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A Note on Arboricity of 2-edge-connected Cubic Graphs

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Abstract: The vertex-arboricity $a(G)$ of a graph G is the minimum number of subsets into which the set of vertices of G can be partitioned so that each subset induces a forest. It is well known that $a(G) \leq 3$ for any planar graph G, and that $a(G) \leq 2$ for any planar graph G of diameter at most 2. The conjecture that every planar graph G without 3-cycles has a vertex partition (V_1, V_2) such that V_1 is an independent set and V_2 induces a forest was given in [*European J. Combin.*, 2008, 29(4): 1064-1075]. In this paper, we prove that a 2-edge-connected cubic graph which satisfies some condition has this partition. As a corollary, we get the result that every up-embeddable 2-edge-connected cubic graph G ($G \neq K_4$) has a vertex partition (V_1, V_2) such that V_1 is an independent set and V_2 induces a forest.

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0 Introduction

Graphs considered in this paper are undirected, finite and without loops. The *vertex-arboricity* $a(G)$ (*linear vertex-arboricity* $\mathrm{la}(G)$, respectively) of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a forest (a linear forest, respectively).

Chartrand, Kronk and Wall[6] first introduced the vertex-arboricity of a graph. They proved that the vertex-arboricity of planar graphs is at most 3. Then the first two authors[5] characterized a planar graph of the vertex-arboricity 3. Yang and Yuan[14] showed that $a(G) \leq 2$ for any planar graphs G of diameter 2. It is proved in [11] that every planar graph G without triangles at distance less than 2 satisfies $a(G) \leq 2$, and that every 2-degenerate graph G has a vertex partition (V_1, V_2) such that V_1 is an independent set and V_2 induces a forest. Catlin et al.[3] proved that if a connected graph G is neither a cycle nor a clique, then there is a coloring of $V(G)$ with at most $\lceil \frac{\triangle(G)}{2} \rceil$ colors, such that each color class induces a forest and one of those induced forests is a maximum induced forest in G. Borodin et al.[2] defined a similar concept for the point arboricity of a graph G. For more results about the vertex-arboricity $a(G)$ of a graph G, the reader is referred to [1, 4, 7, 9, 12–13].

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The *edge-connectivity* $\kappa(G)$ of a graph G is the minimum number of edges whose removal from G results in a disconnected graph or trivial graph. A graph G is *n-edge-connected* ($n \geq 1$), if $\kappa(G) \geq n$. For an integer $k > 0$, a graph G is called *k-regular*, if the degree of every vertex is k. A 3-regular graph is also called a *cubic graph.*

Let S be an orientable surface and G be a connected graph. A graph G is said to be *embedded* in a surface S if it is drawn on S so that edges intersect only at their common end vertices and each component, called a *face*, of $S \setminus G$ is homeomorphic to an open disk.

The maximal genus, $\gamma_M(G)$, of a graph G is the maximum genus among the genera of all surfaces S in which G can be embedded. A graph G is called *up-embeddable* if the maximum genus of G is equal to $\lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G) = |E| - |V| + 1$ is the Beta number of G.

Let H be a cubic graph, $e_1, e_2 \in E(H)$ (e_1 and e_2 may be the same edge). A new vertex is inserted at each of e_1 and e_2, denoted by w_1 and w_2 respectively. Then two new vertices u and v are added. The graph $N = N(u,v)(H)$ obtained from H by adding edges w_1u, w_2v and two double edges joining u and v is called an *N-extension of H* (see Fig. 1).

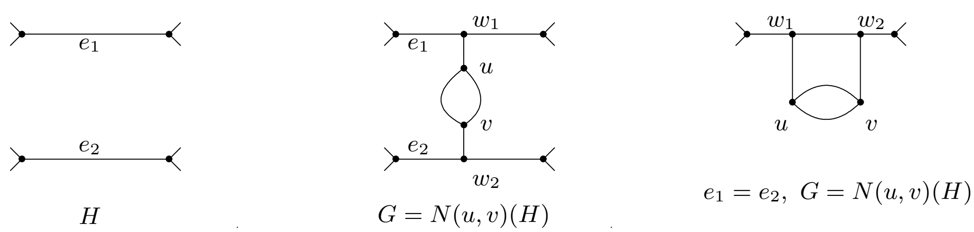

Figure 1 N-extension of the graph H

Let H be a cubic graph, $e_1, e_2, e_3 \in E(H)$ (some of e_1, e_2 and e_3 may be identical). A new vertex is inserted at each of e_1, e_2 and e_3, denoted by w_1, w_2 and w_3 respectively, then a new vertex u is added. The graph $G = M(u)(H)$ obtained from H by adding edges w_1u, w_2u and w_3u is called an *M-extension of H* (see Fig. 2).

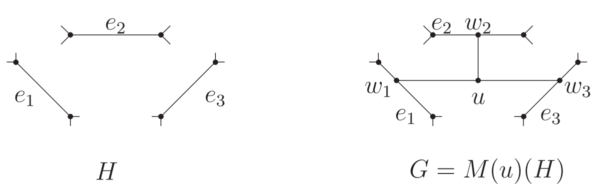

Figure 2 M-extension of the graph H

Let $\theta_i, i = 1, 2, 3$ and a *dumbbell graph* be connected graphs depicted respectively in Fig. 3. A 2-edge-connected cubic graph G is called a *double dumbbell graph* if G contains two vertex disjoint induced subgraphs $G[w_1, w_2, w_3, w_4]$ and $G[v_1, v_2, v_3, v_4]$ such that each of them is a dumbbell graph.

For example, Peterson graph is obtained from θ_1 by two M-extensions. The detail is shown in Fig. 4.

In [11], the following conjecture was given.

Conjecture Every planar graph G without 3-cycles has a vertex partition (V_1, V_2) such that V_1 is an independent set and V_2 induces a forest.

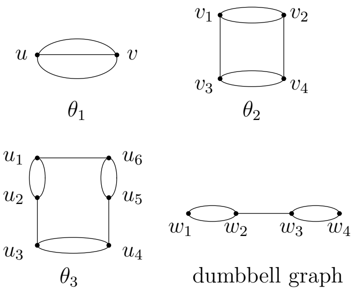

Figure 3 $\theta_i, i = 1, 2, 3$ and dumbbell graph

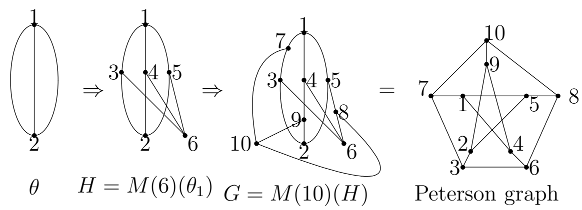

Figure 4 M-extensions to Peterson graph

Although there are some results on $a(G)$ for planar graphs, few characterizations of the vertex arboricity are known for non-planar graphs. In this paper, we investigate arboricity for a class of 2-edge-connected cubic, not necessarily planar graphs. The following results are obtained.

Theorem 0.1 Every 2-edge-connected cubic graph G $(G \neq K_4)$ which is not obtained from any double dumbbell graph by a series of M-extensions or N-extensions has a vertex partition (V_1, V_2) such that V_1 is an independent set and V_2 induces a forest.

The following corollaries follow from Theorem 0.1.

Corollary 0.1 If G is a 2-edge-connected cubic graph which is not obtained from any double dumbbell graph by a series of M-extensions or N-extensions, then $a(G) = 2$.

Corollary 0.2 Every 2-edge-connected cubic up-embeddable graph G $(G \neq K_4)$ has a vertex partition (V_1, V_2) such that V_1 is an independent set and V_2 induces a forest.

Corollary 0.3 If G is a 2-edge-connected cubic up-embeddable graph, then $a(G) = 2$.

1 Proofs of Main Results

For a graph G, if $V(G)$ can be partitioned into an ordered pair (V_1, V_2) such that V_1 is an independent set and $G[V_2]$ is acyclic (i.e., V_2 induces a forest), then the partition pair (V_1, V_2) is called a *good pair of* G, where $G[V_2]$ is an induced subgraph of G by V_2. Note that while K_4 does not have a good pair, we do have $a(K_4) = 2$.

Lemma 1.1 Let G' be a cubic graph that has a good pair (V_1', V_2'), and let G be a cubic graph obtained from G' by an M-extension or by an N-extension. Then G also has a good pair.

Proof First, suppose that $G = M(u)(G')$ is a cubic graph obtained from G' by an M-extension (see Fig. 2). Let $V_1 = V_1' \cup \{u\}$ and $V_2 = V(G) \setminus V_1 = V_2' \cup \{w_1, w_2, w_3\}$. Since (V_1', V_2') is a good pair of G', by the definition of M-extensions, (V_1, V_2) is a good pair of G.

Second, suppose that $G = N(u,v)(G')$ is obtained from G' by an N-extension (see Fig. 1). Let $V_1 = V_1' \cup \{u\}$ and $V_2 = V(G) \setminus V_1 = V_2' \cup \{w_1, w_2, v\}$. Since (V_1', V_2') is a good pair of G', by the definition of N-extensions, (V_1, V_2) is a good pair of G. Thus this lemma is proved.

Lemma 1.2 Let G be a 2-edge-connected graph. If G is obtained from θ_i $(i = 1, 2, 3)$ by a series of M-extensions or N-extensions, then G has a good pair.

Proof By Lemma 1.1, it suffices to verify that each of the θ_i's has a good pair.

Let $V_{11} = \{u\}, V_{12} = \{v\}$, $V_{21} = \{v_1, v_4\}, V_{22} = \{v_2, v_3\}$, $V_{31} = \{u_1, u_3, u_5\}$ and $V_{32} = \{u_2, u_4, u_6\}$. It is easy to see that V_{i1} is an independent set and V_{i2} induces a forest, thus (V_{i1}, V_{i2}) is a good pair of θ_i for each $i \in \{1, 2, 3\}$.

Lemma 1.3 Let G $(G \neq K_4)$ be a 2-edge-connected graph. If G is obtained from K_4 by a series of M-extensions or N-extensions, then G has a good pair.

Proof Since $G \neq K_4$, there exists a sequence of cubic graphs $G_0, G_1, \cdots, G_k$, such that $G_0 = K_4$ and $G_k = G$, and such that for $i = 0, 1, \cdots, k-1$, G_{i+1} is obtained from G_i by an M-extension or an N-extension.

By Lemma 1.1 and the assumption that $G \neq K_4$, it suffices to verify that G_1 has a good pair. We shall consider the following two cases.

Case I $G_1 = M(u)(K_4)$ is a graph obtained from K_4 by an M-extension (see Fig. 2). Let the vertex set of K_4 be $\{v_1, v_2, v_3, v_4\}$ as depicted in Fig. 5. By the definition of an M-extension, three edges e_1, e_2, e_3 must be chosen from $E(K_4)$ to perform the M-extension. By symmetry, G_1 has the following three possibilities.

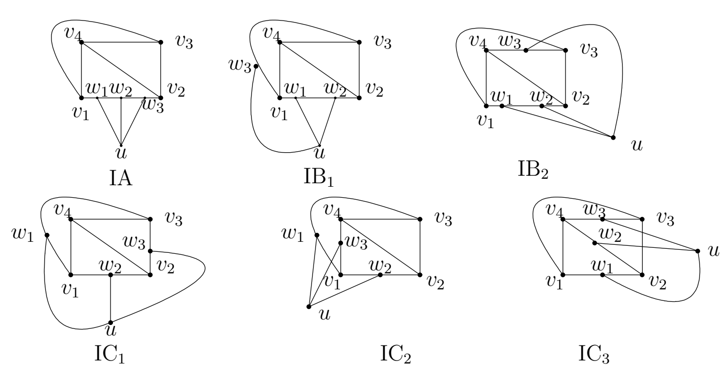

Figure 5 One M-extension of K_4

Case IA Suppose that $e_1 = e_2 = e_3$ in this M-extension of K_4. Without loss of generality, we assume that $e_1 = e_2 = e_3 = v_1v_2$, and the resulting graph G_1 after the M-extension is shown in Fig. 5 IA.

Case IB Suppose that $e_1 = e_2$ and $e_3 \neq e_1$ in this M-extension of K_4. Without loss of generality, we assume that $e_1 = e_2 = v_1v_2$. Depending on whether e_3 is adjacent to e_1 or not, we may further assume that $e_3 = v_1v_3$ (if e_1 and e_3 are adjacent) or $e_3 = v_3v_4$ (if e_1 and e_3 are not adjacent). The resulting graphs G_1 after the M-extension are shown in Fig. 5 IB$_1$ and IB$_2$.

Case IC Suppose that e_1, e_2, e_3 in this M-extension are all distinct edges of K_4. Thus H, the subgraph of K_4 induced by $\{e_1, e_2, e_3\}$, must be either a 3-cycle, or a $K_{1,3}$, or a path of 3 edges. Without loss of generality, we assume that $e_1 = v_3v_1$, $e_2 = v_1v_2$ and $e_3 = v_2v_3$ (if H is a 3-cycle); or $e_1 = v_1v_3,\ e_2 = v_1v_2$ and $e_3 = v_1v_4$ (if H is a $K_{1,3}$); or $e_1 = v_1v_2,\ e_2 = v_2v_4$ and $e_3 = v_4v_3$ (if H is a path of 3 edges). The resulting graphs G_1 after the M-extension are shown in Fig. 5 IC$_1$, IC$_2$ and IC$_3$.

In any of Cases IA, IB$_1$ and IC$_1$, let $V_1 = \{v_4, w_1, w_3\}$ and $V_2 = V(G_1) - V_1$; in any of Cases IB$_2$ and IC$_2$, let $V_1 = \{v_2, w_1, w_3\}$ and $V_2 = V(G_1) - V_1$; in Case IC$_3$, let $V_1 = \{w_1, w_2, w_3\}$ and $V_2 = V(G_1) - V_1$. By inspection, in any of these cases, the ordered pair (V_1, V_2) is always a good pair of G_1.

Case II $G_1 = N(u, v)(K_4)$ is a graph obtained from K_4 by one N-extension (see Fig. 1) and let the vertex set of K_4 be $\{v_1, v_2, v_3, v_4\}$ as depicted in Fig. 6. By the definition of an N-extension, two edges e_1, e_2 must be chosen from $E(K_4)$ to perform the N-extension. By symmetry, G_1 has the following two possibilities (see Fig. 6).

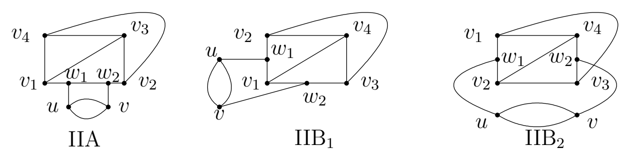

Figure 6 One N-extension of K_4

Case IIA Suppose that $e_1 = e_2$ in this N-extension of K_4. Without loss of generality, we assume that $e_1 = e_2 = v_1v_2$. The resulting graph G_1 after the N-extension is shown in Fig. 6 IIA.

Case IIB Suppose that $e_1 \neq e_2$ in this N-extension of K_4. Depending on whether e_1 and e_2 are adjacent or not adjacent, we may further assume that $e_1 = v_1v_2$ and $e_2 = v_1v_3$ (if e_1 and e_2 are adjacent) or $e_1 = v_1v_2$ and $e_2 = v_3v_4$ (if e_1 and e_2 are not adjacent). The resulting graphs G_1 after the N-extension are shown in Fig. 6 IIB$_1$ and IIB$_2$.

In any of these cases, let $V_1 = \{v, v_4, w_1\}$ and $V_2 = V(G_1) - V_1$. By inspection, the ordered pair (V_1, V_2) is always a good pair of G_1.

As in any case, a graph G_1 obtained from K_4 by an M-extension or an N-extension always has a good pair. It follows by Lemma 1.1 that G has a good pair. The lemma is proved.

The following characterizations for up-embeddable 2-edge-connected cubic graphs were given in [8] and [10] respectively.

Lemma 1.4[8] Let G be a 2-edge-connected cubic graph, G is up-embeddable if and only if G satisfies the following conditions:

(i) G is obtained from θ_1 by a series of M-extensions or N-extensions, if $\beta(G)$ is odd.

(ii) G is obtained from θ_2 or K_4 by a series of M-extensions or N-extensions, if $\beta(G)$ is even.

Lemma 1.5[10] Let G be a 2-edge-connected cubic graph. If G is not up-embeddable, then G is obtained from θ_3 or a double dumbbell graph by a series of M-extensions or N-extensions.

Proof of Theorem 0.1 Let G ($G \neq K_4$) be a 2-edge-connected cubic graph. If G is up-embeddable, by Lemma 1.4, G is obtained from θ_1 or θ_2 or K_4 by a series of M-extensions or N-extensions. By Lemmas 1.2 and 1.3, G has a good pair (V_1, V_2), i.e., G has a vertex partition (V_1, V_2) such that V_1 is an independent set and V_2 induces a forest. If G is not up-embeddable, by

Lemma 1.5 and the assumption of Theorem 0.1, G is obtained from θ_3 by a series of M-extensions or N-extensions, by Lemma 1.2, G has a good pair (V_1, V_2), i.e., G has a vertex partition (V_1, V_2) such that V_1 is an independent set and V_2 induces a forest. Theorem 0.1 is proved.

Corollary 0.1 is obtained from Theorem 0.1 and $a(K_4) = 2$.

Corollary 0.2 is obtained from Lemmas 1.2–1.4.

Corollary 0.3 is a direc corollary of Corollary 0.2 and $a(K_4) = 2$.

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关于 2- 边连通 3 正则图荫度的一个注

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摘要: 图 G 的点荫度 $a(G)$ 是 G 的使得每个子集诱导一个森林的顶点划分中子集的最少个数. 我们熟知对任何平面图 G, $a(G) \leq 3$, 且对任何直径最大是 2 的平面图有 $a(G) \leq 2$. 文献 [*European J. Combin.*, 2008, 29(4): 1064-1075] 中给出下列猜想: 任何没有 3- 圈的平面图都有一个顶点的划分 (V_1, V_2) 使得 V_1 是独立集, V_2 诱导一个森林. 本文证明了任何 2- 边连通上可嵌入的 3- 正则图 G ($G \neq K_4$) 都有一个顶点的划分 (V_1, V_2) 使得 V_1 是独立集, V_2 诱导一个森林.

关键词: 边连通度; 诱导森林; 荫度