Per-spectral characterizations of some edge-deleted subgraphs of a complete graph

Heping Zhang[a], Tingzeng Wu[ab] & Hong-Jian Lai[c]

[a] School of Mathematics and Statistics, Lanzhou University,
Lanzhou, P.R. China.

[b] School of Mathematics and Statistics, Qinghai Nationalities
University, Xining, P.R. China.

[c] Department of Mathematics, West Virginia University,
Morgantown, WV, USA.
Published online: 14 Feb 2014.

PLEASE SCROLL DOWN FOR ARTICLE

Taylor & Francis
Taylor & Francis Group

Per-spectral characterizations of some edge-deleted subgraphs of a complete graph

Heping Zhang[a]*, Tingzeng Wu[ab] and Hong-Jian Lai[c]

[a]*School of Mathematics and Statistics, Lanzhou University, Lanzhou, P.R. China;*
[b]*School of Mathematics and Statistics, Qinghai Nationalities University, Xining, P.R. China;*
[c]*Department of Mathematics, West Virginia University, Morgantown, WV, USA*

Communicated by J.Y. Shao

Let $\mathscr{G}_n$ denote the set of all graphs obtained from K_n by removing five or fewer edges. Cámara and Haemers proved that graphs in $\mathscr{G}_n$ are uniquely determined by their adjacency spectra with the exception for graphs $K_7 - E(K_4 - E(K_2))$ and $K_7 - E(K_{1,4} + K_2)$. In this paper, we show that all graphs in $\mathscr{G}_n$ are uniquely determined by their permanental spectra. We further extend our findings by investigating when a complete graph with a few edges removed is uniquely determined by its permanental spectrum. More precisely, we prove that if $X \subseteq E(K_n)$ induces a star, or a matching, or a disjoint union of a matching and a path P_3, then $K_n - X$ is uniquely determined by its permanental spectrum.

1. Introduction

The permanent of an $n \times n$ matrix M with entries $m_{ij}(i, j = 1, 2, \ldots, n)$ is defined by

$$\mathrm{per}(M) = \sum_{\sigma} \prod_{i=1}^{n} m_{i\sigma(i)},$$

where the sum is taken over all permutations σ of $\{1, 2, \ldots, n\}$. The permanent plays an important role in combinatorics. For example, the permanent of a (0,1)-matrix can enumerate perfect matchings in the corresponding bipartite graphs.[1] However, Valiant [2] has shown that computing the permanent is #P-complete even when restricted to (0, 1)-matrices. Up to now, no efficient algorithm for computing the permanent is known.

We use G to denote a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. For convenience, the complete graph, path, cycle and star on n vertices are denoted by K_n, P_n, C_n and $K_{1,n-1}$, respectively. For a subgraph H of G, let $G - E(H)$ denote the subgraph obtained from G by deleting the edges of H. The *degree*

---

*Corresponding author. Email: zhanghp@lzu.edu.cn

of a vertex $v \in V(G)$ is denoted by $d(v)$. Let $c_i(G)$ and $p_i(G)$ denote, respectively, the numbers of i-cycles and i-vertex paths in G. Let $G \cup H$ be the union of two graphs G and H which have no common vertices. For any positive integer l, let lG denote be the union of l disjoint copies of graph G. Let $A(G)$ be the *adjacency matrix* of G. The characteristic polynomial of graph G, denoted by $\phi(G, x)$, is $\det(xI - A(G))$, where I is the identity matrix of order n. The *adjacency spectrum* of graph G consists of the eigenvalues together with their multiplicities of $A(G)$. The polynomial $\pi(G, x) = \text{per}(xI - A(G))$ is called the permanental polynomial of G. The *per-spectrum* of graph G, denoted by $ps(G)$, is the set of all roots (together with their multiplicities) of $\pi(G, x)$.

Turner [3] first considered a graph polynomial which generalizes both permanental and characteristic polynomials. The permanental polynomials of graphs were first systematically studied by Merris et al. [4], and the study of analogous objects in the chemical literature was started by Kasum et al. [5]. Borowiecki and Jóźwiak [6] posed a problem of characterizing all graphs whose permanental roots are pure imaginary or zeros. Yan and Zhang [7] gave a partial solution to this problem. They proved that if G is a bipartite graph containing no subgraphs which are even subdivisions of $K_{2,3}$, then the permanental roots of G are pure imaginary or zeros. Zhang and Li [8] gave a characterization of bipartite graphs containing no even subdivision of $K_{2,3}$, and presented an approach to compute the permanental polynomials of such graphs by Pfaffian orientation. In addition, Gutman and Cash [9] and Chen [10] obtained some relations between the coefficients of the permanental and characteristic polynomials of some chemical graphs, such as benzenoid hydrocarbons, fullerenes, toroidal fullerenes and coronoid hydrocarbons. Cash [11,12] developed a computer-aided method for the calculation of the permanental polynomials of molecular graphs and applied it to a variety of benzenoid hydrocarbons and fullerenes. Belardo et al. [13,14] gave some formulas for the permanental polynomial of any square matrix (over any field) in terms of the permanental polynomial of weighted digraphs. For more studies on permanental polynomials, see [15–19], among others.

Two graphs with the same adjacency spectrum are called *cospectral*. A graph G is *determined by its adjacency spectrum* (DAS for short) if every graph cospectral with G is isomorphic to G. van Dam and Haemers in [20] proposed the question to determine DAS graphs. This seems difficult in the theory of graph spectrum. By now, only a few types of graphs are proved to be determined by their spectra, such as the complement of the path,[21] T-*shape* trees,[22] lollipop graphs,[23,24] θ-graphs,[25] graphs with index at most $\sqrt{2 + \sqrt{5}}$,[26] graphs K_n^m (see the definition in [27]) and their complements [27] and so on.

Two graphs are *per-cospectral* if they share the same per-spectrum. A graph G is said to be *determined by its per-spectrum* (DPS for short) if for any graph H, $\pi(G, x) = \pi(H, x)$ implies that H is isomorphic to G. Merris et al. [4] indicated that the permanental polynomial seems a little better than the characteristic polynomial when it comes to distinguishing graphs which are not trees, since the permanental polynomial can distinguish the five pairs of cospectral graphs of [28]. Motivated by the statement of Merris et al., Liu and Zhang [29,30] showed that complete graphs, stars, regular complete bipartite graphs, odd cycles and odd lollipop graphs are DPS. They also showed that when restricted to connected graphs, the paths, even cycles C_{4l+2} ($l \geq 1$), lollipop graphs $L_{n,2k+1}$ ($k \geq 1$) and $L_{n,4}$ are DPS. Meanwhile, they found that graphs characterized by the characteristic polynomial are

not necessarily characterized by the permanental polynomial. In this paper, we focus on investigating what other graphs are DPS.

Let $\mathscr{G}_n$ denote the set of graphs each of which is obtained from K_n by removing five or fewer edges. Cámara and Haemers [31] determined all DAS graphs in $\mathscr{G}_n$, and obtained the following result.

THEOREM 1.1 [31] *A graph $G \in \mathscr{G}_n$ is DAS if and only if $G \notin \{K_7 - E(K_4 - E(K_2))$, $K_7 - E(K_{1,4} + K_2)\}$.*

In this paper, we consider which graphs in $\mathscr{G}_n$ are DPS. Surprisingly, we find that all graphs in $\mathscr{G}_n$ are DPS without exceptions. We will prove the following main theorem in Section 3,

THEOREM 1.2 *All graphs in $\mathscr{G}_n$ are DPS.*

As extensions, we investigate when a complete graph K_n with some edges of special structure deleted is DPS. Let H be an edge induced subgraph of K_n with $|E(H)| = l$. We will show that $K_n - E(H)$ is DPS, where (i) $H \cong K_{1,l}$, (ii) $H \cong lP_2$ and (iii) $H \cong (l-2)P_2 \cup P_3$.

The rest of this paper is organized as follows. In Section 2, we present some characterizing properties of the per-spectrum of graphs, and give formulae to compute the numbers of i-cycles ($i = 3, 4, 5$) in $K_n - E(H)$, where H is a subgraph of K_n with l edges. In Section 3, we give the proof of Theorem 1.2. In the final section, we prove that $K_n - E(K_{1,l})$, $K_n - E(lP_2)$ and $K_n - E((l-2)P_2 \cup P_3)$ are DPS.

2. Some preliminaries

It can be seen that there exist 45 non-isomorphic graphs with at most five edges and no isolated vertices; for detail, see Appendix I in [32]. Thus, up to isomorphism there exist exactly 45 graphs in $\mathscr{G}_n$ for $n \geq 10$, which are labelled by G_{ij}, $1 \leq i \leq 5$, $0 \leq j \leq 25$, and illustrated in Figure 1.

A subgraph H of a graph G is said to be a *Sachs subgraph* if each component of H is either a single edge or a cycle.

LEMMA 2.1 [4] *Let G be a graph with $\pi(G, x) = \sum_{k=0}^{n} b_k(G)x^{n-k}$. Then*

$$b_k(G) = (-1)^k \sum_{H} 2^{c(H)}, \quad 1 \leq k \leq n,$$

where the sum is taken over all Sachs subgraphs H of G on k vertices and $c(H)$ is the number of cycles in H.

LEMMA 2.2 *Two graphs G and H are per-cospectral if and only if they have the same permanental polynomials.*

LEMMA 2.3 [29] *Let G be a graph with n vertices and m edges, and let* $(d_1, d_2, \ldots, d_n)$ *be the degree sequence of G. Then*

$$b_0(G) = 1, \quad b_1(G) = 0, \quad b_2(G) = m, \quad b_3(G) = -2c_3(G),$$

$$b_4(G) = \binom{m}{2} - \sum_{i=1}^{n} \binom{d_i}{2} + 2c_4(G).$$

LEMMA 2.4 *Let G be a graph with n vertices and m edges, and let* $t_j(G)$ *denote the degree sum of the three vertices on jth triangle in G. Then*

$$b_5(G) = -2 \left(\sum_{j=1}^{c_3(G)} \left(m + 3 - t_j(G) \right) + c_5(G) \right). \tag{1}$$

Proof By definition, $C_3 \cup P_2$ and C_5 are the only Sachs subgraphs with five vertices. There exist $m + 3 - t_j(G)$ Sachs subgraphs of five vertices containing the *j*th triangle in *G*. By Lemma 2.1, we obtain Equation (1). □

LEMMA 2.5 [29] *The following can be deduced from the permanental polynomial of a graph G:*

   (i)   *The number of vertices.*
  (ii)   *The number of edges.*
 (iii)   *The number of triangles.*
 (iv)   *The length of a shortest odd cycle.*
  (v)   *The number of shortest odd cycles.*
 (vi)   *Whether G is bipartite.*

LEMMA 2.6 [21] *Let* $H \subseteq K_n$ *be a graph with l edges and let* $G = K_n - E(H)$. *Then*

$$c_3(G) = \binom{n}{3} - l(n-2) + \sum_{v \in V(H)} \binom{d(v)}{2} - c_3(H). \tag{2}$$

Using (2), we can calculate the number of triangles of all $G \in \mathscr{G}_n$; see Table 1. By examining Table 1, we observe that some graphs in $\mathscr{G}_n$ have the same number of triangles.

The following results can be derived from the Principle of Inclusion–Exclusion.

LEMMA 2.7 *Let* $H \subseteq K_n$ *be a graph with l edges and let* $G = K_n - E(H)$. *Then*

$$c_4(G) = 3\binom{n}{4} - 2l\binom{n-2}{2} + \left[2\binom{l}{2} + (n-5) \sum_{v \in V(H)} \binom{d(v)}{2} \right] - p_4(H) + c_4(H). \tag{3}$$

Proof Let $E(H) = \{e_1, e_2, \ldots, e_l\}$. Let S_i denote the set of quadrangles of K_n containing e_i $(i = 1, 2, \ldots, l)$. For any four vertices in K_n, there exist exactly three quadrangles containing them. So, K_n contains $3\binom{n}{4}$ quadrangles. By the Inclusion–Exclusion Principle, we have
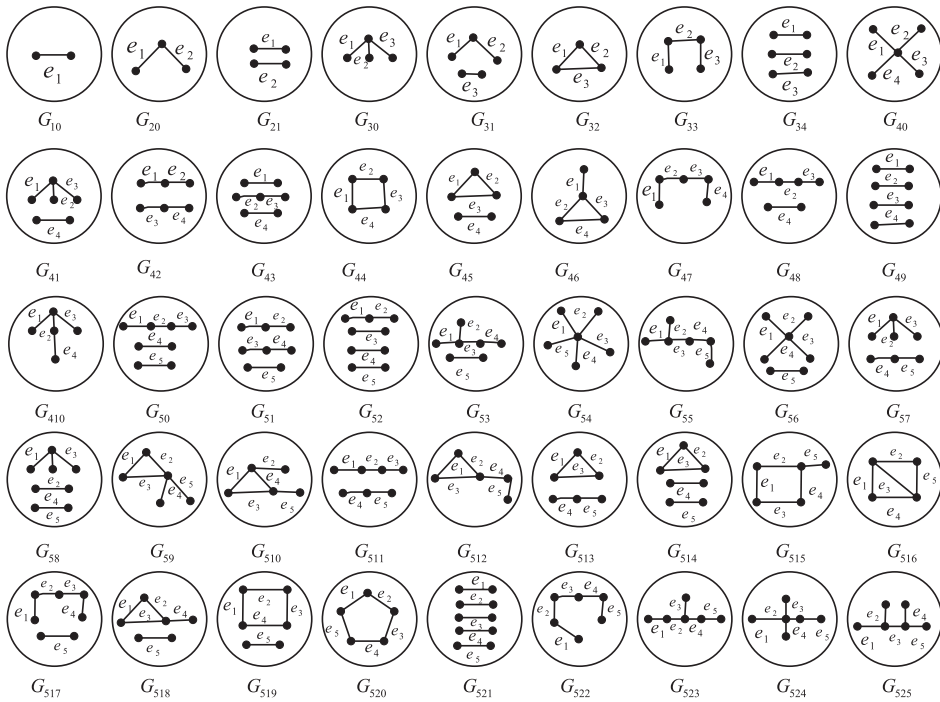
Figure 1. The collection of graphs in $\mathscr{G}_n$ obtained from K_n by deleting five or fewer edges drawn as lines in a disk.

$$c_4(G) = 3\binom{n}{4} - \sum_{i=1}^{l} |S_i| + \sum_{i<j} |S_i \cap S_j| - \sum_{i<j<k} |S_i \cap S_j \cap S_k|$$
$$+ \sum_{i<j<k<s} |S_i \cap S_j \cap S_k \cap S_s|.$$

Since any edge e_i is contained in $2\binom{n-2}{2}$ quadrangles of K_n, we have $\sum_{i=1}^{l} |S_i| = 2l\binom{n-2}{2}$. For any given e_i and e_j, if e_i is adjacent to e_j, then there exist $n-3$ quadrangles in K_n containing them. Otherwise, there exist two quadrangles containing them in K_n. For the graph H, it contains exactly $\sum_{v \in V(H)} \binom{d(v)}{2}$ pairs of adjacent edges and $\binom{l}{2} - \sum_{v \in V(H)} \binom{d(v)}{2}$ pairs of disjoint edges. Thus $\sum_{i<j} |S_i \cap S_j| = 2\left(\binom{l}{2} - \sum_{v \in V(H)} \binom{d(v)}{2}\right) + (n-3)\sum_{v \in V(H)} \binom{d(v)}{2}$. Since any three edges in a quadrangle induce a P_4, $\sum_{i<j<k} |S_i \cap S_j \cap S_k| = p_4(H)$. Similarly, $\sum_{i<j<k<s} |S_i \cap S_j \cap S_k \cap S_s| = c_4(H)$. By the above arguments, we arrive in Equation (3). □

We calculate the number of quadrangles of some graphs in $\mathscr{G}_n$ by applying Equation (3) as shown in Table 2.

For a graph G, let $\mathcal{P}_3(G)$ denote the set of all subgraph of G isomorphic to a path P_3. For a subgraph H of G, and for each $P_3 \in \mathcal{P}_3(G)$, define,

$$x_H(P_3) = \begin{cases} 1, & \text{if } P_3 \text{ is contained in a triangle in } H, \\ 0, & \text{otherwise.} \end{cases}$$

Table 1. The numbers of triangles of graphs in $\mathscr{G}_n$.

Graph	$c_3(G)$	Graph	$c_3(G)$
$3\,G_{10}$	$\binom{n}{3} - n + 2$	G_{521}	$\binom{n}{3} - 5n + 10$
G_{20}	$\binom{n}{3} - 2n + 5$	G_{32}, G_{33}	$\binom{n}{3} - 3n + 8$
G_{21}	$\binom{n}{3} - 2n + 4$	G_{41}, G_{47}	$\binom{n}{3} - 4n + 11$
G_{30}	$\binom{n}{3} - 3n + 9$	G_{59}, G_{524}	$\binom{n}{3} - 5n + 17$
G_{31}	$\binom{n}{3} - 3n + 7$	G_{42}, G_{45}, G_{48}	$\binom{n}{3} - 4n + 10$
G_{34}	$\binom{n}{3} - 3n + 6$	G_{44}, G_{46}, G_{410}	$\binom{n}{3} - 4n + 12$
G_{40}	$\binom{n}{3} - 4n + 14$	G_{50}, G_{51}, G_{514}	$\binom{n}{3} - 5n + 12$
G_{43}	$\binom{n}{3} - 4n + 9$	$G_{55}, G_{512}, G_{520}, G_{523}$	$\binom{n}{3} - 5n + 15$
G_{49}	$\binom{n}{3} - 4n + 8$	$G_{58}, G_{511}, G_{513}, G_{517}$	$\binom{n}{3} - 5n + 13$
G_{52}	$\binom{n}{3} - 5n + 11$	$G_{53}, G_{57}, G_{518}, G_{519}, G_{522}$	$\binom{n}{3} - 5n + 14$
G_{54}	$\binom{n}{3} - 5n + 20$	$G_{56}, G_{510}, G_{515}, G_{516}, G_{525}$	$\binom{n}{3} - 5n + 16$

LEMMA 2.8 *Let $H \subseteq K_n$ have l edges and let $G = K_n - E(H)$. Let $d_j(P_3)$ denote the degree sum of three vertices on the jth P_3 in H, and $q = \sum_{v \in V(H)} \binom{d(v)}{2}$. Then*

$$c_5(G) = 12\binom{n}{5} - 6l\binom{n-2}{3} + 4(n-4)\left(\binom{l}{2} - q\right) + 2q\binom{n-3}{2} - (n-4)p_4(H)$$
$$-2\left(\sum_{i=1}^{q}(l + 2 - d_i(P_3) + x_H(P_3))\right) + p_5(H) - c_5(H). \qquad (4)$$

Proof   Let $E(H) = \{e_1, e_2, \ldots, e_l\}$. For each $i = 1, 2, \ldots, l$, let Q_i denote the set of pentagons (5-cycles) of K_n containing e_i. As K_n contains $12\binom{n}{5}$ pentagons, by the Inclusion–Exclusion Principle, we obtain a formula enumerating pentagons in G as follows,

$$c_5(G) = 12\binom{n}{5} - \sum_{i=1}^{l}|Q_i| + \sum_{i<j}|Q_i \cap Q_j| - \sum_{i<j<k}|Q_i \cap Q_j \cap Q_k|$$
$$+ \sum_{i<j<k<s}|Q_i \cap Q_j \cap Q_k \cap Q_s| - \sum_{i<j<k<s<t}|Q_i \cap Q_j \cap Q_k \cap Q_s \cap Q_t|.$$

Since any edge of K_n is contained in $6\binom{n-2}{3}$ pentagons, we have $\sum_{i=1}^{l}|Q_i| = 6l\binom{n-2}{3}$. For $i \neq j$,

$$|Q_i \cap Q_j| = \begin{cases} 2\binom{n-3}{2}, & \text{if } e_i \text{ is adjacent to } e_j, \\ 4(n-4), & \text{otherwise.} \end{cases}$$

For any graph H, it contains exactly $\binom{l}{2} - \sum_{v \in V(H)} \binom{d(v)}{2}$ pairs of disjoint edges. On the other hand, the number of P_3 in H equals $q = \sum_{v \in V(H)} \binom{d(v)}{2}$. It follows that $\sum_{i<j}|Q_i \cap Q_j| = 4(n-4)(\binom{l}{2} - q) + 2q\binom{n-3}{2}$.

Table 2. The number of quadrangles of some graphs in $\mathscr{G}_n$.

Graph	$c_4(G)$	Graph	$c_4(G)$
G_{32}	$3\binom{n}{4} - 3n^2 + 18n - 27$	G_{510}	$3\binom{n}{4} - 5n^2 + 32n - 50$
G_{33}	$3\binom{n}{4} - 3n^2 + 17n - 23$	G_{511}	$3\binom{n}{4} - 5n^2 + 28n - 26$
G_{42}	$3\binom{n}{4} - 4n^2 + 22n - 22$	G_{515}	$3\binom{n}{4} - 5n^2 + 31n - 45$
G_{44}	$3\binom{n}{4} - 4n^2 + 24n - 35$	G_{516}	$3\binom{n}{4} - 5n^2 + 33n - 55$
G_{46}	$3\binom{n}{4} - 4n^2 + 25n - 39$	G_{517}	$3\binom{n}{4} - 5n^2 + 28n - 27$
G_{47}	$3\binom{n}{4} - 4n^2 + 23n - 29$	G_{518}	$3\binom{n}{4} - 5n^2 + 30n - 37$
G_{48}	$3\binom{n}{4} - 4n^2 + 22n - 23$	G_{520}	$3\binom{n}{4} - 5n^2 + 30n - 40$
G_{410}	$3\binom{n}{4} - 4n^2 + 24n - 34$	G_{523}	$3\binom{n}{4} - 5n^2 + 30n - 39$
G_{50}	$3\binom{n}{4} - 5n^2 + 27n - 21$	G_{524}	$3\binom{n}{4} - 5n^2 + 32n - 48$
G_{51}	$3\binom{n}{4} - 5n^2 + 27n - 20$	G_{41}, G_{45}	$3\binom{n}{4} - 4n^2 + 23n - 27$
G_{53}	$3\binom{n}{4} - 5n^2 + 29n - 32$	G_{57}, G_{513}	$3\binom{n}{4} - 5n^2 + 29n - 30$
G_{55}	$3\binom{n}{4} - 5n^2 + 30n - 38$	G_{58}, G_{514}	$3\binom{n}{4} - 5n^2 + 28n - 25$
G_{56}	$3\binom{n}{4} - 5n^2 + 31n - 40$	G_{512}, G_{525}	$3\binom{n}{4} - 5n^2 + 31n - 44$
G_{59}	$3\binom{n}{4} - 5n^2 + 33n - 54$	G_{519}, G_{522}	$3\binom{n}{4} - 5n^2 + 29n - 33$

Note that any three edges in a C_5 induce either a P_4, or the disjoint union of a P_3 and a P_2. Observe that any P_3 in H is contained in $l + 2 - d(P_3) + x_H(P_3)$ disjoint unions of P_3 and P_2 in H. Further exactly $2(l+2-d_i(P_3)+x_H(P_3))$ pentagons in K_n contain the disjoint union of the i-th P_3 and P_2 in H, and any P_4 is contained in $n - 4$ 5-cycles in K_n. It follows that $\sum_{i<j<k}|Q_i \cap Q_j \cap Q_k| = p_4(H)(n-4) + 2(\sum_{i=1}^{q}(l+2-d_i(P_3)+x_H(P_3)))$. Since every four edges in a C_5 induce a P_5, we have $\sum_{i<j<k<s}|Q_i \cap Q_j \cap Q_k \cap S_s| = p_5(H)$. Similarly, $\sum_{i<j<k<s<t}|Q_i \cap Q_j \cap Q_k \cap Q_s \cap Q_t| = c_5(H)$. Substituting such equations into the expression of $c_5(G)$, we obtain Equation (4). $\square$

Let $K_3(G)$ be the set of all 3-cycles of G. For each $K \in K_3(G)$, define $d_G(K) = \sum_{v \in V(K)} d_G(v)$ and $D(G) = \sum_{K \in K_3}(G)d_G(K)$.

LEMMA 2.9 *Let $G_{519} \cong K_n - E(C_4 \cup K_2)$ with $n \geq 6$. Then*

$$D(G_{519}) = \binom{n}{3}(3n - 3) - 20n^2 + 90n - 80. \tag{5}$$

Proof We use the notation in Figure 1 for G_{519} and let $H = C_4 \cup K_2$ denote a subgraph of K_n. Let $e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_4, e_4 = v_4v_1$ and $e_5 = v_5v_6$ denote the edges H. Direct computation yields $D(K_n) = \binom{n}{3}(3n - 3)$. We will compute $D(G_{519})$ by deleting the edges $e_1, \ldots, e_5$ one edge at a time.

Step 1 We observe that K_n has $n - 2$ triangles containing e_1, and these triangles will be destroyed in $K_n - e_1$. We also note that $K_n - e_1$ has $2\binom{n-2}{2}$ triangles containing exactly one endpoint of e_1. For each of such 3-cycles, its degree sum in $K_n - e_1$ will decrease by 1. As each triangle in K_n has degree sum $3n - 3$, it follows that

$$D(K_n) - D(K_n - e_1) = (n - 2)(3n - 3) + 2\binom{n-2}{2} = 4n^2 - 14n + 12. \quad (6)$$

Step 2 Note that $K_n - e_1$ has $n - 3$ triangles containing e_2, and these triangles will be destroyed in $K_n - \{e_1, e_2\}$, and that the degree sum of each such triangle in $K_n - e_1$ is $3n - 4$. We also note that $K_n - e_1$ has $2\binom{n-3}{2}$ triangles containing exactly one endpoint of e_2 but not v_1, and for each of such 3-cycles, its degree sum in $K_n - \{e_1, e_2\}$ will decrease by 1 from its degree sum in $K_n - e_1$. Moreover, $K_n - e_1$ has $\binom{n-3}{1}$ triangles containing edge v_1v_3, and for each of such 3-cycles, its degree sum in $K_n - \{e_1, e_2\}$ will decrease by 1 from its degree sum in $K_n - e_1$. Thus, after deleting e_2 in $K_n - e_1$, we have

$$D(K_n - e_1) - D(K_n - e_1 - e_2) = (n-3)(3n-4) + 2\binom{n-3}{2} + (n-3) = 4n^2 - 19n + 21. \quad (7)$$

Step 3 Again $K_n - e_1 - e_2$ has $n - 3$ triangles containing e_3. Among these triangles, $v_1v_3v_4$ has degree sum $3n - 5$ in $K_n - e_1 - e_2$, and each of the other 3-cycles has degree sum $3n - 4$ in $K_n - e_1 - e_2$. All these 3-cycles will be destroyed in $K_n - \{e_1, e_2, e_3, e_4\}$. Moreover, $K_n - e_1 - e_2$ has $2\binom{n-4}{2}$ 3-cycles each of which contains exactly one endpoint of e_3 and two vertices in $V(K_n) - \{v_1, v_2, v_3, v_4\}$; and $3\binom{n-4}{1}$ 3-cycles each of which contains exactly one of edges in $\{v_1v_3, v_1v_4, v_2v_4\}$. The degree sum of each of these $2\binom{n-4}{2} + 3\binom{n-4}{1}$ triangles in $K_n - \{e_1, e_2\}$ will be decreased by 1 in $K_n - \{e_1, e_2, e_3\}$. Thus,

$$D(K_n - e_1 - e_2) - D(K_n - e_1 - e_2 - e_3) \quad (8)$$
$$= (3n - 5) + (n - 4)(3n - 4) + 2\binom{n-4}{2} + 3\binom{n-4}{1} = 4n^2 - 19n + 19.$$

Step 4 We again note that $K_n - e_1 - e_2 - e_3$ has $n - 4$ triangles containing e_4, and the degree sum of each of these triangle in $K_n - e_1 - e_2 - e_3$ is $3n - 5$. All these 3-cycles will be destroyed in $K_n - \{e_1, e_2, e_3, e_4\}$. Furthermore, $K_n - e_1 - e_2 - e_3$ has $2\binom{n-4}{2}$ 3-cycles each of which contains exactly one endpoint of e_4 and two vertices in $V(K_n) - \{v_1, v_2, v_3, v_4\}$; and has $2\binom{n-4}{1}$ 3-cycles each of which contain exactly edge in $\{v_1v_3, v_2v_4\}$. The degree sum of each of these $2\binom{n-4}{2} + 2\binom{n-4}{1}$ triangles in $K_n - \{e_1, e_2, e_3\}$ will be decreased by 1 in $K_n - \{e_1, e_2, e_3, e_4\}$. Thus, after deleting e_4 in $K_n - e_1 - e_2 - e_3$, we have

$$D(K_n - e_1 - e_2 - e_3) - D(K_n - e_1 - e_2 - e_3 - e_4) \quad (9)$$
$$= (n - 4)(3n - 5) + 2\binom{n-4}{2} + 2\binom{n-4}{1} = 4n^2 - 24n + 32.$$

Step 5 We observe that $K_n - e_1 - e_2 - e_3 - e_4$ has $n - 2$ triangles containing e_5. Among these triangles, each of $v_1v_5v_6$, $v_2v_5v_6$, $v_3v_5v_6$ and $v_4v_5v_6$ has degree sum $3n - 5$, and each of the $n - 6$ others has degree sum $3n - 3$ in $K_n - e_1 - e_2 - e_3 - e_4$. All these 3-cycles will be destroyed in $K_n - \{e_1, e_2, e_3, e_4, e_5\}$.

Moreover, $K_n - e_1 - e_2 - e_3 - e_4$ has $2\binom{n-6}{2}$ 3-cycles each of which contains exactly one endpoint of e_5 and two vertices in $V(K_n) - \{v_1, v_2, v_3, v_4, v_5, v_6\}$, has $8\binom{n-6}{1}$ triangles each of which contains one of edges in $\{v_1v_5, v_2v_5, v_3v_5, v_4v_5, v_1v_6, v_2v_6, v_3v_6, v_4v_6\}$ and

a vertex in $V(K_n) - \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and the four other 3-cycles in $\{v_5v_1v_3, v_5v_2v_4,$ $v_6v_1v_3, v_6v_2v_4\}$. By direct computation, the degree sum of each of these $2\binom{n-6}{2} + 8\binom{n-6}{1} + 4$ triangles in $K_n - \{e_1, e_2, e_3, e_4\}$ will be decreased by 1 in $K_n - \{e_1, e_2, e_3, e_4, e_5\}$. Thus, after deleting e_5 in $K_n - e_1 - e_2 - e_3 - e_4$, we have

$$D(K_n - e_1 - e_2 - e_3 - e_4) - D(K_n - e_1 - e_2 - e_3 - e_4 - e_5) \tag{10}$$
$$= 4(3n - 5) + (n - 6)(3n - 3) + 2\binom{n-6}{2} + 8\binom{n-6}{1} + 4 = 4n^2 - 14n - 4.$$

Combining (6)–(10), we obtain Equation (5). $\qquad\square$

LEMMA 2.10 *Let $G_{522} \cong K_n - E(P_6)$ with $n \geq 6$. Then*

$$D(G_{522}) = \binom{n}{3}(3n - 3) - 20n^2 + 90n - 78. \tag{11}$$

Proof The proof of Lemma 2.10 is similar to that of Lemma 2.9. Let $H = P_6$ denote a subgraph of K_n. Then $G_{522} = K_n - E(H)$. Let $e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_4, e_4 = v_4v_5$ and $e_5 = v_5v_6$ denote the edges H. With the same Steps 1–3 in the proof of Lemma 2.9 and the similar arguments, we obtain (6)–(8). We only need to modify the proofs in Steps 4 and 5.

Step 4 Note that $K_n - e_1 - e_2 - e_3$ has $n - 3$ triangles containing e_4, and that among these triangles (as subgraphs in $K_n - e_1 - e_2 - e_3$), $v_1v_4v_5$ has degree sum $3n - 5$, $v_2v_4v_5$ has degree sum $3n - 6$ and each of the other $(n - 5)$ 3-cycles has degree sum $3n - 4$ in $K_n - e_1 - e_2 - e_3$. All these 3-cycles will be destroyed in $K_n - \{e_1, e_2, e_3, e_4\}$. Moreover, $K_n - e_1 - e_2 - e_3$ has $2\binom{n-5}{2}$ 3-cycles each of which contains exactly one endpoint of e_4 and two vertices in $V(K_n) - \{v_1, v_2, v_3, v_4, v_5\}$, and has $5\binom{n-5}{1}$ 3-cycles each of which contains exactly one of the edges in $\{v_1v_4, v_2v_4, v_1v_5, v_2v_5, v_3v_5\}$ and a vertex in $V(K_n) - \{v_1, v_2, v_3, v_4, v_5\}$, plus the 3-cycles $v_1v_3v_5$. By direct computation, the degree sum of each of these $2\binom{n-5}{2} + 5\binom{n-5}{1} + 1$ triangles in $K_n - e_1 - e_2 - e_3$ will be reduced by 1 in $K_n - \{e_1, e_2, e_3, e_4\}$. It follows that

$$D(K_n - e_1 - e_2 - e_3) - D(K_n - e_1 - e_2 - e_3 - e_4)$$
$$= (3n - 5) + (3n - 6) + (n - 5)(3n - 4) + 2\binom{n-5}{2}$$
$$+ 5\binom{n-5}{1} + 1 = 4n^2 - 19n + 15. \tag{12}$$

Step 5 We observe that $K_n - e_1 - e_2 - e_3 - e_4$ has $n - 3$ triangles containing e_5. Among these triangles, $v_1v_5v_6$ has degree sum $3n - 5$, both $v_2v_5v_6$ and $v_3v_5v_6$ have degree sum $3n - 6$, and each of the $n - 6$ others has degree sum $3n - 4$ in $K_n - e_1 - e_2 - e_3 - e_4$. All these 3-cycles will be destroyed in $K_n - \{e_1, e_2, e_3, e_4, e_5\}$.

Moreover, $K_n - \{e_1, e_2, e_3, e_4\}$ has $2\binom{n-6}{2}$ 3-cycles each of which contains exactly one endpoint of e_5 and two vertices in $V(K_n) - \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and has $7\binom{n-6}{1}$ triangles each of which contains exactly one of edges in $\{v_1v_5, v_2v_5, v_3v_5, v_1v_6, v_2v_6, v_3v_6, v_4v_6\}$ and a vertex in $V(K_n) - \{v_1, v_2, v_3, v_4, v_5, v_6\}$, plus the 3-cycles in $\{v_5v_1v_3, v_6v_2v_4, v_6v_1v_3, v_6v_1v_4\}$. By direct computation, the degree sum of each of these $2\binom{n-6}{2} + 7\binom{n-6}{1} + 4$ triangles in $K_n - \{e_1, e_2, e_3, e_4\}$ will be reduced by 1 in $K_n - \{e_1, e_2, e_3, e_4, e_5\}$. It follows

that

$$D(K_n - e_1 - e_2 - e_3 - e_4) - D(K_n - e_1 - e_2 - e_3 - e_4 - e_5)$$

$$= (3n - 5) + 2(3n - 6) + (n - 6)(3n - 4) + 2\binom{n - 6}{2}$$

$$+ 7\binom{n - 6}{1} + 4 = 4n^2 - 19n + 11. \tag{13}$$

We now combine (6)–(8), (12) and (13) to conclude that $D(G_{522}) = \binom{n}{3}(3n - 3) - 20n^2 + 90n - 78$. □

3. Proof of theorem 1.2

We first present the main idea in the proof of Theorem 1.2. By Table 1, $\mathscr{G}_n$ is partitioned into different groups according to the number of triangles a graph in $\mathscr{G}_n$ will have. From Lemmas 2.2 and 2.3, we can see that any two graphs in different groups are not per-cospectral since they have either different number of edges or different number of triangles. Further, by Lemmas 2.3 and 2.4 and Table 2, we calculate the fourth and fifth coefficients of the permanental polynomial of graphs in each group above, respectively, and compare the coefficients to determine whether such graphs are per-cospectral or not. So, to prove Theorem 1.2 is sufficient to verify the following lemma.

LEMMA 3.1 *Each of the following holds.*

  (i) *Graphs G_{32} and G_{33} are not per-cospectral.*
 (ii) *Graphs G_{41} and G_{47} are not per-cospectral.*
(iii) *Graphs G_{44}, G_{46} and G_{410} are not pairwise per-cospectral.*
(iv) *Graphs G_{42}, G_{45} and G_{48} are not pairwise per-cospectral.*
 (v) *Graphs G_{50}, G_{51} and G_{514} are not pairwise per-cospectral.*
(vi) *Graphs G_{55}, G_{512}, G_{520} and G_{523} are not pairwise per-cospectral.*
(vii) *Graphs G_{58}, G_{511}, G_{513} and G_{517} are not pairwise per-cospectral.*
(viii) *Graphs G_{59} and G_{524} are not per-cospectral.*
(ix) *Graphs G_{53}, G_{57}, G_{518}, G_{519} and G_{522} are not pairwise per-cospectral.*
 (x) *Graphs G_{56}, G_{510}, G_{515}, G_{516} and G_{525} are not pairwise per-cospectral.*

Proof

  (i) By Table 2, we have $c_4(G_{32}) - c_4(G_{33}) = n - 4$. By Lemma 2.3, $b_4(G_{32}) - b_4(G_{33}) = \sum_{i=1}^n \left(\binom{d_i(G_{33})}{2} - \binom{d_i(G_{32})}{2} \right) + 2(c_4(G_{32}) - c_4(G_{33})) = 2n - 9 \neq 0$. Hence by Lemma 2.2, G_{32} and G_{33} are not per-cospectral.
 (ii) By Lemma 2.3 and Table 2, we have $b_4(G_{41}) - b_4(G_{47}) = \sum_{i=1}^n \left(\binom{d_i(G_{47})}{2} - \binom{d_i(G_{41})}{2} \right) + 2(c_4(G_{41}) - c_4(G_{47})) = 4 \neq 0$. Hence by Lemma 2.2, G_{41} and G_{47} are not per-cospectral.
(iii) For graphs G_{44} and G_{46}, by Lemma 2.3 and Table 2, we have $b_4(G_{44}) - b_4(G_{46}) = \sum_{i=1}^n \left(\binom{d_i(G_{46})}{2} - \binom{d_i(G_{44})}{2} \right) + 2(c_4(G_{44}) - c_4(G_{46})) = 9 - 2n \neq 0$. Hence by Lemma 2.2, G_{44} and G_{46} are not per-cospectral. Similarly, we have that

$b_4(G_{44}) - b_4(G_{410}) = -2$ and $b_4(G_{46}) - b_4(G_{410}) = 2n - 11 \neq 0$, which imply that G_{410} is not per-cospectral with G_{44} or G_{46}.

(iv) We argue similarly to get $b_4(G_{42}) - b_4(G_{45}) = 11 - 2n$, $b_4(G_{42}) - b_4(G_{48}) = 1$ and $b_4(G_{45}) - b_4(G_{48}) = 2n - 9$. These imply, by Lemma 2.2, that none of pairs G_{42} and G_{45}, G_{42} and G_{48} and G_{45} and G_{48} are per-cospectral.

(v) By Lemma 2.3 and Table 2, we have $b_4(G_{50}) - b_4(G_{51}) = -1$, $b_4(G_{50}) - b_4(G_{514}) = 9 - 2n$ and $b_4(G_{51}) - b_4(G_{514}) = 11 - 2n$. By Lemma 2.2, we conclude that any two graphs in $\{G_{50}, G_{51}, G_{514}\}$ are not per-cospectral.

(vi) By Lemma 2.3 and Table 2, we have $b_4(G_{55}) - b_4(G_{512}) = 13 - 2n$, $b_4(G_{55}) - b_4(G_{520}) = 4$, $b_4(G_{55}) - b_4(G_{523}) = 1$, $b_4(G_{512}) - b_4(G_{520}) = 2n - 9$, $b_4(G_{512}) - b_4(G_{523}) = 2n - 11$ and $b_4(G_{520}) - b_4(G_{523}) = -2$. By Lemma 2.2, we conclude that any two graphs in $\{G_{55}, G_{512}, G_{520}, G_{523}\}$ are not per-cospectral.

(vii) By Lemma 2.3 and Table 2, we have $b_4(G_{58}) - b_4(G_{511}) = 2$, $b_4(G_{58}) - b_4(G_{513}) = 11 - 2n$, $b_4(G_{58}) - b_4(G_{517}) = 4$, $b_4(G_{511}) - b_4(G_{517}) = 2$, $b_4(G_{511}) - b_4(G_{513}) = 9 - 2n$ and $b_4(G_{513}) - b_4(G_{517}) = 2n - 7$. These imply, by Lemma 2.2, that G_{58}, G_{511}, G_{513} and G_{517} are not pairwise per-cospectral.

(viii) Assume, by contradiction, that G_{59} and G_{524} are per-cospectral. By Lemma 2.2, we have $b_4(G_{59}) - b_4(G_{524}) = 0$. However, by Lemma 2.3 and Table 2, we have $b_4(G_{59}) - b_4(G_{524}) = 4n - 26 \neq 0$, a contradiction.

(ix) By Lemma 2.3 and Table 2, we observe that $b_4(G_{519}) = b_4(G_{522})$. By (4), we have $c_5(G_{519}) = 12\binom{n}{5} - 30\binom{n-2}{3} + 8\binom{n-3}{2} + 20\binom{n-4}{1} - 8$ and $c_5(G_{522}) = 12\binom{n}{5} - 30\binom{n-2}{3} + 8\binom{n-3}{2} + 21\binom{n-4}{1} - 10$. We assume that G_{519} and G_{522} are per-cospectral. By Lemmas 2.2 and Equations (1), (5) and (11), we have $0 = b_5(G_{519}) - b_5(G_{522}) = 2n - 16$, forcing $n = 8$.

Using Maple 12.0 with $n = 8$, we compute the permanental polynomials of G_{519} and G_{522}, respectively. We found that $\pi(G_{519}, x) = \pi(K_8 - (E(C_4) \cup E(K_2)), x) = x^8 + 23x^6 - 60x^5 + 319x^4 - 936x^3 + 2309x^2 - 3628x + 2812$ and $\pi(G_{522}, x) = \pi(K_8 - E(P_6), x) = x^8 + 23x^6 - 60x^5 + 319x^4 - 936x^3 + 2286x^2 - 3620x + 2909$. So $\pi(G_{519}, x) \neq \pi(G_{522}, x)$, a contradiction, which indicates that G_{519} and G_{522} are not per-cospectral.

Similarly, by Lemma 2.3 and Table 2, we have $b_4(G_{53}) - b_4(G_{57}) = 2$, $b_4(G_{53}) - b_4(G_{518}) = 11 - 2n$, $b_4(G_{53}) - b_4(G_{519}) = -2$, $b_4(G_{53}) - b_4(G_{522}) = -2$, $b_4(G_{57}) - b_4(G_{518}) = 15 - 2n$, $b_4(G_{57}) - b_4(G_{519}) = 6$, $b_4(G_{57}) - b_4(G_{522}) = 6$, $b_4(G_{518}) - b_4(G_{519}) = 2n - 13$ and $b_4(G_{518}) - b_4(G_{522}) = 2n - 13$. These imply, by Lemma 2.2, that G_{53}, G_{57}, G_{518}, G_{519} and G_{522} are not pairwise per-cospectral.

(x) Suppose that G_{56} and G_{516} are per-cospectral, by Lemma 2.3 and Table 2, we have $0 = b_4(G_{56}) - b_4(G_{516}) = 32 - 4n$, and so $n = 8$.

Using Maple 12.0 with $n = 8$, we obtain that $\pi(G_{56}, x) = \pi(K_8 - E(K_{1,4} \cup P_2), x) = x^8 + 23x^6 - 64x^5 + 335x^4 - 980x^3 + 2293x^2 - 3316x + 2252$, $\pi(G_{516}, x) = \pi(K_8 - E(K_4 - E(P_2)), x) = x^8 + 23x^6 - 64x^5 + 335x^4 - 952x^3 + 2161x^2 - 2872x + 1608$. So $\pi(G_{56}, x) \neq \pi(G_{516}, x)$, a contradiction, which shows that G_{56} and G_{516} are not per-cospectral.

Similarly, suppose that $\pi(G_{516}, x) = \pi(G_{525}, x)$. We have $0 = b_4(G_{516}) - b_4(G_{525}) = 4n - 24$, and so $n = 6$. Let W_6 be a graph obtained from the path P_2 by adding two pendant edges at every vertex. By Maple 12.0, we obtain that $\pi(G_{516}, x) = \pi(K_6 - E(K_4 - $

$E(P_2)), x) = x^6 + 10x^4 - 12x^3 + 33x^2 - 20x + 4$, $\pi(G_{525}, x) = \pi(K_6 - E(W_6), x) = x^6 + 10x^4 - 12x^3 + 33x^2 - 28x + 20$. So G_{516} and G_{525} are not per-cospectral.

In addition, by Lemma 2.3 and Table 2, we have $b_4(G_{56}) - b_4(G_{510}) = 42 - 4n$, $b_4(G_{56}) - b_4(G_{515}) = 20$, $b_4(G_{56}) - b_4(G_{525}) = 16$, $b_4(G_{510}) - b_4(G_{515}) = 2n - 11$, $b_4(G_{510}) - b_4(G_{516}) = 2n - 1$, $b_4(G_{510}) - b_4(G_{525}) = 2n - 11$, $b_4(G_{515}) - b_4(G_{516}) = 22 - 4n$ and $b_4(G_{515}) - b_4(G_{525}) = -2$. By Lemma 2.2, we conclude that no two of G_{56}, G_{510}, G_{515}, G_{516} and G_{525} are per-cospectral. □

4. Further characterizations

In this section, we focus on the induced subgraph structures of an edge subset X in K_n such that $K_n - X$ is DPS, as an attempt to extend Theorem 1.2. More precisely, let H be an edge induced subgraph of K_n with $|E(H)| = l$. We will show that when $H \in \{K_{1,l}, lP_2, (l-2)P_2 \cup P_3\}$, $K_n - E(H)$ is DPS.

THEOREM 4.1 *If star $K_{1,l}$ is a subgraph of K_n, then the graph $K_n - E(K_{1,l})$ is DPS.*

Proof Let G be a graph per-cospectral with $K_n - E(K_{1,l})$. Then G must be isomorphic to some $K_n - E(H)$ for an edge induced subgraph H of K_n with $|E(H)| = l$. As $\sum_{v \in V(H)} \binom{d(v)}{2}$ equals the number of P_3's in H, $\sum_{v \in V(H)} \binom{d(v)}{2} \leq \binom{l}{2}$. If $H \not\cong K_{1,l}$, then either there exist at least two edges in H that are not adjacent or H is just a triangle. Hence $\sum_{v \in V(H)} \binom{d(v)}{2} - c_3(H) < \binom{l}{2}$. By Lemma 2.6, we have $c_3(K_n - E(H)) = \binom{n}{3} - l(n-2) + \sum_{v \in V(H)} \binom{d(v)}{2} - c_3(H) < \binom{n}{3} - l(n-2) + \binom{l}{2} = c_3(K_n - E(K_{1,l}))$, contradicting Lemma 2.5 (iii). Hence, we must have $H \cong K_{1,l}$. That is, $G \cong K_n - E(K_{1,l})$. □

Theorem 4.1 immediately implies the following corollary.

COROLLARY 4.2 *The disjoint union of a complete graph and an isolated vertex is DPS.*

LEMMA 4.3 *Let H be a graph with l edges. Then $c_3(H) \leq \frac{1}{3}\sum_{v \in V(H)} \binom{d(v)}{2}$, and equality holds when $H \cong l_1 C_3 \cup (l - 3l_1)P_2$ for some integer $l_1 \geq 0$ and $l - 3l_1 \geq 0$.*

Proof Any vertex $v \in V(H)$ is contained at most $\binom{d(v)}{2}$ triangles in H. Then $c_3(H) \leq \frac{1}{3}\sum_{v \in V(H)} \binom{d(v)}{2}$. Clearly, the equality holds when $H \cong l_1 C_3 \cup (l - 3l_1)P_2$ for some integer $l_1 \geq 0$ and $l - 3l_1 \geq 0$. □

THEOREM 4.4 *If lP_2 is a subgraph of K_n, then the graph $K_n - E(lP_2)$ is DPS.*

Proof Let G be a graph per-cospectral with $K_n - E(lP_2)$. Then G must be isomorphic to some $K_n - E(H)$ for a subgraph H of K_n with $|E(H)| = l$ and no isolated vertices. When $n \leq 2$, the result is trivial. Now we may assume that $n > 2$. Suppose H is not isomorphic to lP_2. Then H has a vertex of degree at least two. By Lemmas 2.6 and 4.3, we have $c_3(G) = c_3(K_n - E(H)) = \binom{n}{3} - l(n-2) + \sum_{v \in V(H)} \binom{d(v)}{2} - c_3(H) > \binom{n}{3} - l(n-2) = c_3(K_n - E(lP_2))$. This is a contradiction with Lemma 2.5 (iii). Thus $G \cong K_n - E(lP_2)$. □

THEOREM 4.5   *If $(l-2)P_2 \cup P_3$ is a subgraph of K_n, then the graph $K_n - E((l-2)P_2 \cup P_3)$ is DPS.*

Proof   Similarly, let G be a graph per-cospectral with $K_n - E((l-2)P_2 \cup P_3)$. Then G must be isomorphic to some $K_n - E(H)$ for a subgraph H of K_n with $|E(H)| = l$ and no isolated vertices. By Lemma 2.5(iii), we have $c_3(G) - c_3(K_n - E((l-2)P_2 \cup P_3)) = 0$. By Lemma 2.6, we simplify this equation to have

$$\sum_{v \in V(H)} \binom{d(v)}{2} - c_3(H) = 1. \tag{14}$$

If H contains at least one triangle, then Lemma 4.3 implies that $\sum_{v \in V(H)} \binom{d(v)}{2} - c_3(H) \geq 2c_3(H) \geq 2$, a contradiction. So, we may assume that H contains no triangles. By (14), we have $\sum_{v \in V(H)} \binom{d(v)}{2} = 1$. This implies that there exists exactly a vertex $v \in V(H)$ such that $d(v)=2$, and the other vertices have degree one. Thus $H \cong (l-2)P_2 \cup P_3$. So $K_n - E((l-2)P_2 \cup P_3)$ is DPS.   □

Acknowledgements

Funding

References

[1] Lovász L, Plummer MD. Matching theory. Providence (RI): American Mathematical Society; 2009.
[2] Valiant LG. The complexity of computing the permanent. Theor. Comput. Sci. 1979;8:189–201.
[3] Turner J. Generalized matrix functions and the graph isomorphism problem. SIAM J. Appl. Math. 1968;16:520–526.
[4] Merris R, Rebman KR, Watkins W. Permanental polynomials of graphs. Linear Algebra Appl. 1981;38:273–288.
[5] Kasum D, Trinajstić N, Gutman I. Chemical graph theory. III. On permanental polynomial. Croat. Chem. Acta. 1981;54:321–328.
[6] Borowiecki M, Jóźwiak T. A note on characteristic and permanental polynomials of multigraphs. In: Borowiecki M, Kennedy JW, Sysło MM, editors. Graph theory. Berlin: Springer-Verlag; 1983. p. 75–78.
[7] Yan W, Zhang F. On the permanental polynomial of some graphs. J. Math. Chem. 2004;35:175–188.
[8] Zhang H, Li W. Computing the permanental polynomials of bipartite graphs by Pfaffian orientation. Discrete Appl. Math. 2012;160:2069–2074.
[9] Gutman I, Cash GG. Relations between the permanental and characteristic polynomials of fullerenes and benzenoid hydrocarbons. MATCH Commun. Math. Comput. Chem. 2002;45:55–70.

[10] Chen R. A note on the relations between the permanental and characteristic polynomials of coronoid hydrocarbons. MATCH Commun. Math. Comput. Chem. 2004;51:137–148.

[11] Cash GG. The permanental polynomial. J. Chem. Inf. Comput. Sci. 2000;40:1203–1206.

[12] Cash GG. Permanental polynomials of smaller fullerenes. J. Chem. Inf. Comput. Sci. 2000;40:1207–1209.

[13] Belardo F, Li Marzi EM, Simić SK. Schwenk-like formulas for weighted digraphs. In: Simos TE, Psihoyios G, Tsitouras C, editors. Numerical analysis and applied mathematics. Vol. 2. CP1168; Crete; 2009. p. 1251–1254.

[14] Belardo F, De Filippis V, Simić SK. Computing the permanental polynomial of a matrix from a combinatorial viewpoint. MATCH Commun. Math. Comput. Chem. 2011;66:381–396.

[15] Huo Y, Liang H, Bai F. An efficient algorithm for computing permanental polynomials of graphs. Comput. Phys. Comm. 2006;175:196–203.

[16] Li W, Zhang H. The permanental polynomials of certain graphs. MATCH Commun. Math. Comput. Chem. 2012;68:871–888.

[17] Liang H, Tong H, Bai F. Computing the permanental polynomial of C_{60} in parallel. MATCH Commun. Math. Comput. Chem. 2008;60:349–358.

[18] Tong H, Liang H, Bai F. Permanental polynomials of the larger fullerenes. MATCH Commun. Math. Comput. Chem. 2006;56:141–152.

[19] Zhang H, Liu S, Li W. A note on the permanental roots of bipartite graphs. Discuss. Math. Graph Theory. Forthcoming. doi:10.7151/dmgt.1704.

[20] van Dam ER, Haemers WH. Which graphs are determined by their spectrum? Linear Algebra Appl. 2003;373:241–272.

[21] Doob M, Haemers WH. The complement of the path is determined by its spectrum. Linear Algebra Appl. 2002;356:57–65.

[22] Wang W, Xu C. On the spectral characterization of T-shape trees. Linear Algebra Appl. 2006;414:492–501.

[23] Boulet R, Jouve B. The lollipop is determined by its spectrum. Electron. J. Combin. 2008;15:#R74.

[24] Haemers WH, Liu XG, Zhang YP. Spectral characterizations of lollipop graphs. Linear Algebra Appl. 2008;428:2415–2423.

[25] Ramezani F, Broojerdian N, Tayfeh-Rezaie B. A note on the spectral characterization of θ-graphs. Linear Algebra Appl. 2009;431:626–632.

[26] Ghareghai N, Omidi GR, Tayfeh-Rezaie B. Spectral characterization of graphs with index at most $\sqrt{2 + \sqrt{5}}$. Linear Algebra Appl. 2007;420:483–489.

[27] Zhang X, Zhang H. Some graphs determined by their spectra. Linear Algebra Appl. 2009;431:1443–1454.

[28] Harary F, King C, Mowshowitz A, Read RC. Cospectal graphs and digraphs. Bull. Lond. Math. Soc. 1971;3:321–328.

[29] Liu S, Zhang H. On the characterizing properties of the permanental polynomials of graphs. Linear Algebra Appl. 2013;438:157–172.

[30] Liu S, Zhang H. Characterizing properties of permanental polynomials of lollipop graphs. Linear Multilinear Algebra. 2013. http://dx.doi.org/10.1080/03081087.2013.779271.

[31] Cámara M, Haemers WH. Spectral characterizations of almost complete graphs. Discrete Appl. Math. Forthcoming 2013. http://dx.doi.org/10.1016/j.dam.2013.08.002

[32] Harary F, Palmer EM. Graphical enumeration. New York (NY): Academinc Press; 1973.