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Degree sequence realizations with given packing and covering of spanning trees

Fengxia Liu [a], Zhao Zhang [b,*], Hong-Jian Lai [a,c], Meng Zhang [c]

[a] College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang, 830046, PR China

[b] College of Mathematics Physics and Information Engineering, Zhejiang Normal University, Jinhua, Zhejiang, 321004, PR China

[c] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

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ABSTRACT

Designing networks in which every processor has a given number of connections often leads to graphic degree sequence realization models. A nonincreasing sequence $d = (d_1, d_2, \ldots, d_n)$ is graphic if there is a simple graph G with degree sequence d. The spanning tree packing number of graph G, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in G. The arboricity of graph G, denoted by $a(G)$, is the minimum number of spanning trees whose union covers $E(G)$. In this paper, it is proved that, given a graphic sequence $d = d_1 \geq d_2 \geq \cdots \geq d_n$ and integers $k_2 \geq k_1 > 0$, there exists a simple graph G with degree sequence d satisfying $k_1 \leq \tau(G) \leq a(G) \leq k_2$ if and only if $d_n \geq k_1$ and $2k_1(n-1) \leq \sum_{i=1}^{n} d_i \leq 2k_2(n-|I|-1) + 2\sum_{i\in I} d_i$, where $I = \{i : d_i < k_2\}$. As corollaries, for any integer $k > 0$, we obtain a characterization of graphic sequences with at least one realization G satisfying $a(G) \leq k$, and a characterization of graphic sequences with at least one realization G satisfying $\tau(G) = a(G) = k$.

1. Introduction

We consider simple and finite graphs. Undefined terms and notations will follow [1]. For a graph G and a vertex $v \in V(G)$, $d_G(v)$ denotes the degree of v in G and $N_G(v)$ denotes the vertices in G that are adjacent to v. If $U \subset V(G)$, then $N_G(U) = \bigcup_{v\in U} N_G(v) - U$. If K is a subgraph of G, then we also write $N_G(K)$ for $N_G(V(K))$. When the graph G is understood from the context, we often omit the subscript G in these notations. Following [1], $c(G)$ denotes the number of components of a graph G. An integral sequence $d = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a simple graph G with degree sequence d. Let (d) denote the set of all simple graphs with degree sequence d. Any graph $G \in (d)$ is called a realization of d, or simply a d-realization.

The problem of designing networks with n processors each of which has a given number of connections and with a certain level of expected network strength is often modeled as a problem of finding graph realizations with certain graphical properties for a given degree sequence. For more on the literature on the degree sequence realization with given properties, see a resourceful survey by Li [6].

The *spanning tree packing number* of G (see [12]), denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in G. There have been many studies on the behavior of $\tau(G)$, see [3,4,9,10,13], among others. In a recent paper [5], the authors characterized the degree sequences d for which there exists a graph $G \in (d)$ with $\tau(G) \geq k$.

Theorem 1.1 (*Theorem 1.1 in [5]*). *Let $k > 0$ be an integer. For a graphic sequence $d = (d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n$ with $n \geq 2$, there exists $G \in (d)$ such that $\tau(G) \geq k$ if and only if both $d_n \geq k$ and $\sum_{i=1}^{n} d_i \geq 2k(n-1)$.*

* Corresponding author.
E-mail address: hxhzz@163.com (Z. Zhang).

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The *arboricity* of G, denoted by $a(G)$, is the minimum number of spanning trees whose union equals $E(G)$. By definition, $\tau(G) \le a(G)$. The main result of this paper is the following. (Any empty summation is considered to have value zero.)

Theorem 1.2. *Let $k_2 \ge k_1 \ge 0$ and $n > 1$ be integers. Let $d = (d_1, d_2, \cdots, d_n)$ with $d_1 \ge d_2 \ge \ldots \ge d_n$ be a graphic sequence and let $I = \{i : d_i < k_2\}$. Then there exists a graph $G \in (d)$ such that $k_2 \ge a(G) \ge \tau(G) \ge k_1$ if and only if each of the following holds.*

(i) $d_n \ge k_1$.
(ii) $2k_2(n - |I| - 1) + 2\sum_{i\in I} d_i \ge \sum_{i=1}^{n} d_i \ge 2k_1(n-1)$.

Theorem 1.2 has two immediate corollaries, as stated below, by letting $k_1 = 0$ and $k_2 = k$ in Corollary 1.3 and let $k_1 = k_2 = k$ in Corollary 1.4.

Corollary 1.3. *Let $n \ge 2$ and $k > 0$ be integers. For a graphic sequence $d = (d_1, d_2, \ldots, d_n)$ with $d_1 \ge d_2 \ge \cdots \ge d_n$, the following is equivalent.*

(i) *There exists a d-realization G such that $a(G) \le k$.*
(ii) $\sum_{i=1}^{n} d_i \le 2k(n - |I| - 1) + 2\sum_{i\in I} d_i$, *where $I = \{i : d_i < k\}$.*

Corollary 1.4. *Let $n \ge 2$ and $k > 0$ be integers. For a graphic sequence $d = (d_1, d_2, \ldots, d_n)$ with $d_1 \ge d_2 \ge \cdots \ge d_n$, there exists $G \in (d)$ such that $a(G) = \tau(G) = k$ if and only if $d_n \ge k$ and $\sum_{i=1}^{n} d_i = 2k(n-1)$.*

We shall utilize the properties related to uniformly dense graphs (see [2]) together with a decomposition (introduced in [9]) based on subgraph densities in the proofs of the main result. In the next section, we present the preliminaries on uniformly dense graphs and the related decomposition, which will be deployed in the proof arguments of our main result. The proof of Theorem 1.2 and the corollaries will be given in the last section.

2. Preliminaries

In this section, we introduce some notations and results that will be needed in the proofs of our main results. For a simple connected graph G, let V_1, V_2 be two subsets of $V(G)$. Following [1], define $E_G[V_1, V_2] = \{uv \in E(G) : u \in V_1, v \in V_2\}$. When H and H' are two subgraphs of G, we also use $E_G[H, H']$ for $E_G[V(H), V(H')]$. When the graph G is understood from the context, we often omit the subscript G. For a vertex subset $V_1 \subseteq V(G)$, define $E[V_1] = \{uv \in E(G) : u, v \in V_1\}$. If $X \subseteq E(G)$, then $G[X]$ is the subgraph of G induced by the edge subset X. For an edge subset X, the contraction of G by contracting edges in X, denoted by G/X, is the graph obtained first from G by identifying the two ends of each edge in X, and then by deleting all the resulting loops.

Recall that $\tau(G)$ is the maximum number of edge-disjoint spanning trees of G. For an integer $r \ge 1$, let $\mathcal{T}_r$ denote the family of all graphs G with $\tau(G) \ge r$. Let G be a connected graph. For any natural number $r \in \mathbb{N}$, a subgraph H of G is called *r-maximal* if $H \in \mathcal{T}_r$ and if there is no subgraph K of G, such that K contains H properly and $K \in \mathcal{T}_r$. An r-maximal subgraph H of G is called an *r-region* if $\tau(H) = r$. A subgraph H of G is a *region* if H is an r-region for some integer r. Define $\xi(G) = \max\{r |\ G \text{ has a subgraph as an } r\text{-region}\}$.

Let H be a graph with $|V(H)| > 1$. The *density* of H is

$$d(H) = \frac{|E(H)|}{|V(H)| - 1}.$$

It should be indicated that when H is a graph, $d(H)$ denotes the density of H, but when $v \in V(G)$ is a vertex of a graph G, $d_G(v)$ denotes the degree of v in G. In the following, we list some known results which will be used in Section 3.

Theorem 2.1 (*Nash-Williams [11]*). *Let G be a graph. Then*

$$a(G) = \max_{H\subseteq G} \lceil d(H) \rceil,$$

where the maximum is taken over all induced subgraphs H of G with $|V(H)| \ge 2$.

By definition and by Theorem 2.1, for a connected graph,

$$a(G) \ge d(G) = \frac{|E(G)|}{|V(G)| - 1} \ge \tau(G). \quad (1)$$

Theorem 2.2 (*Theorem 6 of [2]*). *If $a(G) > \tau(G)$, then $d(G) > \tau(G)$.*

Theorem 2.3 (*Liu et al. [9]*). *Let G be a nontrivial connected graph. Then*

(i) *there exist an integer $m \in \mathbb{N}$, and an m-tuple $(i_1, i_2, \ldots, i_m)$ of integers in $\mathbb{N}$ with $\tau(G) = i_1 < i_2 < \cdots < i_m = \xi(G)$, and a sequence of edge subsets $E_m \subset \cdots \subset E_2 \subset E_1 = E(G)$ such that each component of the spanning subgraph of G induced by E_j is an r-region of G for some $r \in \mathbb{N}$ with $r \ge i_j$ $(1 \le j \le m)$, and such that at least one component H in $G[E_j]$ is an i_j-region of G;*
(ii) *if H is a subgraph of G with $\tau(H) \ge i_j$, then $E(H) \subseteq E_j$;*
(iii) *the integer m and the sequences in* (i) *are uniquely determined by G.*

Theorem 2.4 (*Liu et al., Corollary 3.2 of [9]*). *$a(G) \geq \xi(G) \geq a(G) - 1$.*

Lemma 2.5 (*Lemma 2.5 of [5]*). *Let $k \geq 1$ be an integer, G be a graph with $\xi(G) \geq k$. Then each of the following statements holds.*
(i) *The graph G has a unique edge subset $X_k \subseteq E(G)$, such that every component H of $G[X_k]$ is a maximal subgraph with $\tau(H) \geq k$. In particular, $G \notin \mathcal{T}_k$ if and only if $E(G) \neq X_k$.*
(ii) *If $G \notin \mathcal{T}_k$, then G/X_k contains no nontrivial subgraph H' with $\tau(H') \geq k$.*
(iii) *If $G \notin \mathcal{T}_k$, then $d(H') < k$ for any nontrivial subgraph H' of G/X_k.*

Remark 2.6. By Theorem 2.4 and by $\xi(G) = i_m$, we deduce that the same conclusions of Lemma 2.5 also hold if the condition $\xi(G) \geq k$ in Lemma 2.5 is replaced by the condition $a(G) > k$.

Lemma 2.7 (*Lemma 2.6 of [5]*). *Let G be a graph with $d(G) \geq k$ and let $X_k \subset E(G)$ be the edge subset defined in Lemma 2.5 (i). If $G[X_k]$ has at least two components, then for any nontrivial component H of $G[X_k]$, $d(H) \geq k$, and $G[X_k]$ has at least one component H with $d(H) > k$.*

Next, we shall show that the same conclusions of Lemma 2.7 hold if we replace the condition $d(G) \geq k$ in Lemma 2.7 by the condition $a(G) > k$. For this purpose, the following result is needed.

Theorem 2.8 (*Liu et al., Lemma 2.1 of [9]*). *Let G be a connected graph, and let r, r' be integers with $r' \geq r > 0$. Each of the following holds.*
(i) *If $\tau(G) \geq r$, then for any $e \in E(G)$, $\tau(G/e) \geq r$.*
(ii) *If H is a subgraph of G with $\tau(H) \geq r'$, then $\tau(G/H) \geq r$ if and only if $\tau(G) \geq r$.*

Theorem 2.9. *Let G be a graph with $a(G) > k$ and let $X_k \subset E(G)$ be the edge subset defined in Lemma 2.5(i). Then $G[X_k]$ has at least one component H with $d(H) > k$.*

Proof. Since $a(G) > k$, by Theorem 2.1, there exists $G_0 \subseteq G$ with $d(G_0) > k$.

Let $X_k' \subset E(G_0)$ be the edge subset defined in Lemma 2.5(i). If $G_0[X_k']$ has only one component, then $G_0[X_k'] = G_0$ and $d(G_0[X_k']) = d(G_0) > k$. If $G_0[X_k']$ has at least two components, then by Lemma 2.7, $G_0[X_k']$ has at least one component K with $d(K) > k$. In both cases, we use K to denote a component of $G_0[X_k']$ with $\tau(K) \geq k$ and $d(K) > k$.

Let $X_k \subset E(G)$ be the edge subset defined in Lemma 2.5(i). Then $X_k' \subseteq X_k$, and there exists a component H of $G[X_k]$ such that $K \subseteq H$ and $\tau(H) \geq k$. By Theorem 2.8, we have $\tau(H/K) \geq k$, and so $|E(H/K)| \geq k(|V(H/K)| - 1)$. Since $d(K) > k$, $|E(K)| > k(|V(K)| - 1)$. By $|V(H)| = |V(H/K)| + |V(K)| - 1$, $|E(H)| = |E(H/K)| + |E(K)|$, we have

$$d(H) = \frac{|E(H)|}{|V(H)| - 1} = \frac{|E(H/K)| + |E(K)|}{|V(H/K)| + |V(K)| - 2}$$
$$> \frac{k(|V(H/K)| - 1) + k(|V(K)| - 1)}{|V(H/K)| + |V(K)| - 2} = k.$$

This completes the proof. □

The following lemma is useful in the proof of the main result. The related matroidal extensions can be found in [7,8].

Lemma 2.10 (*Lemma 2.12 of [5]*). *Let G be a graph and let $X_k \subset E(G)$ be the edge subset defined in Lemma 2.5(i). If H' and H'' are two components of $G[X_k]$, then each of the following holds.*
(i) *$|E(H', H'')| < k$.*
(ii) *If $d(H') > k$, then H' has a subgraph K such that $d(K) > k$ and $\tau(K - e) \geq k$ for any $e \in E(K)$.*
(iii) *If $d(H') > k$, then H' has an edge e' such that $\tau(H' - e') \geq k$, and $E(G) - X_k$ has at most one edge joining the ends of e' to H''.*

3. The proofs

Throughout this section, we assume that $k_1, k_2 > 0$ and $n > 1$ are integers and that $d = (d_1, d_2, \ldots, d_n)$ is a nonincreasing graphic sequence. For this degree sequence d, define $I = \{i : d_i < k_2\}$ and $t = |I|$. For a graph $G \in (d)$, define $V_I = \{v \in V(G) : d(v) < k_2\}$ and $V_{II} = \{v \in V(G) : d(v) \geq k_2\}$. Thus $|V_I| = t$ and $|V_{II}| = n - t$.

Lemma 3.1. *If some $G \in (d)$ has $a(G) \leq k_2$, then*

$$\sum_{i=1}^{n} d_i \leq 2k_2(n - |I| - 1) + 2\sum_{i \in I} d_i.$$

Proof. Since $a(G[V_{II}]) \leq a(G) \leq k_2$, so by (1), $|E[V_{II}]| \leq k_2((n - t) - 1)$. By counting the incidences of vertices in V_I, we have $|E[V_I, V_{II}] \cup E[V_I]| \leq \sum_{i \in I} d_i$. It follows that $\sum_{i=1}^{n} d_i = 2|E[V_{II}]| + 2|E[V_I, V_{II}] \cup E[V_I]| \leq 2k_2(n - t - 1) + 2\sum_{i \in I} d_i$, and so the lemma follows. □

Together with Theorem 1.1, Lemma 3.1 justifies the necessity of Theorem 1.2. In the following, we assume that $d = (d_1, d_2, \ldots, d_n)$ satisfies Theorem 1.2(i) and (ii).

Since d satisfies Theorem 1.2(ii), by the definition of I, we have $\sum_{i\in I} d_i < \sum_{i\in I} k_2$. As $\sum_{i=1}^{n} d_i \le 2k_2(n-|I|-1) + 2\sum_{i\in I} d_i = 2k_2(n-1) - 2(k_2|I| - \sum_{i\in I} d_i)$, it follows that

$$\sum_{i=1}^{n} d_i \le 2k_2(n-1). \tag{2}$$

The next lemma will be needed in the proof of Theorem 1.2.

Lemma 3.2. *Let $k' > k > 0$ and $r \ge k$ be integers. Let G be a graph with $a(G) \ge k'$ and $\tau(G) \ge k$, and let H be an r-region of G such that for some $e = uv \in E(H)$ with $\tau(H-e) \ge r$. For any edge $e' = xy \in E(G-H)$, if $f = ux, f' = vy \notin E(G)$, then*

$$\tau((G - \{e, e'\}) \cup \{f, f'\}) \ge k.$$

Proof. Let $G' = G/H$ and $G'' = (G - \{e, e'\}) \cup \{f, f'\}$. Since $\tau(G) \ge k$, by Theorem 2.8(i), $\tau(G') \ge k$. Let $T_1', T_2', \ldots, T_k'$ be k edge-disjoint spanning trees of G'. Since $\tau(H-e) \ge r \ge k$, $H-e$ has k edge-disjoint spanning trees $L_1, L_2, \ldots, L_k$. Since $G' = G/H$ and since $e \in E(H)$, $e \notin E(G')$. For each i with $1 \le i \le k$, if $e' \notin E(T_i')$, then $E(L_i) \cup E(T_i') \subseteq E(G'')$. Let

$$T_i = G''[E(L_i) \cup E(T_i')].$$

Then, T_i is a spanning tree of G''. In particular, if $e' \notin \bigcup_{i=1}^{k} E(T_i')$, then $T_1, T_2, \ldots, T_k$ are k edge-disjoint spanning trees of G'', and so $G'' \in \mathcal{T}_k$.

Thus we assume that $e' \in E(T_1')$. Let T_{11}' and T_{12}' be the two components of $T_1' - e'$ in G'. We may assume that T_{11}' contains the vertex v_H in G' onto which the subgraph H is contracted. Since $e' = xy$, we may also assume that $x \in V(T_{11}')$ and $y \in V(T_{12}')$. Let $T_1'' = G''[E(L_1) \cup E(T_1' - e') \cup \{f'\}]$. Then T_1'' is a spanning tree of G''. It follows that $T_1'', T_2, \ldots, T_k$ are k edge-disjoint spanning trees of G'', and so $G'' \in \mathcal{T}_k$. □

Proof of Theorem 1.2. By Theorem 1.1 and Lemma 3.1, the necessity of Theorem 1.2 follows immediately. It remains to prove the sufficiency.

Let $(d)_1 = \{G \in (d) : \tau(G) \ge k_1\}$. By Theorem 1.1, $(d)_1 \ne \emptyset$. To prove the sufficiency, we argue by contradiction and assume that

$$\text{for any } G \in (d)_1, \quad a(G) > k_2. \tag{3}$$

Thus by (2), for any $G \in (d)_1$,

$$\sum_{i=1}^{n} d_i \le 2k_2(n-1) < 2a(G)(n-1). \tag{4}$$

By Theorem 2.3, there exists a sequence of positive integers $\tau(G) = i_1 < i_2 < \cdots < i_m = \xi(G)$.

Claim 1. *For any $G \in (d)_1$, $m \ge 2$.*

By contradiction, we assume that for some $G \in (d)_1$, $m = 1$. By Theorem 2.4, $\tau(G) = a(G)$ or $\tau(G) = a(G)-1$. If $\tau(G) = a(G)$, then by (1), $2a(G) = 2|E(G)|/(n-1)$, and so $2a(G)(n-1) = \sum_{i=1}^{n} d_i \le 2k_2(n-1)$, contrary to (4). Thus we must have $\tau(G) = a(G) - 1$. By Theorem 2.2, $2\tau(G) < 2d(G) = 2|E(G)|/(n-1)$, and so by (2), $2\tau(G)(n-1) < 2|E(G)| = \sum_{i=1}^{n} d_i \le 2k_2(n-1)$. It follows that $\tau(G) \le k_2 - 1$, and so $a(G) = \tau(G) + 1 \le k_2$, contrary to (3). This proves Claim 1.

By Claim 1, $m \ge 2$. By (3) and by Theorem 2.3, there exists an m-tuple $(i_1, i_2, \ldots, i_m)$ of integers as stated in Theorem 2.3 with $k_2 \le a(G) - 1 \le i_m \le a(G)$. Thus there exists a smallest index i_j such that $i_j \ge k_2$. By Theorem 2.3, G has a unique edge subset $E_{i_j} \subseteq E(G)$ such that each component of $G[E_{i_j}]$ is a k_2-maximal subgraph of G.

Claim 2. *For any $G \in (d)_1$, $E_{i_j} \ne E(G)$.*

By contradiction, assume that for some $G \in (d)_1$, $E_{i_j} = E(G)$. By Theorem 2.3, $E(G)$ has i_j edge-disjoint spanning trees, and so $2k_2(n-1) \le 2i_j(n-1) \le 2|E(G)| = \sum_{i=1}^{n} d_i$. By (2), we have $i_j = k_2$ and $|E(G)| = k_2(n-1)$. It follows that $E(G)$ is a disjoint union of k_2 spanning trees, and so by definition, $a(G) = k_2$, contrary to (3). This proves Claim 2.

By Theorem 2.3 with a given value k_2, for any $G \in (d)_1$, E_{i_j} is uniquely determined by G. Throughout this paper, we define $X(G) = E(G) - E_{i_j}$, and when G is understood from the context, we also use X for $X(G)$. By Claim 2, $X \ne \emptyset$. Let $c = c(G-X)$. (Thus $c = c(G[E_{i_j}])$ as well.) Label the components of $G - X$ as $H_1, H_2, \ldots, H_c$ so that

$$d(H_1) \ge d(H_2) \ge \cdots \ge d(H_s) \ge i_j, \quad \text{and} \quad H_{s+1} = \cdots = H_c = K_1. \tag{5}$$

Notice that $H_1, H_2, \ldots, H_s$ are all the nontrivial k_2-maximal subgraphs of G. Since $X = X(G)$ is uniquely determined by G, it follows that the components of $G - X$ and the value of $s = s(G)$ satisfying (5) are also uniquely determined by G. Since $G - X$ is spanning in G and by Claim 2, we have $c \ge 2$. By (3) and by Theorem 2.9,

$$\text{for any } G \in (d)_1, \text{ we always have } d(H_1) \ge k_2. \tag{6}$$

Throughout the rest of the proof in this section, we choose $G \in (d)_1$ such that

$$c = c(G[E_{i_j}]) \text{ is minimized,} \tag{7}$$

and subject to (7),

$$|X(G)| \text{ is maximized.} \tag{8}$$

Claim 3. *If $s \geq 2$, then $d(H_2) \leq k_2$.*

Suppose that $s \geq 2$ and $d(H_2) > k_2$. By Lemma 2.10(iii), there exist $e_1 = u_1v_1 \in E(H_1)$ and $e_2 = u_2v_2 \in E(H_2)$ such that $\tau(H_1 - e_1) \geq k_2$ and $\tau(H_2 - e_2) \geq k_2$, and there exists at most one edge in X joining the ends of e_1 and e_2. Without loss of generality, assume $u_1u_2, v_1v_2 \notin E(G)$ and let

$$G_1' = (G - \{u_1v_1, u_2v_2\}) \cup \{u_1u_2, v_1v_2\} \quad \text{and} \quad X_1 = X \cup \{u_1u_2, v_1v_2\}. \tag{9}$$

It follows from Lemma 3.2 that $G_1' \in (d)_1$. For each $i \in \{1, 2\}$, by the choice of $e_i = u_iv_i$, $H_i - u_iv_i$ is contained in a k_2-maximal subgraph of G_1'. It follows by (7) that $G_1' - X(G_1') = (H_1 - u_1v_1) \cup (H_2 - u_2v_2) \cup H_3 \cup \cdots \cup H_c$, and so $|X(G_1')| = |X(G)| + 2$, contrary to (8). This proves Claim 3.

By Claim 3 and by Lemma 2.9, there exists $G \in (d)_1$ such that

$$G \text{ has a unique } k_2 \text{ -maximal subgraph } H_1 \text{ with } d(H_1) > k_2. \tag{10}$$

Among all such graphs in $(d)_1$ satisfying (10), choose G so that

$$|V(H_1)| \text{ is maximized,} \tag{11}$$

and subject to (11),

$$|X(G)| \text{ is maximized.} \tag{12}$$

Throughout the rest of the proof, we shall assume that $G \in (d)_1$ satisfies (10), as well as (11) and (12).

Claim 4. *$s = 1$.*

Suppose $s \geq 2$. By Lemma 2.10, H_1 has an edge $e_1 = uv \in E(H_1)$ with

$$d(H_1 - e_1) \geq \tau(H_1 - e_1) \geq i_j \quad \textit{and} \quad |E[H_1[\{e_1\}], H_2]| \leq 1. \tag{13}$$

By (13), H_2 has an edge $e_2 = xy$ such that $xu, yv \notin E(G)$. Let $G_1 = (G - \{xy, uv\}) \cup \{xu, yv\}$. By Lemma 3.2, $G_1 \in (d)_1$.

By Claim 3, $d(H_2) = k_2$ and so $\tau(H_2 - e_2) < k_2$. Let $H_{2,1}, \ldots, H_{2,l}$ be the k_2-maximal subgraphs of $H_2 - e_2$. Thus for each $z \in \{1, 2, \ldots, l\}$, either $d(H_{2,z}) = k_2$ or $H_{2,z} = K_1$. By the choice of e_1, $\tau(H_1 - e_1) \geq k_2$. If $\tau(H_1 - e_1) = k_2$, then by Claim 3, and by the fact that either $d(H_{2,z}) = k_2$ or $H_{2,z} = K_1$, we must have $a(G_1) \leq k_2$, contrary to (3). Hence $d(H_1 - e_1) > k_2$, and so $H_1 - e_1, H_{2,1}, \ldots, H_{2,l}$ are the k_2-maximal subgraphs of $G_1[(H_1 - e_1) \cup (H_2 - e_2)]$. It follows that $X \subseteq X' - \{xu, yv\}$, and so $|X'| \geq |X| + 2$, contrary to (12). This proves Claim 4.

By Claim 4, $s = 1$. If $c = 2$, then $|V(H_1)| = n - 1$. Let $V(H_2) = \{x\}$. By the definition of $X(G)$ and i_j, $\tau(H_1) \geq i_j \geq k_2$. By Theorem 2.9, we have

$$\sum_{i=1}^{n} d_i = 2|E(H_1)| + 2|E(x, H_1)| > 2k_2(n-2) + 2d_G(x). \tag{14}$$

If $d_G(x) \geq k_2$, then $\sum_{i=1}^{n} d_i > 2k_2(n-1)$, contrary to (4). Hence $d_n \leq d_G(x) < k_2$. For any $v \in V(H_1)$, we have $d_G(v) \geq d_{H_1}(v) \geq \tau(H_1) \geq k_2$. It follows that $t = 1$, that is, there is a unique vertex whose degree is smaller than k_2. By (14), we have $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n-1} d_i + d_n > 2k_2(n - t - 1) + 2\sum_{i \in I} d_i$, contrary to Theorem 1.2(ii).

Thus for the rest of the proof, we shall assume that $s = 1$ and $c > 2$. Since $s = 1$, for each i with $2 \leq i \leq c$, denote $V(H_i) = \{x_i\}$. Since $\tau(H_1) \geq k_2$, if for some i, $|N_G(x_i) \cap V(H_1)| \geq k_2$, then $G[V(H_1) \cup \{x_i\}]$ should have been in a k_2-maximal subgraph of G, contrary to the choice of E_{i_j}. Hence we have

$$\text{for any } i \text{ with } 2 \leq i \leq c,\ |N_G(x_i) \cap V(H_1)| < k_2. \tag{15}$$

Claim 5. *For some $i \neq j$, $x_ix_j \in E(G)$.*

By contradiction, we assume that $\{x_2, x_3, , \ldots, x_c\}$ is an independent set of G. Then for any x_i with $i \geq 2$, $N_G(x_i) \subseteq V(H_1)$. By (15), $d_G(x_i) < k_2$. Since for any $v \in V(H_1)$, $d_G(v) \geq d_{H_1}(v) \geq \tau(H_1) \geq k_2$, it follows that $t = |I| = c - 1$ and

$$\begin{aligned}\sum_{i=1}^{n} d_i &= 2|E(H_1)| + 2\sum_{i=2}^{c} |E\{x_i\}, V(H_1)| \\ &> 2k_2[n - (c-1) - 1] + 2\sum_{i \in I} d_i = 2k_2(n - t - 1) + 2\sum_{i \in I} d_i,\end{aligned}$$

contrary to the assumption in Theorem 1.2(ii). This proves Claim 5.

By Claim 5, we may assume $e' = x_2x_3 \in E(G)$. By Lemma 2.10(ii), H_1 has a subgraph K such that $d(K) > i_j$, $\tau(H) \geq i_j$, and such that $\tau(K - e) \geq i_j$ for any $e \in E(K)$. As G is a simple graph,

$$|V(K)| \geq i_j \geq k_2. \tag{16}$$

Claim 6. *For any edge $e = uv \in E(K)$, if $ux_2 \notin E(G)$, then $vx_3 \in E(G)$; if $ux_3 \notin E(G)$, then $vx_2 \in E(G)$.*

By contradiction, suppose for some edge $e = uv \in V(K)$ such that $ux_2, vx_3 \notin E(G)$. Define $G_2 = (G - \{uv, x_2x_3\}) \cup \{ux_2, vx_3\}$. By Lemma 3.2, $G_2 \in (d)_1$. Since $\tau(K - e) \geq i_j \geq k_2$, it follows by Theorem 2.8(ii)that $\tau(H_1 - e) \geq k_2$, and so $H_1 - e$ belongs to a k_2-maximal subgraph H_1'' of G_2. By Claim 4, H_1'' is the only nontrivial k_2-maximal subgraph of G_2. It follows by (11) that $V(H_1'') = V(H_1)$ and so $H_1'' = H_1 - e_1$. Hence $X(G) \subseteq X(G_2) - \{ux_2, vx_3\}$, and so $|X(G)| < |X(G_2)$, contrary to (12). This proves Claim 6.

Define

$$S_1 = N_G(x_2) \cap N_G(x_3) \cap V(K).$$
$$S_2 = (N_G(x_2) - N_G(x_3)) \cap V(K).$$
$$S_3 = (N_G(x_3) - N_G(x_2)) \cap V(K).$$
$$S_4 = V(K) - (S_1 \cup S_2 \cup S_3).$$

Claim 7. *$S_4 \neq \emptyset$.*

By (15) and by (16), we have $V(K) - N_G(x_3) \neq \emptyset$, and so $S_2 \cup S_4 \neq \emptyset$. Assume by contradiction that $S_4 = \emptyset$. Then $S_2 \neq \emptyset$. By Claim 6, $N_K(S_2) \subseteq S_1 \cup S_2$, and so $|S_1 \cup S_2| \geq |N_K(S_2)| \geq \delta(K) \geq i_j \geq k_2$. On the other hand, it follows by (15) that $|S_1 \cup S_2| = |E_G(V(K), \{x_2\})| \leq |N_G(x_2) \cap V(H_1)| < k_2$. This contradiction establishes Claim 7.

By Claim 7, $S_4 \neq \emptyset$. By Claim 6, $N_K(S_4) \subseteq S_1$. It follows that $|S_1| \geq |N_K(S_4)| \geq \delta(K) \geq i_j$. On the other hand, we have $|S_1| \leq |V(K) \cap N_G(x_2)| \leq |N_G(x_2) \cap V(H_1)| < i_j$. This contradiction proves the theorem. □

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References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[2] P.A. Catlin, J.W. Grossman, A.M. Hobbs, H.-J. Lai, Fractional arboricity, strength, and principal partition in graphs and matroids, Discrete Appl. Math. 40 (1992) 285–302.
[3] W.H. Cunningham, Optimal attack and reinforcement of a network, J. Assoc. Comput. Mach. 32 (1985) 549–561.
[4] D. Gusfield, Connectivity and edge-disjoint spanning trees, Inform. Process. Lett. 16 (1983) 87–89.
[5] H.-J. Lai, Y.T. Liang, P. Li, Degree sequences and graphs with disjoint spanning trees, Discrete Appl. Math. 159 (2011) 1447–1452.
[6] J. Li, Degree sequences of graphs, Adv. Math. 23 (1994) 193–204.
[7] P. Li, Cycles and bases in matroids and graphs (PhD Dissertation), West Virginia University, 2012.
[8] P. Li, H.-J. Lai, Y. Liang, Characterization of removable elements with respect to having *k* disjoint bases in a matroid, Discrete Appl. Math. 160 (16–17) (2012) 2445–2451.
[9] D. Liu, H.-J. Lai, Z.H. Chen, Reinforcing the number of disjoint spanning trees, Ars Combin. 93 (2009) 113–127.
[10] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. Lond. Math. Soc. 36 (1961) 445–450.
[11] C.St.J.A. Nash-Williams, Decompositions of finite graphs into forests, J. Lond. Math. Soc. 39 (1964) 12.
[12] E.M. Palmer, On the spanning tree packing number of a graph, a survey, Discrete Math. 230 (2001) 13–21.
[13] W.T. Tutte, On the problem of decomposing a graph into *n* connected factors, J. Lond. Math. Soc. 36 (1961) 221–230.