjecture 1.3 for all planar graphs with girth at least 6. Bu and Zhu in [2] proved the special case when $r = \Delta$ of Theorem 1.4, and so Theorem 1.4 is a generalization of this former result in [2].

2. Notations and terminology

Let G denote a planar graph embedded on the plane and $k > 0$ be an integer. We use $F(G)$ to denote the set of all faces of this plane graph G. For a face $f \in F(G)$, if v is a vertex on f (or if e is an edge on f, respectively), then we say that v (or e, respectively) is incident with f. The number of edges incident with f is denoted by $d_G(f)$, where each cut edge counts twice. A face f of G is called a k-*face* (or a k^+-*face*, respectively) if $d_G(f) = k$ (or $d_G(f) \geq k$, respectively). A vertex of degree k (at least k, at most k, respectively) in G is called a k-*vertex* (k^+-*vertex*, k^--*vertex*, respectively). We use $n_i(v)$ to denote the number of i-vertices adjacent to v.

For two vertices $u, w \in V(G)$, we say that u and w are *weak-adjacent* if there is a 2-vertex v such that $u, w \in N_G(v)$. A 3-vertex v is a *weak 3-vertex* if v is adjacent to a 2-vertex. The neighbors of a weak 3-vertex are called *star-adjacent*. If a 5-vertex is weak-adjacent to five 5-vertices, we call it a *bad vertex*. (As an example, see the vertex v in H_4 of Fig. 2). If a 5-vertex is adjacent to one weak 3-vertex and is weak-adjacent to four other 5-vertices, we call it a *semi-bad* type vertex. As Fig. 2 demonstrates, the vertex v in H_5 is a semi-bad type vertex.

Let G be a graph with $V = V(G)$, and let $V' \subseteq V$ be a vertex subset. As in [1], $G[V']$ is the subgraph of G induced by V'. A mapping $c : V' \to [k]$ is a *partial* (k, r)-*coloring* of G if c is a (k, r)-coloring of $G[V']$. The subset V' is the *support* of the partial (k, r)-coloring c. The support of c is denoted by $S(c)$. If c_1, c_2 are two partial (k, r)-colorings of G such that $S(c_1) \subseteq S(c_2)$ and such that for any $v \in S(c_1)$, $c_1(v) = c_2(v)$, then we say that c_2 is an *extension* of c_1. Given a partial (k, r)-coloring c on $V' \subset V(G)$, for each $v \in V - V'$, define $\{c(v)\} = \emptyset$; and for every vertex $v \in V$, we extend the definition of $c(N_G(v))$ by setting $c(N_G(v)) = \cup_{z \in N_G(v)} \{c(z)\}$, and define

$$c[v] = \begin{cases} \{c(v)\}, & \text{if } |c(N_G(v))| \geq r; \\ \{c(v)\} \cup c(N_G(v)), & \text{otherwise.} \end{cases} \tag{1}$$

By (1), $|c[v]| \le r$. We have the following observation.

Observation 2.1. *Let c be a partial (k, r)-coloring of G with support $S(c)$. For any $u \notin S(c)$, and for any $v \in N_G(u)$, by the definition of $c[v]$, we have $|c[v]| \le \min\{d(v), r\}$ and $c[v]$ represents the colors that cannot be used as $c(u)$ if one wants to extend the support of c to include u. In other words, the colors in $[k] - \bigcup_{v \in N(u)} c[v]$ are available colors to define $c(u)$ in extending the support of c from $S(c)$ to $S(c) \cup \{u\}$.*

3. Proof of Theorem 1.4

Theorem 1.1 of [3] proved Theorem 1.4 for $r \in \{1, 2\}$. So we assume that $r \ge 3$. Let $k = r + 5$. Then $k \ge 8$. We shall argue by contradiction to prove Theorem 1.4, and assume that there exists a planar graph with girth at least 6 and without any (k, r)-coloring. Throughout the rest of this section, we assume that

$$G \text{ is a counterexample to Theorem 1.4 such that } |V(G)| \text{ is minimized.} \quad (2)$$

By (2), for any non-empty proper subset $S \subset V(G)$, $G - S$ has a (k, r)-coloring. In the following two subsections, we first investigate the structure of this minimum counterexample G, and then use charge and discharge method to obtain a contradiction to complete the proof.

3.1. Structure and properties of G

Since $\chi_r(G) = \chi_\Delta(G)$ for all $r \ge \Delta(G)$, we shall always assume that $r \le \Delta(G)$. We investigate the structure of this minimum counterexample G via a sequence of lemmas.

Lemma 3.1. *Each of the following holds.*

(i) *G is 2-connected.*
(ii) *G has no adjacent 2-vertices.*
(iii) *G has no path $v_0v_1v_2v_3$ such that in G, $d(v_1) = 2$, $d(v_2) = 3$, $d(v_3) \le 3$.*

Proof. (i) If G is disconnected, then by (2), every component of G has a (k, r)-coloring, and so G has a (k, r)-coloring, contrary to (2). Hence G is connected. Assume that G has a cut-vertex v and so G has two nontrivial connected subgraphs G_1 and G_2 satisfying $V(G_1) \cap V(G_2) = \{v\}$ and $G = G_1 \cup G_2$. As for $i \in \{1, 2\}$, $|V(G_i)| < |V(G)|$, it follows by (2) that G_i has a (k, r)-coloring c_i. Permuting the colors in $c_2(V(G_2))$ such that $c_1(v) = c_2(v)$ and such that $|c_1(N_{G_1}(v)) \cup c_2(N_{G_2}(v))| \ge \min\{d_G(v), r\}$. Since $r \le \Delta(G)$, the permutation of colors in G_2 can be done to satisfy the requirements. Now define $c : V(G) \to [k]$ by $c(x) = c_i(x)$ if $v \in V(G_i)$, for $1 \le i \le 2$. It follows that c is a (k, r)-coloring of G, contrary to (2). This justifies (i).

(ii) By contradiction, we assume that G has a path $wuvx$ such that $d_G(v) = d_G(u) = 2$. By (2), $G - \{u, v\}$ has a (k, r)-coloring c. As $|c[w] \bigcup \{c(x)\}| \le r + 1 < k$, we can extend c to c_1 by letting $c_1(u) \in [k] - c[w] \bigcup \{c(x)\}$. Thus c_1 is a partial (k, r)-coloring with $S(c_1) = V(G) - \{v\}$ and $c_1(u) \ne c_1(x)$. As $d(u) = 2$, we have $|c_1[u] \bigcup c_1[x]| \le r + 2 < k$, which allows c_1 be further extended to a (k, r)-coloring c_2 of G by choosing $c_2(v) \in [k] - (c_1[u] \bigcup c_1[x])$, contrary to (2). This proves (ii).

(iii) By contradiction, we assume G contains a path $P = v_0v_1v_2v_3$ with $d_G(v_1) = 2$, $d_G(v_2) = 3$ and $d_G(v_3) \le 3$. Let $N(v_2) = \{v_1, v_3, v_4\}$. By (2), $G - \{v_1\}$ has a (k, r)-coloring c. Let c_0 denote the restriction of c to $V(G) - \{v_1, v_2\}$. Since $d_G(v_3) \le 3$, we have $|c_0[v_3] \cup c_0[v_4] \cup \{c_0(v_0)\}| \le 3 + r + 1 < k$, and so we can extend c_0 to c_1 by taking $c_1(v_2) \in [k] - \{c_0(v_0)\} \bigcup c_0[v_4] \bigcup c_0[v_3]$. This results in a (k, r)-coloring c_1 of $G - \{v_1\}$ satisfying $c_1(v_0) \ne c_1(v_2)$. Since $d_G(v_2) = 3$, we have $|c_1[v_0] \bigcup c_1[v_2]| \le r + 3 < k$, and so c_1 can be extended to a (k, r)-coloring c_2 of G by defining $c_2(v_1) \in [k] - c_1[v_0] \bigcup c_1[v_2]$, contrary to (2). This completes the proof of the lemma. □

Lemma 3.2. *Suppose v is a 2-vertex of G with $N_G(v) = \{u, w\}$. Let c be a partial (k, r)-coloring of G with $v \notin S(c)$, $u, w \in S(c)$ such that $c(u) \ne c(w)$. If $|c[u] \bigcup c[w]| < k$, then G has a partial (k, r)-coloring c' such that $S(c) \cup \{v\} \subseteq S(c')$ and such that for any $z \in S(c)$, $c(z) = c'(z)$. (We call that c' is a partial (k, r)-coloring extending c, or an extension of c.)*

Proof. Since $|c[u] \bigcup c[w]| < k$, one can define $c'(v) \in [k] - c[u] \bigcup c[w]$, and $c'(z) = c(z)$ for all $z \in S(c)$. □

Lemma 3.3. *Each of the following holds.*

(i) *Any 4-vertex v of G is adjacent to at most two 2-vertices.*
(ii) *If a 4-vertex v of G is adjacent to two 2-vertices, then v cannot be adjacent to any weak 3-vertex.*
(iii) *If a 4-vertex v of G is adjacent to one 2-vertex, then v cannot be adjacent to three weak 3-vertices.*

Proof. (i) By contradiction, we assume that G has a 4-vertex v adjacent to at least three 2-vertices. Thus G has H_1 (as depicted in Fig. 1) as a subgraph. The neighbors of v are v_1, v_2, v_3, v_4 with $d(v_1) = d(v_2) = d(v_3) = 2$. By (2), $G - v_1$ has a (k, r)-coloring. Choose a (k, r)-coloring c of $G - v_1$ such that $|\{c(v), c(x)\}|$ is maximized. We claim that $c(v) \ne c(x)$. Assume that, to the contrary, we have $c(v) = c(x)$. Since c is a (k, r)-coloring with $S(c) = V(G) - \{v_1\}$, the r-hued condition (C2) holds

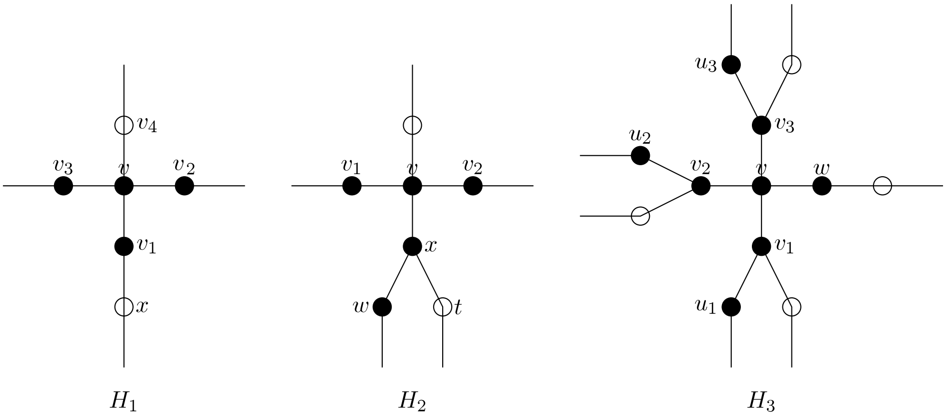

Fig. 1. A vertex is represented by a solid point if all of its incident edges are drawn, otherwise it is represented by a hollow point.

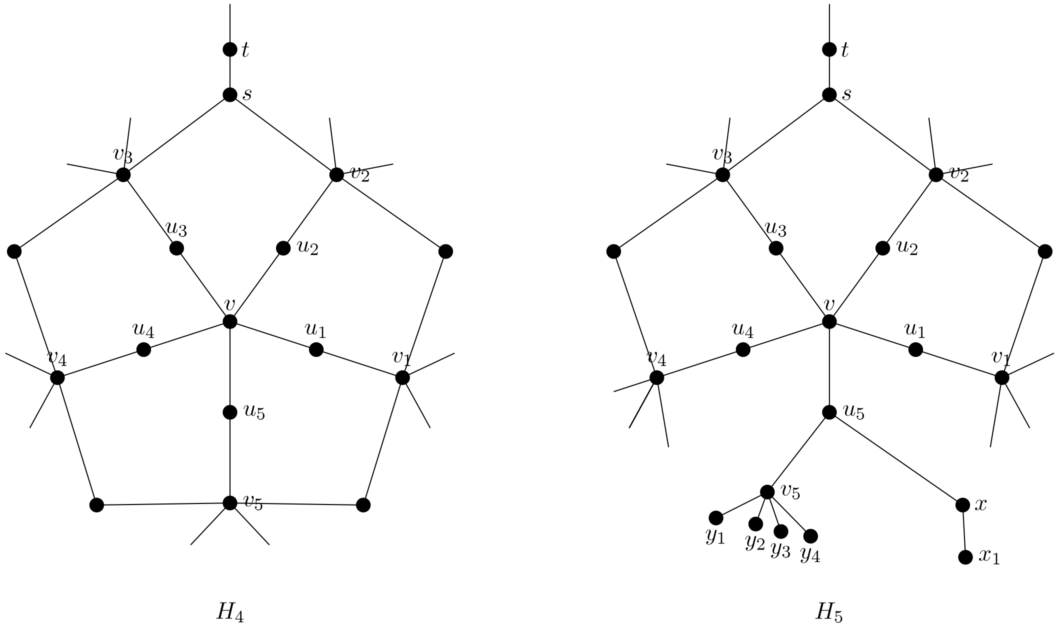

Fig. 2. A bad vertex v in H_4 (left); a semi-bad type vertex v in H_5 (right).

for each of v_2, v_3 and v if $r = 3$; and if $r \geq 4$, then $|c(N_{G-v_1}(v))| = 3$. Let c_0 be the restriction of c to $V(G) - \{v_1, v_2, v_3, v\}$. Then c_0 is a partial (k, r)-coloring with $S(c_0) = V(G) - \{v_1, v_2, v_3, v\}$. We first extend c_0 by recoloring v. By Observation 2.1, the colors in $[k] - \bigcup_{1\leq i\leq 4} c_0[v_i]$ can be used to color v. Since $c_0[v_1] = \{c_0(x)\}$ and $|c_0[v_i]| = 1$ for $2 \leq i \leq 3$, we have $|\bigcup_{1\leq i\leq 4} c_0[v_i]| \leq r + 3 < k$. We define $c_1(v) \in [k] - \bigcup_{1\leq i\leq 4} c_0[v_i]$ and $c_1(z) = c_0(z)$ for all $z \in V(G) - \{v_1, v_2, v_3, v\}$. Hence c_1 is a partial (k, r)-coloring with $S(c_1) = V(G) - \{v_1, v_2, v_3\}$. Let v and w_j be the two neighbors of v_j in G, for $2 \leq j \leq 3$. With a similar argument, since $|c_1[w_3] \cup c_1[v]| \leq r + 2 < k$, it follows by Lemma 3.2 that there exists a (k, r)-coloring c_2 of $G - \{v_1, v_2\}$, extending c_1. Since $|c_2[w_2] \cup c_2[v]| \leq r + 3 < k$, it follows by Lemma 3.2 that there exists a (k, r)-coloring c_3 of $G - \{v_1\}$, extending c_2. But $c_3(x) = c_1(x) \neq c_1(v) = c_3(v)$, this leads to a contradiction to the maximality of $|\{c(v), c(x)\}|$. Hence we must have $c(v) \neq c(x)$. Since $|c[x] \cup c[v]| \leq r + 4 < k$, it follows by Lemma 3.2 that there exists a (k, r)-coloring c_4 of G, contrary to (2). This proves (i).

(ii) By contradiction, we assume that G has a 4-vertex v adjacent to two 2-vertices and at least a weak 3-vertex. Thus G has H_2 (as depicted in Fig. 1) as a subgraph. We shall adopt the notation of H_2 in Fig. 1, and let v_1, v_2, v_3, x denote the neighbors of v in G such that v_1, v_2 are 2-vertices and x is a weak 3-vertex with $N_G(x) = \{v, w, t\}$. By the definition of weak 3-vertices, we may assume that $N_G(w) = \{w', x\}$. By (2), $G - x$ has a (k, r)-coloring c. Let c_0 be the restriction of c to $V(G) - \{v_1, v_2, v, w, x\}$. Thus each of v_1, v_2, v, w satisfies the r-hued condition (C2) under the coloring c_0. As $|c_0[v_i]| = 1$ for $1 \leq i \leq 2$ and $c_0[x] = \{c(t)\}$, $|\cup_{1\leq i\leq 3} c_0[v_i] \cup \{c_0(t)\}| \leq r + 3 < k$, we can extend c_0 to c_1 by setting $c_1(v) \in [k] - (\cup_{1\leq i\leq 3} c_0[v_i] \cup \{c_0(t)\})$ with $S(c_1) = V(G) - \{v_1, v_2, w, x\}$. Let $\{v, w_i\}$ be the neighbor set of v_i for $1 \leq i \leq 2$. As $|c_1[v] \cup c_1[w_1]| \leq 2 + r < k$, by Lemma 3.2, c_1 can be extended to c_2 with $c_2(v_1) \in [k] - (c_1[v] \cup c_1[w_1])$ and $S(c_2) = V(G) - \{v_2, w, x\}$. As w is a 2-vertex of G and as $w, v_2 \notin S(c_2)$, we have $|c_2[v] \cup c_2[w] \cup c_2[t]| \leq 3 + 1 + r < k$. Thus c_2 can be extended to c_3 with $c_3(x) \in [k] - (c_2[v] \cup c_2[w] \cup c_2[t])$ and $S(c_3) = V(G) - \{w, v_2\}$. As $|c_3[v] \cup c_3[w_2]| \leq 4 + r < k$, it follows by Lemma 3.2 that c_3 can be extended to c_4 with $c_4(v_2) \in [k] - (c_3[v] \cup c_3[w_2])$ and $S(c_4) = V(G) - \{w\}$. As $N_G(w) = \{w', x\}$, we have $|c_4[x] \cup c_4[w']| \leq 3 + r < k$. By Lemma 3.2, c_4 can be extended to a (k, r)-coloring c_5 of G by defining $c_5(w) \in [k] - (c_4[x] \cup c_4[w'])$, contrary to (2).

(iii) By contradiction, we assume that G has a 4-vertex v adjacent to one 2-vertex w and three weak 3-vertices v_1, v_2, v_3. Thus G has H_3 (as depicted in Fig. 1) as a subgraph. We will adopt the notations in H_3 of Fig. 1. By (2), $G - w$ has a (k, r)-coloring c. Let c_0 be the restriction of c to $V(G) - \{v, u_1, u_2, u_3, w\}$. For $i = 1, 2, 3$, let $\{v, u_i, u_i'\}$ denote the neighbor set of v_i, and $\{u_i'', v_i\}$ denote the neighbor set of u_i, and let $\{v, w'\}$ be the neighbor set of w.

As $k \geq 8$, c_0 can be extended to a (k, r)-coloring c_1 by defining

$c_1(v) \in [k] - \{c_0(v_1), c_0(v_2), c_0(v_3), c_0(u_1'), c_0(u_2'), c_0(u_3'), c_0(w')\}$ with $S(c_1) = V(G) - \{u_1, u_2, u_3, w\}$.

For $i = 1, 2, 3$, as $|c_1[u_i''] \cup c_1[v_i]| \leq r + 3 < k$, by Lemma 3.2, c_1 can be extended to a (k, r)-coloring c_2 such that $c_2(u_i) \in [k] - (c_1[u_i''] \cup c_1[v_i])$ and $S(c_2) = V(G) - \{w\}$. As $|c_2[v] \cup c_2[w']| \leq 4 + r < k$, by Lemma 3.2, c_2 can be extended to a (k, r)-coloring c_3 of G such that $c_3(w) \in [k] - (c_2[v] \cup c_2[w'])$, contrary to (2). This completes the proof of the lemma. □

Lemma 3.4. *If $r \neq 5$, any 5-vertex of G is adjacent to at most four 2-vertices; Furthermore, if it is adjacent to four 2-vertices, then it is not adjacent to a weak 3-vertex.*

Proof. We argue by contradiction and assume that $r \neq 5$ and G has a 5-vertex v with $N_G(v) = \{u_1, u_2, u_3, u_4, u_5\}$, such that u_1, u_2, u_3, u_4 are 2-vertices and u_5 is either a 2-vertex or a weak 3-vertex. For each i with $1 \leq i \leq 4$, let $N_G(u_i) = \{v, v_i\}$; and let v, v_5 be two vertices adjacent to u_5. By Lemma 3.1(ii), $d(v_i) \geq 3$ for $1 \leq i \leq 4$. If u_5 is a weak 3-vertex, then denoting $N_G(u_5) = \{v, v_5, x\}$ where $d(x) = 2$, we apply Lemma 3.1(iii) to the path xu_5v_5 to conclude that $d(v_5) \geq 4$.

If $3 \leq r \leq 4$, we have $2r \leq r + 4$. By (2), $G - \{v, u_1, u_2, u_3, u_4\}$ has a (k, r)-coloring c_1. Since $|c_1[u_5] \cup \{c_1(v_1), c_1(v_2), c_1(v_3), c_1(v_4)\}| \leq r + 4 < k$, we extend c_1 to a (k, r)-coloring c_2 with $S(c_2) = V(G) - \{u_1, u_2, u_3, u_4\}$ by defining $c_2(v) \in [k] - (c_1[u_5] \cup \{c_1(v_1), c_1(v_2), c_1(v_3), c_1(v_4)\})$. For $1 \leq i \leq 4$, as $|c[v_i] \bigcup c[v]| \leq 2r \leq r + 4 < k$, it follows by Lemma 3.2 that c_2 can be extended to a (k, r)-coloring c of G, contrary to (2).

Therefore, we assume that $r \geq 6$, and so $k = r + 5 \geq 11$. If u_5 is 2-vertex, then by (2), $G - v$ has a (k, r)-coloring c. Let c_1 be the restriction of c to $V(G) - \{v, u_2, u_3, u_4, u_5\}$. As $|c_1[v_2] \cup \{c_1(u_1)\}| \leq r + 1 < k$, we extend c_1 to c_2 by defining $c_2(u_2) \in [k] - (c_1[v_2] \cup \{c_1(u_1)\})$. For $i = 2, 3, 4$, as $|c_i[v_{i+1}] \cup \{c_i(u_1), \ldots, c_i(u_i)\}| \leq r + 4 < k$, the coloring c_i can be extended to c_{i+1} by defining $c_{i+1}(u_{i+1}) \in [k] - (c_i[v_{i+1}] \cup \{c_i(u_1), \ldots, c_i(u_i)\})$. Hence $S(c_5) = V(G) - \{v\}$, and $c_5(u_1), c_5(u_2), c_5(u_3), c_5(u_4), c_5(u_5)$ are mutually distinct. Since every u_i is a 2-vertex, $|\bigcup_{i=1}^{5} c_5[u_i]| \leq 10 < 6 + 5 \leq k$, this coloring c_5 can be extended to a (k, r)-coloring c_6 by defining $c_6(v) \in [k] - \bigcup_{i=1}^{5} c_5[u_i]$. As $S(c_6) = V(G)$, this is a contradiction to (2).

Hence u_5 must be a weak 3-vertex, By (2), $G - v$ has a (k, r)-coloring c. Let c_1 be the restriction of c to $V(G) - \{v, u_1, u_2, u_3, u_4, x\}$. As $|c_1[v_1] \cup \{c_1(u_5)\}| \leq r + 1 < k$, one can extend c_1 to c_2 by defining $c_2(u_1) \in [k] - (c_1[v_1] \cup \{c_1(u_5)\})$. For $i = 2, 3, 4$, as $|c_i[v_i] \cup \{c_i(u_5), c_i(u_1), \ldots, c_i(u_{i-1})\}| \leq r + 4 < k$, one can extend c_i to c_{i+1} by defining $c_{i+1}(u_i) \in [k] - (c_i[v_i] \cup \{c_i(u_5), c_i(u_1), \cdots, c_i(u_{i-1})\})$. Hence $S(c_5) = V(G) - \{v, x\}$, and $c_5(u_1), c_5(u_2), c_5(u_3), c_5(u_4), c_5(u_5)$ are mutually distinct. Note that in $G[S(c_5) \bigcup \{v\}]$, each u_i, $(1 \leq i \leq 5)$, is a 2-vertex. Therefore, $|\bigcup_{i=1}^{5} c_5[u_i]| \leq 10 < 6 + 5 \leq k$, and so c_5 can be extended to a (k, r)-coloring c_6 by defining $c_6(v) \in [k] - \bigcup_{i=1}^{5} c_5[u_i]$ with $S(c_6) = V(G) - \{x\}$. Denote $N_G(x) = \{u_5, x_1\}$. Since $|c_6[u_5] \cup c_6[x_1]| \leq 3 + r < k$, this coloring c_6 can be extended to a (k, r)-coloring c_7 of G by defining $c_7(x) \in [k] - (c_6[u_5] \cup c_6[x_1])$, contrary to (2). This justifies (iii) and proves the lemma. □

Lemma 3.5. *If a 5-vertex v of G is adjacent to at least four 2-vertices, then any one of its weak-adjacent neighbors must be an r-vertex.*

Proof. Denote $N_G(v) = \{u_1, u_2, u_3, u_4, u_5\}$. We assume that u_1, u_2, u_3, u_4 are 2-vertices. Let $N_G(u_i) = \{v, v_i\}$, $1 \leq i \leq 4$. By definition, each v_i is a weak-adjacent neighbor of v. By contradiction, we assume that v_4 is not an r-vertex. By (2), $G - u_4$ has a (k, r)-coloring c.

Let $G_0 = G - \{u_1, u_2, u_3, u_4, v\}$, and c_0 be the restriction of c to $V(G_0)$. Since $|\cup_{i=1}^{5} c_0[u_i]| \leq r + 4 < k$, we extend c_0 to a (k, r)-coloring c_1 with $S(c_1) = S(c_0) \cup \{v\} = V(G_0) \cup \{v\}$ by defining $c_1(v) \in [k] - (\cup_{i=1}^{5} c_0[u_i])$. Let $G_1 = G - \{u_1, u_2, u_3, u_4\}$. For each $i = 1, 2, 3, 4$, we inductively define $G_{i+1} = G[V(G_i) \cup \{u_i\}]$, and extend c_i to c_{i+1} with $S(c_{i+1}) = V(G_{i+1})$ as follows.

For $i = 1, 2, 3$, $|c_i[v] \cup c_i[v_i]| \leq i + 1 + r < r + 5$. Recall that $d_G(v_4) \neq r$. If $d_G(v_4) \geq r + 1$, then by the definition of (k, r)-coloring, $|c_4(N_{G_4}(v_4))| = |c(N_{G_4}(v_4))| \geq r$, and so by (1), $|c_4[v_4]| = 1$. If $d_G(v_4) \leq r - 1$, then $d_{G_4}(v_4) \leq r - 2$, and so by (1), $|c_4[v_4]| \leq d_{G_4}(v_4) + 1 \leq r - 1$. Hence we always have $|c_4[v_4] \cup c_4[v]| \leq r - 1 + 5 < r + 5$. For all $i = 1, 2, 3, 4$, the discussion above implies that $|c_i[v] \cup c_i[v_i]| < r + 5$, and so $c_i(v) \neq c_i(v_i)$. By Lemma 3.2, c_i can be extended to c_{i+1} with $S(c_{i+1}) = V(G_i) \cup \{u_i\} = V(G_{i+1})$. Since $G_5 = G$, c_5 is a (k, r)-coloring of G, contrary to (2). □

Lemma 3.6. *Suppose that $r = 5$ and G has a bad vertex or a semi-bad type vertex v, with $N_G(v) = \{u_1, u_2, u_3, u_4, u_5\}$ as depicted in Fig. 2. (We shall adopt the notations in Fig. 2.) Then $G - v$ has a (k, r)-coloring c satisfying each of the following.*

(i) *If v is a bad vertex, then for each i with $1 \leq i \leq 5$, we have*

$$c(N[v_i] - \{u_i\}) = \{c(v_1), c(v_2), c(v_3), c(v_4), c(v_5)\} \quad \text{and} \quad |\{c(v_1), c(v_2), c(v_3), c(v_4), c(v_5)\}| = 5.$$

(ii) *If v is a semi-bad type vertex, then for each i with $1 \le i \le 4$, we have*

$$c(N[v_i] - \{u_i\}) = \{c(v_1), c(v_2), c(v_3), c(v_4), c(v_5)\} \quad \text{and} \quad |\{c(v_1), c(v_2), c(v_3), c(v_4), c(v_5)\}| = 5.$$

(Thus we may assume that $c(v_5) = 5$, $c(v_i) = i$ and $c(N[v_i] - \{u_i\}) = \{1, 2, 3, 4, 5\}$, $1 \le i \le 4$.) Moreover, we have $4 \le d(v_5) \le 5$ and one of the following must hold.

(ii-1) *If $d(v_5) = 4$, then $c(\{x_1, y_1, y_2, y_3\}) = \{1, 2, 3, 4\}$, and for any $i \in \{1, 2, 3\}$, y_i is not a 2-vertex.*
(ii-2) *If $d(v_5) = 5$, then $\{1, 2, 3, 4\} \subseteq c(\{x_1, y_1, y_2, y_3, y_4\})$.*

Proof. (i) By (2), $G - v$ has a (k, r)-coloring c. Let $A = \{c(v_1), c(v_2), c(v_3), c(v_4), c(v_5)\}$. Choose $a \in [k] - A$ such that if $c[v_5] - A \ne \emptyset$, then $a \in c[v_5] - A$. Let c_0 be the restriction of c with $S(c_0) = V(G) - \{u_1, u_2, u_3, u_4, u_5, v\}$.

Claim 1. $|A| = 5$.

By contradiction, we assume that there exist $i, j \in \{1, 2, 3, 4, 5\}$ such that $i < j$ and $c_0(v_i) = c_0(v_j)$. Then we first extend c_0 to a partial (k, r)-coloring c_1 by letting $c_1(v) = a$. Next apply Lemma 3.2 to extend c_1 to a partial (k, r)-coloring c_2 by coloring u_1 with $c_2(u_1) \in [k] - (c_1[v_1] \cup \{c_1(v)\})$ and $S(c_2) = V(G) - \{u_2, u_3, u_4, u_5\}$. For $2 \le i \le 4$, apply Lemma 3.2 repeatedly to extend c_i to a partial (k, r)-coloring c_{i+1} by defining $c_{i+1}(u_i) \in [k] - (c_i[v_i] \cup \{c_i(v), c_i(u_1), \cdots, c_i(u_{i-1})\})$ with $S(c_{i+1}) = S(c_i) \cup \{u_i\}$. Hence $S(c_5) = V(G) - \{u_5\}$. If $c[v_5] \subseteq A$, then $|c_5[v_5] \bigcup c_5[v]| \le 4 + 5 < 10$, and so c_5 can be extended to a (k, r)-coloring c_6 of G by letting $c_6(u_5) \in [k] - (c_5[v_5] \bigcup c_5[v])$. If $c[v_5] - A \ne \emptyset$, then as $c_5(v) = c_1(v) = a \in c_5[v_5]$, we again have $|c_5[v_5] \bigcup c_5[v]| < 10$, and so c_5 can always be extended to a (k, r)-coloring c_6 of G, contrary to (2). This proves Claim 1.

By Claim 1, we have $|A| = 5$. By permuting the colors, we assume that in $G - v$ has a (k, r)-coloring c such that $c(v_i) = i$ for $1 \le i \le 5$. Thus $A = \{1, 2, 3, 4, 5\}$. Again let c_0 be the restriction of c with $S(c_0) = V(G) - \{u_1, u_2, u_3, u_4, u_5, v\}$. Note that $|c(N[v_i] - \{u_i\})| \le d_G(v_i)$ for all $i = 1, \ldots, 5$. If v is a bad vertex, $d_G(v_i) = 5$ for all $i = 1, \ldots, 5$. Thus to prove (i), it suffices to justify the claim below.

Claim 2. For any i with $1 \le i \le 5$, $A \subseteq c(N[v_i] - \{u_i\})$.

By contradiction and by symmetry, we assume that there exists a color $a' \in A - c(N[v_1] - \{u_1\})$. Then we extend c_0 to a (k, r)-coloring c_1 by choosing $c_1(u_1) = a'$ with $S(c_1) = S(c_0) \cup \{u_1\}$. For each $i = 2, 3, 4, 5$, as $|c_{i-1}[v_i] \cup \{c_{i-1}(u_1), \cdots, c_{i-1}(u_{i-1})\}| \le r + i - 1 < k$, we can extend c_{i-1} to a (k, r)-coloring c_i by defining $c_i(u_i) \in [k] - (c_{i-1}[v_i] \cup \{c_{i-1}(u_1), \cdots, c_{i-1}(u_{i-1})\})$ with $S(c_i) = S(c_{i-1}) \cup \{u_i\}$. Since $c_5(u_1) = c_1(u_1) = a' \in A$, it follows that $|\{c_5(u_1), \cdots, c_5(u_5)\} \cup A| < 10 = k$. Since $S(c_5) = V(G) - \{v\}$, we can extend c_5 to a (k, r)-coloring c_6 of G by letting $c_6(v) \in [k] - (\{c_5(u_1), \cdots, c_5(u_5)\} \cup A)$, contrary to (2). This proves Claim 2. Now Lemma 3.6(i) follows from Claims 1 and 2.

(ii) Assume that v is a semi-bad type vertex. Then $d_G(v_5) \ge 4$ by Lemma 3.1(iii). We make the following claims.

Claim 3. $4 \le d_G(v_5) \le 5$.

By contradiction, we assume that $d_G(v_5) \ge 6$. By (2), $G - \{u_5, x\}$ has a (k, r)-coloring c. As $d_{G-\{u_5,x\}}(v_5) \ge 5 = r$, v_5 satisfies the r-hued condition (C2) under this coloring c, and so $c[v_5] = \{c(v_5)\}$. Let c_0 be the restriction of c to $S(c) - \{v\}$. Extend c_0 to c_1 by letting $c_1(v) \in [k] - (\cup_{i=1}^{4}\{c_0(u_i)\}) \cup (\cup_{j=1}^{5}\{c_0(v_j)\})$. Thus $c_1(v) \ne c_1(v_5)$ and

$$|c_1[v] \cup c_1[v_5] \cup c_1[x]| = |\{c_1(u_1), c_1(u_2), c_1(u_3), c_1(u_4), c_1(v), c_1(v_5), c_1(x_1)\}| \le 7 < k,$$

and so we can extend c_1 to c_2 by defining $c_2(u_5) \in [k] - (c_1[v] \cup c_1[v_5] \cup c_1[x])$, with $S(c_2) = V(G) - \{x\}$. Since $|c_2[x_1] \cup c_2[u_5]| \le r + 3 < k$, we can further extend c_2 to a (k, r)-coloring c_3 of G by letting $c_3(x) \in [k] - (c_2[x_1] \cup c_2[u_5])$, contrary to (2). This justifies Claim 3.

By (2), $G - v$ has a (k, r)-coloring c. In the rest of the proof of this lemma, we let c_0 denote the restriction of c to $V(G) - \{u_1, u_2, u_3, u_4, u_5, x, v\}$, and let $A = c(\{v_1, v_2, v_3, v_4, v_5\})$.

Claim 4. $|A| = 5$. (Thus we shall assume that $A = \{1, 2, 3, 4, 5\}$ in the rest of the proof of this lemma.)

Suppose that $|A| < 5$. As $S(c_0) = V(G) - \{u_1, u_2, u_3, u_4, u_5, x, v\}$, we have $|c_0[v_5] \cup \{c_0(x_1)\}| < k$ and so c_0 can be extended to c_1 by defining $c_1(u_5) \in [k] - (c_0[v_5] \cup \{c_0(x_1)\})$. Define $u_0 = u_5$. For $i = 1, 2, 3, 4$, as $|c_i[v_i] \cup \{c_i(u_0), c_i(u_1), \cdots, c_i(u_{i-1})\}| \le r + 4 < k$, c_i can be extended to c_{i+1} by defining $c_{i+1}(u_i) \in [k] - (c_i[v_i] \cup \{c_i(u_0), c_i(u_1), \cdots, c_i(u_{i-1})\})$. Now $S(c_5) = V(G) - \{v, x\}$. Since $|A| \le 4$, $|c_5(\{u_1, u_2, u_3, u_4, u_5\}) \cup A| \le 5 + 4 = 9 < k$, we extend c_5 to c_6 by defining $c_6(v) \in [k] - (c_5(\{u_1, u_2, u_3, u_4, u_5\}) \cup A)$. Since $c_6(u_5) = c_1(u_5) \ne c_0(x_1) = c_6(x_1)$, and since $|c_6[x_1] \cup c_6[u_5]| \le r + 3 < k$, it follows by Lemma 3.2, c_6 can be extended to a (k, r)-coloring of G, contrary to (2). This proves Claim 4.

Claim 5. For $1 \le i \le 4$, we have $c(N[v_i] - \{u_i\}) = A = \{1, 2, 3, 4, 5\}$.

By contradiction, we may assume that there exists a $j \in A - c(N[v_1] - \{u_1\})$. First extend c_0 to c_1 by defining $c_1(u_1) = j$. As $|c_1[v_5] \cup c_1(\{u_1, x_1\})| \le 5 + 2 < k$, we extend c_1 to c_2 by defining $c_2(u_5) \in [k] - (c_1[v_5] \cup c_1(\{u_1, x_1\}))$. For $i = 2, 3, 4$, as $|c_i[v_i] \cup \{c_i(u_5), c_i(u_1), \cdots, c_i(u_{i-1})\}| \le r + 4 < k$, c_i can be extended to c_{i+1} by defining $c_{i+1}(u_i) \in [k] - (c_i[v_i] \cup \{c_i(u_5), c_i(u_1), \cdots, c_i(u_{i-1})\})$. Now $S(c_5) = V(G) - \{v, x\}$. Since $c_5(u_1) = c_1(u_1) = j \in A$, we have $|c_5(\{u_1, u_2, u_3, u_4, u_5\}) \cup A| < 10 = k$. Hence c_5 can be extended to c_6 by defining $c_6(v) \in [k] - (c_5(\{u_1, u_2, u_3, u_4, u_5\}) \cup A)$. Since $c_6(u_5) = c_2(u_5) \ne c_1(x_1) = c_6(x_1)$ and since $|c_6[x_1] \cup c_6[u_5]| \le r + 3 < k$, it follows by Lemma 3.2 that c_6 can be extended to a (k, r)-coloring of G, contrary to (2). This proves Claim 5.

By Claim 3, $d_G(v_5) \in \{4, 5\}$. Thus we will proceed our proof by discussing each of these two possibilities. As noted before, we have a (k, r)-coloring of $G - v$ with $c(v_i) = i$, $(1 \le i \le 5)$, $A = c(\{v_1, v_2, v_3, v_4, v_5\})$ and c_0 is its restriction with

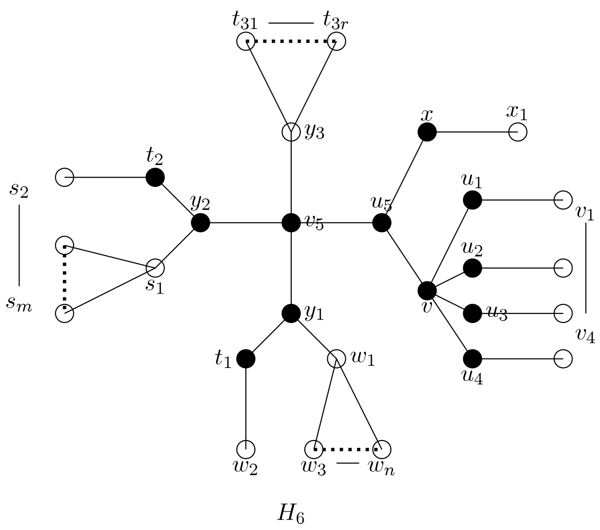

Fig. 3. v is a semi-bad type vertex and v_5 is adjacent to three weak-3-vertices.

$S(c_0) = V(G) - \{u_1, u_2, u_3, u_4, u_5, x, v\}$. We will continue using the notations of H_5 in Fig. 2 for our discussions below, except that y_4 will be removed in the proof of Case 1.

Case 1. $d(v_5) = 4$.

We shall show that (ii-1) holds. As $c(v_5) = 5$, we first claim that $c(\{x_1, y_1, y_2, y_3\}) = \{1, 2, 3, 4\}$. Assume that the claim is false and there exists a color $a \in \{1, 2, 3, 4\} - c(\{x_1, y_1, y_2, y_3\})$. Then we extend c_0 to c_1 by assigning $c_1(u_5) = a$. Let $u_0 = u_5$. For $1 \le i \le 4$, as $|c_i[v_i] \cup c_i(\{u_0, u_1, \cdots, u_{i-1}\})| \le r + 4 < k$, we can extend c_i to c_{i+1} by defining $c_{i+1}(u_i) \in [k] - (c_i[v_i] \cup c_i(\{u_0, u_1, \cdots, u_{i-1}\}))$. Note that $S(c_5) = V(G) - \{v, x\}$. As $c_5(u_5) = c_1(u_5) = a \in A$, we have $|c_5(\{u_1, u_2, u_3, u_4, u_5\}) \cup A| < 10 = k$. Hence we can extend c_5 to c_6 by letting $c_6(v) \in [k] - c_5(\{u_1, u_2, u_3, u_4, u_5\}) \cup A$. As $|c_6[x_1] \cup c_6[u_5]| \le r+3 < k$ and $c_6(u_5) = c_1(u_5) = a \ne c_0(x_1) = c_6(x_1)$, by Lemma 3.2, c_6 can be extended to a (k, r)-coloring c_7 of G by letting $c_7(x) \in [k] - (c_6[x_1] \cup c_6[u_5])$, contrary to (2). This justifies the claim that $c(\{x_1, y_1, y_2, y_3\}) = \{1, 2, 3, 4\}$.

We claim next that for any i with $1 \le i \le 3$, y_i cannot be a 2-vertex. If not, we may assume that y_1 is a 2-vertex. Let $a' = c(y_1)$. Let c_0' be the restriction of c_0 with $S(c_0') = S(c_0) - \{y_1\} = V(G) - \{u_1, u_2, u_3, u_4, u_5, y_1, v, x\}$. Extend c_0' to c_1' by defining $c_1'(u_5) = a' \in \{1, 2, 3, 4\}$. Similar to the arguments above, c_1' can be extended to c_5' with $S(c_5') = V(G) - \{v, x, y_1\}$. Since $c(N[v_i] - \{u_i\}) = A$ for $1 \le i \le 4$, $c_5'(\{u_1, u_2, u_3, u_4\}) \subset \{6, 7, 8, 9, 10\}$. As $c_5'(u_5) = c_1'(u_5) \in A$, we have $|c_5'(\{u_1, u_2, u_3, u_4, u_5\}) \cup A| < 10 = k$. Hence we can extend c_5' to c_6' by letting $c_6'(v) \in [k] - (c_5'(\{u_1, u_2, u_3, u_4, u_5\}) \cup A)$. Let $N_G(y_1) = \{w, v_5\}$. For $|c_6'[w] \cup c_6'[v_5]| \le r + 4 < k$ and $|c_6'[x_1] \cup c_6'[u_5]| \le r + 3 < k$, we extend c_6' to a (k, r)-coloring c_7' of G by letting $c_7'(y_1) \in [k] - (c_6'[w] \cup c_6'[v_5])$ and $c_7'(x) \in [k] - (c_6'[x_1] \cup c_6'[u_5])$, contrary to (2). Thus by symmetry, for any $1 \le i \le 3$, y_i is not a 2-vertex.

Case 2. $d(v_5) = 5$.

We shall show that (ii-2) holds. By contradiction, we assume that there exists a color $a \in \{1, 2, 3, 4\} - c(\{x_1, y_1, y_2, y_3, y_4\})$. Then we extend c_0 to c_1 by assigning $c_1(u_5) = a$. Let $u_0 = u_5$. For $1 \le i \le 4$, as $|c_i[v_i] \cup c_i(\{u_0, u_1, \cdots, u_{i-1}\})| \le r + 4 < k$, we can extend c_i to c_{i+1} by defining $c_{i+1}(u_i) \in [k] - (c_i[v_i] \cup c_i(\{u_0, u_1, \cdots, u_{i-1}\}))$. Note that $S(c_5) = V(G) - \{v, x\}$. As $c_5(u_5) = c_1(u_5) = a \in A$, we have $|c_5(\{u_1, u_2, u_3, u_4, u_5\}) \cup A| < 10 = k$. Hence we can extend c_5 to c_6 by letting $c_6(v) \in [k] - c_5(\{u_1, u_2, u_3, u_4, u_5\}) \cup A$. As $|c_6[x_1] \cup c_6[u_5]| \le r + 3 < k$ and $c_6(u_5) \ne c_6(x_1)$, we finally extend c_6 to a (k, r)-coloring c_7 of G by letting $c_7(x) \in [k] - (c_6[x_1] \cup c_6[u_5])$, contrary to (2). This completes the proof for Case 2, as well as the proof for the lemma. □

Lemma 3.7. *Suppose that $r = 5$ and G has a semi-bad type vertex v. Let $N_G(v) = \{u_1, u_2, u_3, u_4, u_5\}$ such that u_5 is the weak 3-vertex which is adjacent to v with $N_G(u_5) = \{v, v_5, x\}$. If $d(v_5) = 4$, then v_5 is adjacent to at most two weak 3-vertices.*

Proof. By contradiction, we assume that G, v and v_5 satisfy the hypothesis of the lemma with $d(v_5) = 4$, and v_5 is adjacent to three weak 3-vertices y_1, y_2, u_5, (see Fig. 3). Hence H_6 depicted in Fig. 3 is a subgraph of G. We shall use the notations in Fig. 3 in the proof of this lemma.

By (2), $G - v$ has a (k, r)-coloring c. By Lemma 3.6, we assume that

$$c(v_i) = i, \quad (1 \le i \le 5), \qquad c(x_1) = 4 \quad \text{and} \quad c(y_j) = j, \ (1 \le j \le 3). \tag{3}$$

Let c denote the restriction of c itself to $V(G) - \{v, t_1, t_2, x, u_1, u_2, u_3, u_4, u_5\}$. By Lemma 3.6(ii), we may assume (by recoloring) that $c(u_i) = i + 5$, for $i = 1, 2, 3, 4$. Extend this recolored c with $S(c) = V(G) - \{v, t_1, t_2, x, u_5\}$ to c_1 by defining $c_1(v) = 10$. By Lemma 3.1(3), w_1, s_1 must be 4^+-vertices.

Claim 1. $\{4, 6, 7, 8, 9, 10\} \subseteq c_1(N[w_1] \cup \{w_2\} - \{y_1\}) \cap c_1(N[s_1] \cup \{s_2\} - \{y_2\})$.

By symmetry, it suffices to prove that $\{4, 6, 7, 8, 9, 10\} \subseteq c_1(N[w_1] \cup \{w_2\} - \{y_1\})$. By contradiction, assume that there exists a color $a' \in \{4, 6, 7, 8, 9, 10\} - c_1(N[w_1] \cup \{w_2\} - \{y_1\})$. Recall that we have $c_1(y_1) = c(y_1) = 1$. Define

$$c_2'(z) = \begin{cases} c_1(z) & \text{if } z \in S(c_1) - \{y_1\} \\ a' & \text{if } z = y_1 \\ 1 & \text{if } z = u_5. \end{cases}$$

As $a' \in \{4, 6, 7, 8, 9, 10\} - c_2'(N[w_1] \cup \{w_2\} - \{y_1\})$, we note that both $c_2'(y_1) = a' \notin c_2'[w_1] \cup c_2'[t_1] \cup c_2'[v_5] - \{c_2'(y_1)\}$ and $c_2'(u_5) = 1 \notin c_2'(N_G[v] \cup N_G[v_5] \cup \{x_1\} - \{u_5\})$. Therefore by definition, c_2' is a partial (k, r)-coloring with $S(c_2') = V(G) - \{x, t_1, t_2\}$.

As $c_2'(u_5) = 1 \neq 4 = c_2'(x_1)$, $c_2'(y_1) = a' \neq c_2'(w_2)$, $c_2'(y_2) = c(y_2) \neq c(s_2) = c_2'(s_2)$, it follows by Lemma 3.2 that c_2' can be extended to a (k, r)-coloring of G, contrary to (2). Hence we must have $\{4, 6, 7, 8, 9, 10\} \subseteq c_1(N[w_1] \cup \{w_2\} - \{y_1\})$. By symmetry, we also have $\{4, 6, 7, 8, 9, 10\} \subseteq c_1(N[s_1] \cup \{s_2\} - \{y_2\})$. This proves Claim 1.

Claim 2. $c_1(N[w_1] \cup \{w_2\} - \{y_1\}) = \{4, 6, 7, 8, 9, 10\}$ and $c_1(N[s_1] \cup \{s_2\} - \{y_2\}) = \{4, 6, 7, 8, 9, 10\}$.

By contradiction and Claim 1, assume that $c_1(N[w_1] \cup \{w_2\} - \{y_1\}) \supset \{4, 6, 7, 8, 9, 10\}$. Thus $|c_1(N(w_1) - \{y_1\})| \geq 5$, and so the forbidden color set of y_1 is $c_1(\{w_1, w_2, v_5, y_2, y_3\})$. Let $a'' \in ([k] - \{1\}) - c_1(\{w_1, w_2, v_5, y_2, y_3\})$. Define

$$c_2''(z) = \begin{cases} c_1(z) & \text{if } z \in S(c_1) - \{y_1\} \\ a'' & \text{if } z = y_1 \\ 1 & \text{if } z = u_5. \end{cases}$$

With a similar analysis as in Claim 1, c_2'' is a partial (k, r)-coloring with $S(c_2'') = V(G) - \{x, t_1, t_2\}$. By Lemma 3.2, c_2'' can be extended to (k, r)-coloring of G, contrary to (2). Hence we must have $c_1(N[w_1] \cup \{w_2\} - \{y_1\}) = \{4, 6, 7, 8, 9, 10\}$. By symmetry, we also have $c_1(N[s_1] \cup \{s_2\} - \{y_2\}) = \{4, 6, 7, 8, 9, 10\}$. This proves Claim 2.

We now continue the proof of the lemma. Define

$$c_2(z) = \begin{cases} c_1(z) & \text{if } z \in S(c_1) - \{v, v_5, y_1\} \\ 5 & \text{if } z \in \{v, y_1\}. \end{cases}$$

By Claim 2, (3) and since c_1 is a partial (k, r)-coloring of G, we conclude that c_2 is also a partial (k, r)-coloring of G with $S(c_2) = S(c_1) - \{v_5\} = V(G) - \{x, t_1, t_2, u_5, v_5\}$. Since $c_2[y_1] = \{c_2(y_1), c_2(w_1)\}$, $c_2[y_2] = \{c_2(y_2), c_2(s_1)\}$, $c_2[u_5] = \{c_2(v)\}$ and $c_2(y_1) = c_2(v)$, we have $|c_2[y_1] \cup c_2[y_2] \cup c_2[u_5] \cup c_2[y_3]| \leq 4 + r < k$, and so there exists a color $a \in [k] - (c_2[y_1] \cup c_2[y_2] \cup c_2[u_5] \cup c_2[y_3])$. Extend c_2 to c_3 by defining $c_3(v_5) = a$. By the choice of a, c_3 is a partial (k, r)-coloring with $S(c_3) = V(G) - \{x, t_1, t_2, u_5\}$. Since $c_3(v) = c_3(y_1) \in c_3[v_5] \cap c_3[v]$, we have $|c_3[v_5] \cup c_3[v] \cup c_3[x] \leq 8 + 1 < k$. Extend c_3 to c_4 by defining $c_4(u_5) \in [k] - (c_3[v_5] \cup c_3[v] \cup c_3[x])$. Thus c_4 is a partial (k, r)-coloring of G with $S(c_4) = V(G) - \{x, t_1, t_2\}$. As $c_4(u_5) \neq c_4(x_1)$, $c_4(y_1) = 5 \neq c_4(w_2)$, $c_4(y_2) = c(y_2) \neq c(s_2) = c_4(s_2)$, it follows by Lemma 3.2 that c_4 can be extended to a (k, r)-coloring of G, contrary to (2). This proves the lemma. □

Lemma 3.8. *Suppose that $r = 5$ (and so $k = 10$). Each of the following holds for G.*

(i) *Any two bad vertices cannot be weak-adjacent.*
(ii) *Any two semi-bad type vertices cannot be star-adjacent.*
(iii) *Any two semi-bad type vertices cannot be weak-adjacent.*
(iv) *A bad vertex cannot be weak-adjacent to a semi-bad type vertex.*

Proof. (i) Assume that G has two bad vertices u and v which are weak-adjacent. By definition, G has a 2-vertex x adjacent to both u and v. Denote $N_G(u) = \{x, u_1, u_2, u_3, u_4\}$ and $N_G(v) = \{x, v_1, v_2, v_3, v_4\}$, where each u_i is a 2-vertex and each v_j is a 2-vertex. Then G has a subgraph isomorphic to H_7 as depicted in Fig. 4. We shall adopt the notations in Fig. 4 in our arguments below. For $1 \leq i \leq 4$, denote $N_G(u_i) = \{u, u_i'\}$ and $N_G(v_i) = \{v, v_i'\}$.

By (2), $G - v$ has a (k, r)-coloring c. By Lemma 3.6(i), we may assume that,

$$c(u) = 5, \quad \text{for } 1 \leq i \leq 4, \qquad c(u_i) = i, \qquad c(v_i') = i \quad \text{and} \quad c(N[v_i'] - \{v_i\}) = \{1, 2, 3, 4, 5\}. \tag{4}$$

Let c_0 be the restriction of c to $V(G) - \{u, v, v_1, v_2, v_3, v_4, x\}$. Pick a color $a \in \{6, 7, 8, 9, 10\} - c(\{u_1', u_2', u_3', u_4'\})$. Denote $\{6, 7, 8, 9, 10\} = \{a, a', a_2, a_3, a_4\}$. Define

$$c_1(z) = \begin{cases} c_0(z) & \text{if } z \in S(c_0) \\ 5 & \text{if } z = x \\ a & \text{if } z \in \{u, v_1\} \\ a' & \text{if } z = v \\ a_i & \text{if } z = v_i, i \in \{2, 3, 4\}. \end{cases}$$

By (4), c_1 is a (k, r)-coloring of G, contrary to (2). This justifies (i).

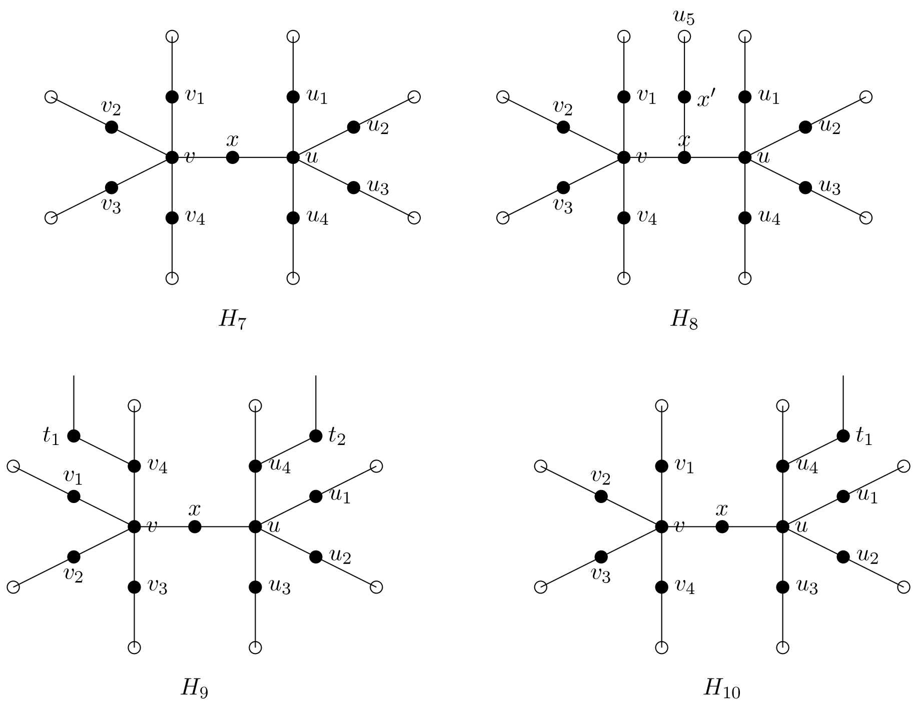

Fig. 4. Four cases of weak-adjacency and star-adjacency.

(ii) Assume that G has two semi-bad type vertices u and v which are star-adjacent. By definition, G has a 3-vertex x adjacent to a 2-vertex as well as to both u and v. Denote $N_G(x) = \{u, v, x'\}$, $N_G(x') = \{u_5, x\}$, $N_G(u) = \{x, u_1, u_2, u_3, u_4\}$ and $N_G(v) = \{x, v_1, v_2, v_3, v_4\}$, where for $1 \le i, j \le 4$, each u_i is a 2-vertex and each v_j is a 2-vertex. Then G has a subgraph isomorphic to H_7 as depicted in Fig. 4. We shall adopt the notation in Fig. 4 in our argument below. For $1 \le i \le 4$, let u_i' (v_i', respectively) denote the other neighbor of u_i (v_i, respectively).

By (2), $G - v$ has a (k, r)-coloring c. By Lemma 3.6(ii), we may assume that,

$$c(u) = 5, \quad \text{for } 1 \le i \le 4, \quad c(v_i') = i, \quad c(N[v_i'] - \{v_i\}) = \{1, 2, 3, 4, 5\} \quad \text{and } \{1, 2, 3, 4\} \subseteq c(\{u_1, u_2, u_3, u_4, u_5\}). \tag{5}$$

Let c_0 be the restriction of c to $V(G) - \{u, v, x, x', v_1, v_2, v_3, v_4\}$.

Case (ii)-1. $c(u_5) \ge 5$, and so by (5) $c(\{u_1, u_2, u_3, u_4\}) = \{1, 2, 3, 4\}$.

Choose colors $a \in \{6, 7, 8, 9, 10\} - c(\{u_1', u_2', u_3', u_4'\})$ and $a' \in \{6, 7, 8, 9, 10\} - \{a, c(u_5)\}$. Denote $\{6, 7, 8, 9, 10\} - \{a', a\} = \{a_2, a_3, a_4\}$. Define

$$c_1(z) = \begin{cases} c_0(z) & \text{if } z \in S(c_0) \\ 5 & \text{if } z = v \\ a & \text{if } z \in \{u, v_1\} \\ a' & \text{if } z = x \\ a_i & \text{if } z = v_i, i \in \{2, 3, 4\}. \end{cases}$$

By (5), c_1 is a partial (k, r)-coloring with $S(c_1) = V(G) - \{x'\}$ such that $c_1(x) \ne c_1(u_5)$. By Lemma 3.2, c_1 can be extended to a (k, r)-coloring of G, contrary to (2). This proves Case (ii)-1.

Case (ii)-2. $c(u_5) \in \{1, 2, 3, 4\}$. By symmetry, we assume that $c(u_5) = 1$.

By Lemma 3.6(ii), $\{2, 3, 4\} \subseteq c(\{u_1, u_2, u_3, u_4\})$, and so we may assume that $c(u_i) = i$, $(2 \le i \le 4)$, and $c(u_1) \in \{1, 6, 7, 8, 9, 10\}$.

Case (ii)-2.1. $c(u_1) = 1$.

Choose $a \in \{6, 7, 8, 9, 10\} - c(\{u_1', u_2', u_3', u_4'\})$ and $a' \in \{6, 7, 8, 9, 10\} - \{a\}$. Denote $\{6, 7, 8, 9, 10\} - \{a', a\} = \{a_2, a_3, a_4\}$. Define

$$c_1(z) = \begin{cases} c_0(z) & \text{if } z \in S(c_0) \\ 5 & \text{if } z = x \\ a & \text{if } z \in \{u, v_1\} \\ a' & \text{if } z = v \\ a_i & \text{if } z = v_i, i \in \{2, 3, 4\}. \end{cases}$$

By (5), c_1 is a partial (k, r)-coloring with $S(c_1) = V(G) - \{x'\}$, such that $c_1(x) = 5 \neq 1 = c_1(u_5)$. By Lemma 3.2, c_1 can be extended to a (k, r)-coloring of G, contrary to (2). This proves Case (ii)-2.1.

Case (ii)-2.2. $c(u_1) \in \{6, 7, 8, 9, 10\}$.

Choose a color $a \in \{1, 6, 7, 8, 9, 10\} - c(\{u_1, u_1', u_2', u_3', u_4'\})$. If $a = 1$, denote $\{6, 7, 8, 9, 10\} = \{a', a_1, a_2, a_3, a_4\}$. Define

$$c_1(z) = \begin{cases} c_0(z) & \text{if } z \in S(c_0) \\ 5 & \text{if } z = x \\ a & \text{if } z = u \\ a' & \text{if } z = v \\ a_i & \text{if } z = v_i, i \in \{1, 2, 3, 4\}. \end{cases}$$

If $a \in \{6, 7, 8, 9, 10\}$, denote $\{6, 7, 8, 9, 10\} = \{a, a', a_2, a_3, a_4\}$. Define

$$c_1(z) = \begin{cases} c_0(z) & \text{if } z \in S(c_0) \\ 5 & \text{if } z = x \\ a & \text{if } z \in \{u, v_1\} \\ a' & \text{if } z = v \\ a_i & \text{if } z = v_i, i \in \{2, 3, 4\}. \end{cases}$$

By (5), c_1 is a partial (k, r)-coloring with $S(c_1) = V(G) - \{x'\}$ such that $c_1(x) \neq c_1(u_5)$. By Lemma 3.2, c_1 can be extended to a (k, r)-coloring of G, contrary to (2). This proves Case (ii)-2.2, and completes the proof of (ii).

(iii) By contradiction, assume that G has two semi-bad type vertices u and v which are weak-adjacent. By definition, G has a 2-vertex x adjacent to both u and v. Denote $N_G(u) = \{x, u_1, u_2, u_3, u_4\}$ and $N_G(v) = \{x, v_1, v_2, v_3, v_4\}$. By definition, we assume that u_1, u_2, u_3 and v_1, v_2, v_3 are 2-vertices, u_4 is a 3-vertex with $N_G(u_4) = \{u, u_4', t_2\}$, and v_4 is a 3-vertex with $N_G(v_4) = \{v, v_4', t_1\}$. Also denote $N_G(t_1) = \{v_4, t_1'\}$ and $N_G(t_2) = \{u_4, t_2'\}$. For each $1 \leq i \leq 3$, let $N_G(u_i) = \{u, u_i'\}$ and $N_G(v_i) = \{v, v_i'\}$. Then G has a subgraph isomorphic to H_9 as depicted in Fig. 4. We shall adopt the notations in Fig. 4 in our argument below.

By (2), $G - v$ has a (k, r)-coloring c. By Lemma 3.6(ii), we may assume that, for some color a with $1 \leq a \leq 10$,

$$\begin{aligned} &c(u) = 5, \quad \text{and for } 1 \leq i \leq 4,\ c(u_i) = i, \quad \text{for } 1 \leq j \leq 3, \quad c(N[v_j'] - \{v_j\}) = \{1, 2, 3, 4, 5\}, \\ &\text{and} \quad c((N(v_4') - \{v_4\}) \bigcup \{t_1'\}) = \{1, 2, 3, 4, a\}. \end{aligned} \tag{6}$$

Let c_0 be the restriction of c to $V(G) - \{u, v, v_1, v_2, v_3, v_4, x, t_1, t_2\}$. Choose $a_1 \in \{6, 7, 8, 9, 10\} - c(\{u_1', u_2', u_3', u_4'\})$. Define

$$c_1(z) = \begin{cases} c_0(z) & \text{if } z \in S(c_0) \\ 5 & \text{if } z = x \\ a_1 & \text{if } z \in \{u, v_1\}. \end{cases}$$

By (6), c_1 is a partial (k, r)-coloring with $S(c_1) = V(G) - \{v, v_2, v_3, v_4, t_1, t_2\}$.

Case (iii)-1. $a \in \{1, 2, 3, 4, 5\}$. Thus by (6), $c_1(t_1') \in \{1, 2, 3, 4, 5\}$.

Denote $\{6, 7, 8, 9, 10\} = \{a_1, a', a_2, a_3, a_4\}$. Define

$$c_2(z) = \begin{cases} c_0(z) & \text{if } z \in S(c_1) \\ a' & \text{if } z = v \\ a_i & \text{if } z = v_i, i \in \{2, 3, 4\}. \end{cases}$$

Case (iii)-2. $a \in \{6, 7, 8, 9, 10\}$

Choose $a_4 \in \{6, 7, 8, 9, 10\} - \{a, a_1\}$. Denote $\{6, 7, 8, 9, 10\} - \{a_1, a_4\} = \{a', a_2, a_3\}$. Define

$$c_2(z) = \begin{cases} c_0(z) & \text{if } z \in S(c_1) \\ a' & \text{if } z = v \\ a_i & \text{if } z = v_i, i \in \{2, 3, 4\}. \end{cases}$$

By (5), c_2 is a partial (k, r)-coloring with $S(c_2) = V(G) - \{t_1, t_2\}$ such that $c_2(t_1') \neq c_2(v_4)$ and $c_2(t_2') \neq c_2(u_4)$. By Lemma 3.2, c_2 can be extended to a (k, r)-coloring of G, contrary to (2). This proves Case (iii).

(iv) By Contradiction, we assume that a semi-bad type vertex u is weak-adjacent to a bad vertex v in G. Denote $N_G(u) = \{x, u_1, u_2, u_3, u_4\}$ and $N_G(v) = \{x, v_1, v_2, v_3, v_4\}$. By definition, we assume that u_1, u_2, u_3 and v_1, v_2, v_3, v_4 are 2-vertices, u_4 is a 3-vertex with $N_G(u_4) = \{u, u_4', t_1\}$, and $N_G(t_1) = \{u_4, t_1'\}$. Then G has a subgraph isomorphic to H_{10} as depicted in Fig. 4. We shall adopt the notations in Fig. 4 in our arguments below. For $1 \le i \le 3$, denote $N_G(u_i) = \{u, u_i'\}$; and for $1 \le j \le 4$, denote $N_G(v_j) = \{v, v_j'\}$.

By (2), $G - v$ has a (k, r)-coloring c. By Lemma 3.6(i), we may assume that

$$c(u) = 5, \quad \text{for } 1 \le i \le 4, \qquad c(u_i) = c(v_i') = i \quad \text{and} \quad c(N[v_i'] - \{v_i\}) = \{1, 2, 3, 4, 5\}. \tag{7}$$

Let c_0 be the restriction of c to $V(G) - \{u, v, v_1, v_2, v_3, v_4, x, t_1\}$. Choose $a \in \{6, 7, 8, 9, 10\} - c(\{u_1', u_2', u_3', u_4'\})$, and let $\{6, 7, 8, 9, 10\} = \{a, a', a_2, a_3, a_4\}$.

Define

$$c_1(z) = \begin{cases} c_0(z) & \text{if } z \in S(c_0) \\ 5 & \text{if } z = x \\ a & \text{if } z = u, v_1 \\ a' & \text{if } z = v \\ a_i & \text{if } z = v_i, i \in \{2, 3, 4\}. \end{cases}$$

By (7), c_1 is a partial (k, r)-coloring with $S(c_1) = V(G) - \{t_1\}$ such that $c_1(t_1') \neq c_1(u_4)$. By Lemma 3.2, c_1 can be extended to a (k, r)-coloring of G, contrary to (2). This completes the proof of (iv). □

Lemma 3.9. *Suppose that $r = 5$. Let $F_1 = \{f_1, f_2, f_3, f_4, f_5\}$ be the set of faces incident with a bad vertex v of G, as shown in the graph H_4 depicted in Fig. 2; and $F_2 = \{f_1, f_2, f_3\}$ be the subset set of faces incident with a semi-bad type vertex v of G, as shown in the graph H_5 depicted in Fig. 2. Let s and t be the vertices as shown in H_4 or in H_5 in Fig. 2. Suppose that $f = v_2u_2vu_3v_3s$ is a 6-face which is in F_1 or in F_2. Then each of the following holds.*

(i) *$d_G(s) \ge 3$, and*

(ii) *if $d_G(s) = 3$, then $d_G(t) \ge 3$.*

Proof. We shall argue using the notations in Fig. 2. By (2), $G - v$ has a (k, r)-coloring c. By Lemma 3.6, we may assume that $c(v_i) = i$ for $1 \le i \le 5$, $c(s) \in \{1, 2, 3, 4, 5\}$, and for $1 \le j \le 4$, $c(N[v_j] - \{u_j\}) = \{1, 2, 3, 4, 5\}$. Furthermore, if v is a bad vertex, then $c(N[v_5] - \{u_5\}) = \{1, 2, 3, 4, 5\}$, and if v is a semi-bad type vertex, then $\{1, 2, 3, 4\} \subseteq c(N(v_5) - \{u_5\}) \cup \{x_1\}$. Thus $c(u_5) \in \{6, 7, 8, 9, 10\}$.

(i) Assume first by contradiction that $d_G(s) = 2$ and $N_G(s) = \{v_2, v_3\}$. Let c_1 be the restriction of c to $V(G) - \{s, u_1, u_2, u_3, u_4, v\}$. Denote $\{6, 7, 8, 9, 10\} = \{a, c(u_5), a_1, a_3, a_4\}$. Extend c_1 to a (k, r)-coloring c_2 by defining $c_2(u_2) = c(s)$, $c_2(v) = a$, and $c_2(u_i) = a_i$ for $i = 1, 3, 4$. Now $S(c_2) = V(G) - \{s\}$, $c_2(v_2) \neq c_2(v_3)$ and $c_2[v_2] \cup c_2[v_3] = \{1, 2, 3, 4, 5, a_3\}$. By Lemma 3.2, c_2 can be extended to a (k, r)-coloring of G by coloring s, contrary to (2).

(ii) Now assume that $d_G(s) = 3$ and $N_G(s) = \{t, v_2, v_3\}$. By contradiction, assume that $d_G(t) = 2$, let $t' \neq s$ be another neighbor of t. Let c_1 be the restriction of c to $V(G) - \{s, t, u_1, u_2, u_3, u_4, v\}$. Denote $\{6, 7, 8, 9, 10\} = \{a, c(u_5), a_1, a_3, a_4\}$. Extend c_1 to a (k, r)-coloring c_2 by defining $c_2(u_2) = c(s)$, $c_2(v) = a$, and $c_2(u_i) = a_i$ for $i = 1, 3, 4$. Now $S(c_2) = V(G) - \{s, t\}$. As $c_2(v_2) \neq c_2(v_3)$, $\{c_2(t)\} = \phi$ and as $|c_2[v_2] \cup c_2[v_3] \cup c_2[t]| \le 7$, we conclude that c_2 can be extended to a partial (k, r)-coloring c_3 by defining $c_3(s) \in [k] - (c_2[v_2] \cup c_2[v_3] \cup c_2[t])$, with $S(c_3) = V(G) - \{t\}$. Since $c_3(s) \neq c_3(t')$ and since $|c_3[t'] \cup c_3[s]| \le r + 3 < k$, by Lemma 3.2, c_3 can be extended to a (k, r)-coloring of G by coloring t, contrary to (2). □

3.2. Discharging

We will complete the proof of Theorem 1.4 in this subsection. Throughout this section, G always denotes a 2-connected plane graph embedded on the plane with girth at least 6. Let $F = F(G)$ denote the set of all faces of G. We will use $V = V(G)$ and $E = E(G)$. We assign the initial charges to the vertices and faces of G as a weight function w defined as follows

$$w(x) = \begin{cases} 2d_G(x) - 6 & \text{if } x \in V \\ d_G(x) - 6 & \text{if } x \in F. \end{cases}$$

By Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and by the relation $\sum_{v \in V} d(v) = \sum_{f \in F} d(f) = 2|E|$ (Theorem 10.10 of [1]), it follows that

$$\sum_{x \in V(G) \cup F(G)} w(x) = \sum_{v \in V} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -12. \tag{8}$$

Discharging Rules We will recharge the vertices and faces of G with certain charge and discharge rules. The resulting new charge will be denoted as a new weight function w'. A contradiction to (8) will then be obtained if the new charge w'

satisfies $w'(x) \geq 0$ for all $x \in V \bigcup F$. This contradiction then will establish Theorem 1.4. In the following, we will describe our recharge and discharging rules based on the different cases. Depending whether $r = 5$ or not, we use different rules. In the discharge rules (R1) and (R2) defined below, for all unmentioned vertex or face $x \in V \cup F$, we do not change the charge of x. That is, $w'(x) = w(x)$.

(R1) Suppose that $r \neq 5$. For a vertex v, and for each $i \geq 0$, let $n_i(v)$ be the number of i-vertices in $N_G(v)$, and define $n_{i^+}(v) = \sum_{j \geq i} n_j(v)$.

(i) If a 2-vertex v is adjacent to two 4^+-vertices v_1, v_2, then increase the charge of v by 2, and for $i = 1, 2$, reduced the charge of v_i by 1.

(ii) If a 2-vertex v is adjacent to one 4^+-vertex v_1, and one 3-vertex v_2 such that $N_G(v_2) = \{v, v_2^1, v_2^2\}$, then increase the charge of v by 2, reduced the charge of v_1 by 1, and for $i = 1, 2$, reduced the charge of v_2^i by $\frac{1}{2}$.

(iii) If a 2-vertex v is adjacent to two 3-vertices v_1, v_2 such that for $1 \leq j \leq 2$, $N_G(v_j) = \{v, v_j^1, v_j^2\}$, (as girth of G is at least 6, $N_G(v_1) \cap N_G(v_2) = \{v\}$,) then increase the charge of v by 2, and for $1 \leq i, j \leq 2$, decrease the charge of v_j^i by $\frac{1}{2}$.

Claim 1. *Let $w'(x)$ denote the new charge of each $x \in V \cup F$ after the applications of* (R1). *Then for any $x \in V \cup F$, we have $w'(x) \geq 0$.*

Proof of Claim 1. Since the girth of G is at least 6, if follows that for any $f \in F$, we have $w'(f) = w(f) = d(f) - 6 \geq 0$. Let $v \in V$ be a d-vertex and $N_G(v) = \{v_1, v_2, \cdots, v_d\}$.

Case 1.1 $d_G(v) = 2$. By Lemma 3.1, $n_2(v) = 0$ and each 3-vertex incident with v must be adjacent to two other 4^+-vertices. Thus either $n_{4^+}(v) = 2$, whence by (R1)(i), $w'(v) = 2 \times 2 - 6 + 2 = 0$; or $n_{4^+}(v) = 1$, whence by (R1)(ii), $w'(v) = 2 \times 2 - 6 + 2 = 0$; or $n_{4^+}(v) = 0$, whence by (R1)(iii), $w'(v) = 2 \times 2 - 6 + 2 = 0$.

Case 1.2 $d_G(v) = 3$. By (R1), we conclude that $w'(v) = w(v) = 2 \times 3 - 6 = 0$.

Case 1.3 $d_G(v) = 4$. By Lemma 3.3(i), $n_2(v) \leq 2$. If $n_2(v) = 0$, then by (R1), for each weak-3-neighbor of v, v will discharge $\frac{1}{2}$ through this weak-3-neighbor to a 2-vertex. Since $d_G(v) = 4$, we have $w'(v) \geq 2 \times 4 - 6 - 4 \times \frac{1}{2} = 0$. Now we assume that $n_2(v) > 0$. Thus by (R1), if $n_2(v) = 2$, then by Lemma 3.3(ii) v cannot be adjacent to any weak 3-vertex, and so $w'(v) = 2 \times 4 - 6 - 2 \times 1 = 0$; and if $n_2(v) = 1$, then by Lemma 3.3(iii) v is adjacent to at most two weak-3-vertices, and so $w'(v) \geq 2 \times 4 - 6 - 1 - 2 \times \frac{1}{2} = 0$.

Case 1.4 $d_G(v) = 5$. By Lemma 3.4, either $n_2(v) = 4$ and $n_{4^+}(v) = 1$, whence by (R1), $w'(v) \geq 2 \times 5 - 6 - 4 \times 1 = 0$; or $n_2(v) \leq 3$, whence by (R1), $w'(v) \geq 2 \times 5 - 6 - n_2(v) - \frac{1}{2} \times (5 - n_2(v)) = \frac{3}{2} - \frac{n_2(v)}{2} \geq 0$.

Case 1.5 $d_G(v) \geq 6$. Then $n_2(v) + n_3(v) \leq d_G(v)$, and so $w'(v) \geq 2 \times d(v) - 6 - d(v) = d(v) - 6 \geq 0$. This completes the proof of Claim 1.

(R2) Suppose that $r = 5$. For a vertex v, let $n_2^*(v)$ be the number of 2-vertices star-adjacent to v and $n_3^*(v)$ be the number of semi-bad type vertices star-adjacent to v.

(i) If a 4^+-vertex v is adjacent to 2-vertices $v_1, v_2, \ldots, v_{d_1}$, then reduce the charge of v by d_1, and for $1 \leq i \leq d_1$, increase the charge of v_i by 1.

(ii) If a 4^+-vertex v is star-adjacent to 2-vertices $v_1, v_2, \ldots, v_{d_2}$, then reduce the charge of v by $\frac{d_2}{2}$, and for $1 \leq i \leq d_2$, increase the charge of v_i by $\frac{1}{2}$.

(iii) If a 4-vertex v is star-adjacent to semi-bad type vertices $v_1, v_2, \ldots, v_{d_3}$, then reduce the charge of v by $\frac{d_3}{2}$, and for $1 \leq i \leq d_3$, increase the charge of v_i by $\frac{1}{2}$.

(iv) If a 7^+-face f is incident with bad or semi-bad type vertices $v_1, v_2, \ldots, v_{d_4}$, then reduce the charge of f by $\frac{4d_4}{7}$, and for $1 \leq i \leq d_4$, increase the charge of v_i by $\frac{4}{7}$.

(v) If a 5-vertex v is weak-adjacent to bad or semi-bad type vertices $v_1, v_2, \ldots, v_{d_5}$, then reduce the charge of v by $\frac{d_5 \times (2 \times 5 - 6 - n_2(v) - \frac{1}{2} n_2^*(v))}{n_2(v)}$, and for $1 \leq i \leq d_5$, increase the charge of v_i by $\frac{2 \times 5 - 6 - n_2(v) - \frac{1}{2} n_2^*(v)}{n_2(v)}$.

Claim 2. *Let F_1, F_2 be the two sets of faces defined in Lemma 3.9, as shown in the graphs H_4 and H_5 in Fig. 2, respectively, and use the notations in Fig. 2. Each of the following holds.*

(i) *If F_1 has at least four 6-faces, then there exist at least three vertices in the 5-vertices v_1, v_2, v_3, v_4, v_5, each of which is adjacent to at most three 2-vertices.*

(ii) *If all faces in F_2 are all 6-faces, then each of the two 5-vertices v_2, v_3 is adjacent to at most three 2-vertices.*

Proof of Claim 2. As defined in Lemma 3.9, the faces in F_1 are all incident with a bad vertex v, with $N_G(v) = \{u_1, u_2, u_3, u_4, u_5\}$. By the definition of a bad vertex, for each $1 \leq i \leq 5$, u_i is a 2-vertex and v_i is a 5-vertex adjacent to u_i. Let f_i denote the face in F_1 incident with v_{i-1} and v_i, for all integer i (mod 5). Let $N' = \{v_i \mid f_i \text{ and } f_{i+1} \text{ are 6-faces}\}$. Therefore if F_1 contains four 6-faces, then $|N'| \geq 3$, (see H_4 in Fig. 2). Without lose of generality, we assume that $v_2 \in N'$, and $s \in N_G(v_2) \cap N_G(v_3)$. Since $v_2 \in N'$, both f_2 and f_3 are 6-faces. By Lemma 3.9, s must be a 3^+-vertex, and furthermore, s is not a weak 3-vertex. Thus we conclude that each vertex in N' is adjacent to at most three 2-vertices. This justifies Claim 2(i). The proof for Claim 2(ii) is similar and will be omitted. □

Claim 3. *Let f be a face. Let $w'(f)$ denote the new charge after performing* (R2).

(i) *If f is a 6-face, then $w'(f) = 0$.*
(ii) *If f is a 7^+-face, then $w'(f) \geq 0$.*

Proof of Claim 3. By (R2), any 6-face neither receives charges from other vertices, nor does it discharge to other vertices, and so $w'(f) = w(f) = d(f) - 6 = 0$. Thus (i) follows. If $d(f) \geq 7$, then by Lemma 3.8, f is incident with at most $\lfloor \frac{d(f)}{4} \rfloor$ bad or semi-bad type vertices. It follows by (R2)(iv) that $w'(f) \geq w(f) - \frac{4}{7} \times \frac{d(f)}{4} = d(f) - 6 - \frac{d(f)}{7} \geq 0$. □

Claim 4. *For any $v \in V(G)$, let $w'(v)$ denote the new charge after performing recharge rule* (R2). *Then $w'(v) \geq 0$.*

Proof of Claim 4. We examine the value of $w'(v)$ based on the degree of v. By Lemma 3.1(i), $d_G(v) \geq 2$.

Case 2.1 $2 \leq d_G(v) \leq 3$. The justification for this case is identical to those of Cases 1.1 and 1.2 in the proof of Claim 1, with (R1) replaced by (R2). Thus it is omitted.

Case 2.2 $d_G(v) = 4$. By Lemma 3.3(i), $n_2(v) \leq 2$.

Assume first that $n_3^*(v) = 0$. If $n_2(v) = 0$, then by (R2)(ii), for each weak-3-neighbor of v, v will discharge $\frac{1}{2}$ through this weak-3-neighbor to a 2-vertex. Since $d_G(v) = 4$, we have $w'(v) \geq 2 \times 4 - 6 - 4 \times \frac{1}{2} = 0$. Now we assume that $n_2(v) > 0$. If $n_2(v) = 2$, then by Lemma 3.3(ii) v cannot be adjacent to any weak 3-vertex, and so by (R2)(i) $w'(v) = 2 \times 4 - 6 - 2 \times 1 = 0$; If $n_2(v) = 1$, then by Lemma 3.3(iii) v is adjacent to at most two weak 3-vertices, and so by (R2)(i) and (ii), $w'(v) \geq 2 \times 4 - 6 - 1 - 2 \times \frac{1}{2} = 0$.

Now assume that $n_3^*(v) \geq 1$. By Lemma 3.6(ii-1), $n_2(v) = 0$; and by Lemma 3.7, v is adjacent to at most two weak 3-vertices. Hence by definition, $n_3^*(v) \leq 2$. It follows that either $n_3^*(v) = 2$, and so by (R2)(iii), $w'(v) = 2 \times 4 - 6 - 2 \times 2 \times \frac{1}{2} = 0$; or $n_3^*(v) = 1 \leq n_2^*(v) \leq 2$, and so by (R2)(ii) and (iii), $w'(v) \geq 2 \times 4 - 6 - 2 \times \frac{1}{2} - \frac{1}{2} = \frac{1}{2}$.

Case 2.3 $d_G(v) = 5$. Let F_1 and F_2 be the sets of faces defined in Lemma 3.9.

Suppose first that v is a bad vertex with F_1 being the set of faces incident with v, such that F_1 has $t \geq 0$ 7^+-faces and $5 - t$ 6-faces. It follows by (R2)(iv) and (v) that if $t \geq 2$, then $w'(v) \geq 2 \times 5 - 6 - 5 + t \times \frac{4}{7} \geq \frac{1}{7}$; and if $t \leq 1$, then by Claim 2(i), v receives at least $\frac{1}{3}$ from each weak-adjacent 5-vertex, and so $w'(v) \geq 2 \times 5 - 6 - 5 + 3 \times \frac{1}{3} = 0$.

Suppose that v is a semi-bad type vertex with F_2 being a subset of faces incident with v, such that F_2 has $t \geq 0$ 7^+-faces and $3 - t$ 6-faces. It follows by (R2)(iv) and (v) that if $t \geq 1$, then $w'(v) \geq 2 \times 5 - 6 - 4 - \frac{1}{2} + t \times \frac{4}{7} > 0$; and if $t = 0$, then by Claim 2(ii), v receives at least $\frac{1}{3}$ from each weak-adjacent 5-vertex, and so $w'(v) \geq 2 \times 5 - 6 - 4 - \frac{1}{2} + 2 \times \frac{1}{3} = \frac{1}{6} > 0$.

Finally we assume that v is neither a bad vertex nor a semi-bad type vertex. Then by Lemma 3.5, $n_2(v) \leq 4$. It follows by (R2)(i), (ii) and (v) that either $n_2(v) = 4$, whence $w'(v) \geq 2 \times 5 - 6 - 4 \times 1 = 0$; or $n_2(v) \leq 3$, whence $w'(v) \geq 2 \times 5 - 6 - 3 - 2 \times \frac{1}{2} = 0$.

Case 2.4 $d_G(v) \geq 6$.

It follows by (R2)(i) and (ii) that $w'(v) \geq 2 \times d(v) - 6 - d(v) = d(v) - 6 \geq 0$. This completes the proof of Claim 4.

By (R1) and (R2), after the recharge process, we obtain a new charge w' satisfying $\sum_{x \in F \cup V} w'(x) = \sum_{x \in F \cup V} w(x)$. By Claims 1, 3 and 4, $w'(x) \geq 0$ for any $x \in V(G) \cup F(G)$. It follows by (8) that $0 \leq \sum_{x \in F \cup V} w'(x) = \sum_{x \in F \cup V} w(x) = -12 < 0$. This contradiction establishes Theorem 1.4.

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