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Abstract With graphs considered as natural models for many network design problems, edge connectivity $\kappa'(G)$ and maximum number of edge-disjoint spanning trees $\tau(G)$ of a graph G have been used as measures for reliability and strength in communication networks modeled as graph G (see Cunningham, in J ACM 32:549–561, 1985; Matula, in Proceedings of 28th Symposium Foundations of Computer Science, pp 249–251, 1987, among others). Mader (Math Ann 191:21–28, 1971) and Matula (J Appl Math 22:459–480, 1972) introduced the maximum subgraph edge connectivity $\overline{\kappa'}(G) = \max\{\kappa'(H) : H \text{ is a subgraph of } G\}$. Motivated by their applications in network design and by the established inequalities

$$\overline{\kappa'}(G) \geq \kappa'(G) \geq \tau(G),$$

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we present the following in this paper:

1. For each integer $k > 0$, a characterization for graphs G with the property that $\overline{\kappa'}(G) \leq k$ but for any edge e not in G, $\overline{\kappa'}(G+e) \geq k+1$.
2. For any integer $n > 0$, a characterization for graphs G with $|V(G)| = n$ such that $\kappa'(G) = \tau(G)$ with $|E(G)|$ minimized.

Keywords Edge connectivity · Edge-disjoint spanning trees · k-Maximal graphs · Network strength · Network reliability

1 Introduction

With graphs considered as natural models for many network design problems, edge connectivity and maximum number of edge-disjoint spanning trees of a graph have been used as measures for reliability and strength in communication networks modeled as a graph (see [4,13], among others).

We consider finite graphs with possible multiple edges, and follow notations of Bondy and Murty [2], unless otherwise defined. Thus for a graph G, $\omega(G)$ denotes the number of components of G, and $\kappa'(G)$ denotes the edge connectivity of G. For a connected graph G, $\tau(G)$ denotes the maximum number of edge-disjoint spanning trees in G. A survey on $\tau(G)$ can be found in [16]. By definition, $\tau(K_1) = \infty$. A graph G is *nontrivial* if $|E(G)| \neq \emptyset$.

For any graph G, we further define $\overline{\kappa'}(G) = \max\{\kappa'(H) : H \text{ is a subgraph of } G\}$. The invariant $\overline{\kappa'}(G)$, first introduced by Matula [12], has been studied by Boesch and McHugh [1], Lai [6], Matula [12,13], Mitchem [14] and implicitly by Mader [11]. In [13], Matula gave a polynomial algorithm to determine $\overline{\kappa'}(G)$.

Throughout the paper, k and n denote positive integers, unless otherwise defined.

Mader [11] first introduced k-maximal graphs. A graph G is *k-maximal* if $\overline{\kappa'}(G) \leq k$ but for any edge $e \notin E(G)$, $\overline{\kappa'}(G+e) \geq k+1$. The k-maximal graphs have been studied in [1,6,11–14], among others.

Simple k-maximal graphs have been well studied. In [11], Mader proved that the maximum number of edges in a simple k-maximal graph with n vertices is $(n-k)k+\binom{k}{2}$ and characterized all the extremal graphs. In 1990, Lai [6] showed that the minimum number of edges in a simple k-maximal graph with n vertices is $(n-1)k - \binom{k}{2}\lfloor \frac{n}{k+2} \rfloor$. In the same paper, Lai also characterized all extremal graphs and all simple k-maximal graphs.

In this paper, we mainly focus on multiple k-maximal graphs, and show that the number of edges in a k-maximal graph with n vertices is $k(n-1)$ and give a complete characterization of all k-maximal graphs as well as show several equivalent graph families.

As it is known that for any connected graph G, $\kappa'(G) \geq \tau(G)$, it is natural to ask when the equality holds. Motivated by this question, we characterize all graphs G satisfying $\kappa'(G) = \tau(G)$ with minimum number of possible edges for a fixed number of vertices. We also investigate necessary and sufficient conditions for a graph to have a spanning subgraph with this property or to be a spanning subgraph of another graph with this property.

In Sect. 2, we display some preliminaries. In Sect. 3, we will characterize all k-maximal graphs. The characterizations of minimal graphs with $\kappa' = \tau$ and reinforcement problems will be discussed in Sects. 4 and 5, respectively.

In this paper, an edge-cut always means a minimal edge-cut.

2 Preliminaries

Let G be a nontrivial graph. The *density* of G is defined by

$$d(G) = \frac{|E(G)|}{|V(G)| - \omega(G)}. \tag{1}$$

Hence, if G is connected, then $d(G) = \frac{|E(G)|}{|V(G)|-1}$. Following the terminology in [3], we define $\eta(G)$ and $\gamma(G)$ as follows:

$$\eta(G) = \min \frac{|X|}{\omega(G - X) - \omega(G)} \text{ and } \gamma(G) = \max\{d(H)\},$$

where the minimum or maximum is taken over all edge subsets X or subgraph H whenever the denominator is non-zero. From the definitions of $d(G)$, $\eta(G)$ and $\gamma(G)$, we have, for any nontrivial graph G,

$$\eta(G) \leq d(G) \leq \gamma(G). \tag{2}$$

As in [3], a graph G satisfying $d(G) = \gamma(G)$ is said to be *uniformly dense*. The following theorems are well known.

Theorem 2.1 (Nash-Williams [15], Tutte [17])
Let G be a connected graph with $E(G) \neq \emptyset$, and let $k > 0$ be an integer. Then $\tau(G) \geq k$ if and only if for any $X \subseteq E(G)$, $|X| \geq k(\omega(G - X) - 1)$.

Theorem 2.1 indicates that for a connected graph G

$$\tau(G) = \lfloor \eta(G) \rfloor. \tag{3}$$

Theorem 2.2 (Catlin et al. [3])
Let G be a graph. The following statements are equivalent.

(i) $\eta(G) = d(G)$.
(ii) $d(G) = \gamma(G)$.
(iii) $\eta(G) = \gamma(G)$.

For a connected graph G with $\tau(G) \geq k$, we define $E_k(G) = \{e \in E(G) : \tau(G - e) \geq k\}$.

Lemma 2.3 (Li et al. [9], Li [8])
Let G be a connected graph with $\tau(G) \geq k$. Then $E_k(G) = \emptyset$ if and only if $d(G) = k$.

Lemma 2.4 (Haas [5], Lai et al. [7] and Liu et al. [10])
Let G be a graph, then the following statements are equivalent.

(i) $\gamma(G) \leq k$.
(ii) *There exist $k(|V(G)|-1)-|E(G)|$ edges whose addition to G results in a graph that can be decomposed into k edge-disjoint spanning trees.*

3 Characterizations of k-Maximal Graphs

In this section, we are to present a structural characterization of k-maximal graphs as well as several equivalent conditions, as shown in Theorem 3.1.

Let $F(n,k)$ be the maximum number of edges in a graph G on n vertices with $\overline{\kappa'}(G) \leq k$. We define $\mathcal{F}(n,k) = \{G : |E(G)| = F(n,k), |V(G)| = n, \overline{\kappa'}(G) \leq k\}$.

Let G_1 and G_2 be connected graphs such that $V(G_1) \cap V(G_2) = \emptyset$. Let K be a set of k edges each of which has one vertex in $V(G_1)$ and the other vertex in $V(G_2)$. The *K-edge-join* $G_1 *_K G_2$ is defined to be the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup K$. When the set K is not emphasized, we use $G_1 *_k G_2$ for $G_1 *_K G_2$, and refer to $G_1 *_k G_2$ as a k-edge-join.

Let $\mathcal{G}_k$ be a family of graphs such that for any $G_1, G_2 \in \mathcal{G}_k \cup \{K_1\}$, $G_1 *_k G_2 \in \mathcal{G}_k$. Let $\overline{\tau}(G) = \max\{\tau(H) : H \text{ is a subgraph of } G\}$. The main theorem in this section is stated below.

Theorem 3.1 *Let G be a graph on n vertices. The following statements are equivalent.*

(i) $G \in \mathcal{F}(n,k)$;
(ii) *G is k-maximal;*
(iii) $\eta(G) = \overline{\kappa'}(G) = k$;
(iv) $\tau(G) = \overline{\kappa'}(G) = k$;
(v) $\tau(G) = \overline{\tau}(G) = \kappa'(G) = \overline{\kappa'}(G) = k$;
(vi) $G \in \mathcal{G}_k$.

In order to prove Theorem 3.1, we need some lemmas.

Lemma 3.2 *Let X be a k-edge cut of a graph G. If H is a subgraph of G with $\kappa'(H) > k$, then $E(H) \cap X = \emptyset$.*

Proof If $E(H) \cap X \neq \emptyset$, then $\kappa'(H) \leq |E(H) \cap X| \leq |X| = k < \kappa'(H)$, a contradiction. □

Lemma 3.3 *If a graph G is k-maximal, then $\kappa'(G) = \overline{\kappa'}(G) = k$.*

Proof Since G is k-maximal, $\kappa'(G) \leq \overline{\kappa'}(G) \leq k$. It suffices to show that $\kappa'(G) = k$. We assume that $\kappa'(G) < k$ and prove it by contradiction. Let X be an edge cut with $|X| < k$ and suppose that $G = G_1 *_X G_2$. Let $e \notin E(G)$ be an edge with one end in $V(G_1)$ and the other end in $V(G_2)$. By the definition of k-maximal graphs, $\overline{\kappa'}(G+e) \geq k+1$. Thus $G+e$ has a subgraph H with $\kappa'(H) \geq k+1$. Then it must be the case that $e \in E(H)$, otherwise H is a subgraph of G, contrary to $\overline{\kappa'}(G) \leq k$. Since $X \cup \{e\}$ is an edge cut of $G+e$ with $|X \cup \{e\}| \leq k$ and H is a subgraph of $G+e$ with $\kappa'(H) \geq k+1$, by Lemma 3.2, $E(H) \cap (X \cup \{e\}) = \emptyset$, contrary to $e \in E(H)$. □

Lemma 3.4 *If a graph G is k-maximal, then $G = G_1 *_k G_2$ where either $G_i = K_1$ or G_i is k-maximal for $i = 1, 2$.*

Proof By Lemma 3.3, G has a k-edge cut X, and so $G = G_1 *_k G_2$. For $i = 1, 2$, suppose that $G_i \neq K_1$, we want to prove that G_i is k-maximal. Since G is k-maximal, $\overline{\kappa'}(G) \leq k$, whence $\overline{\kappa'}(G_i) \leq k$. For any edge $e \notin E(G_i)$, $\overline{\kappa'}(G + e) \geq k + 1$. Thus $G + e$ has a subgraph H with $\kappa'(H) \geq k + 1$. Since $\overline{\kappa'}(G) \leq k$, H is not a subgraph of G, and so $e \in E(H)$. Since X is a k-edge cut of $G + e$, by Lemma 3.2, $E(H) \cap X = \emptyset$. Hence H is a subgraph of $G_i + e$ with $\kappa'(H) \geq k + 1$, whence $\overline{\kappa'}(G_i) \geq k + 1$. Thus G_i is k-maximal. □

Lemma 3.5 *Let G be a graph on n vertices. Then $G \in \mathcal{F}(n, k)$ if and only if G is k-maximal.*

Proof By the definition of $\mathcal{F}(n, k)$, if $G \in \mathcal{F}(n, k)$, then $|E(G)| = F(n, k)$ and $\overline{\kappa'}(G) \leq k$. Then for any edge $e \notin E(G)$, $|E(G + e)| = |E(G)| + 1 > F(n, k)$, and so $\overline{\kappa'}(G + e) \geq k + 1$. By the definition of k-maximal graphs, G is k-maximal.

Now we assume that G is k-maximal to prove that $G \in \mathcal{F}(n, k)$. It suffices to show that any k-maximal graph G has the property $\overline{\kappa'}(G) \leq k$ with the maximum number of edges. We will prove that for any k-maximal graph G, $|E(G)| = F(n, k) = k(n - 1)$. We use induction on n. When $n = 2$, G is kK_2, which is the graph with 2 vertices and k multiple edges, and so $|E(G)| = k$. We assume that $|E(G)| = F(n, k) = k(n - 1)$ holds for smaller values of $n > 2$. By Lemma 3.4, $G = G_1 *_k G_2$ where G_i is k-maximal or k_1 for $i = 1, 2$. Let $|V(G_i)| = n_i$. By inductive hypothesis, $|E(G_i)| = k(n_i - 1)$. Thus $|E(G)| = k(n_1 - 1) + k(n_2 - 1) + k = k(n - 1)$. □

Corollary 3.6 $F(n, k) = k(n - 1)$.

Lemma 3.7 *Suppose $\tau(G) = \overline{\tau}(G) = \kappa'(G) = \overline{\kappa'}(G) = k$. Then $G = G_1 *_k G_2$ where either $G_i = K_1$ or G_i satisfies $\tau(G_i) = \overline{\tau}(G_i) = \kappa'(G_i) = \overline{\kappa'}(G_i) = k$ for $i = 1, 2$.*

Proof Since $\kappa'(G) = k$, there must be an edge-cut of size k. Hence there exist graphs G_1 and G_2 such that $G = G_1 *_k G_2$. If $G_i \neq K_1$, we will prove $\tau(G_i) = \overline{\tau}(G_i) = \kappa'(G_i) = \overline{\kappa'}(G_i) = k$, for $i = 1, 2$. First, by the definition of $\overline{\tau}$, $\tau(G_i) \leq \overline{\tau}(G_i) \leq \overline{\tau}(G) = k$ for $i = 1, 2$. Since G has k disjoint spanning trees, we have $\tau(G_i) \geq k$ for $i = 1, 2$. Thus $\tau(G_i) = \overline{\tau}(G_i) = k$ for $i = 1, 2$. Now we prove $\kappa'(G_i) = \overline{\kappa'}(G_i) = k$ for $i = 1, 2$. Since $\overline{\kappa'}(G) = k$, $\kappa'(G_i) \leq \overline{\kappa'}(G_i) \leq k$. But $\kappa'(G_i) \geq \tau(G_i) = k$ for $i = 1, 2$. Hence we have $\tau(G_i) = \overline{\tau}(G_i) = \kappa'(G_i) = \overline{\kappa'}(G_i) = k$ for $i = 1, 2$. □

Lemma 3.8 *Let $G = G_1 *_k G_2$ where $G_i = K_1$ or G_i satisfies $\tau(G_i) = \overline{\tau}(G_i) = \kappa'(G_i) = \overline{\kappa'}(G_i) = k$ for $i = 1, 2$. Then $\tau(G) = \overline{\tau}(G) = \kappa'(G) = \overline{\kappa'}(G) = k$.*

Proof Since $G = G_1 *_k G_2$ and $\kappa'(G_1) = \kappa'(G_2) = k$, we have $\tau(G) \leq \kappa'(G) = k$ and there exists an edge-cut $X = \{x_1, x_2, \ldots, x_k\}$ such that $G = G_1 *_X G_2$. Let $T_{1,i}, T_{2,i}, \ldots, T_{k,i}$ be edge-disjoint spanning trees of G_i, for $i = 1, 2$. Then $T_{1,1} + x_1 + T_{1,2}, T_{2,1} + x_2 + T_{2,2}, \ldots, T_{k,1} + x_k + T_{k,2}$ are k edge-disjoint spanning trees of G. Thus $\tau(G) = \kappa'(G) = k$. Now we need to prove that for any subgraph H

of G, $\tau(H) \le k$ and $\kappa'(H) \le k$. If $E(H) \cap X \ne \emptyset$, then $E(H) \cap X$ is an edge cut of H and thus $\tau(H) \le \kappa'(H) \le k$. If $E(H) \cap X = \emptyset$, then H is a spanning subgraph of either G_1 or G_2, whence $\tau(H) \le \kappa'(H) \le k$. □

Now we present the proof of Theorem 3.1.

Proof of Theorem 3.1 By Lemma 3.5, (i) and (ii) are equivalent. By (3), (iii)⇒(iv).

(i)⇒(iii): By Corollary 3.6, $|E(G)| = k(n-1)$. By the definition of $d(G)$, $d(G) = k$. Since $\overline{\kappa'}(G) \le k$, for any subgraph H of G, $\overline{\kappa'}(H) \le k$. By Corollary 3.6, $|E(H)| \le k(|V(H)| - 1)$, whence $d(H) \le k$. By the definition of $\gamma(G)$, we have $\gamma(G) \le k$. Thus $d(G) = \gamma(G) = k$. By Theorem 2.2, $\eta(G) = k$. Hence $k = \eta(G) = \tau(G) \le \overline{\kappa'}(G) \le k$, i.e., $\eta(G) = \overline{\kappa'}(G) = k$.

(iv)⇒(i): Since $\overline{\kappa'}(G) = k$, by Corollary 3.6, $|E(G)| \le k(n-1)$. Since $\tau(G) = k$, G has k edge-disjoint spanning trees, and so $|E(G)| \ge k(n-1)$. Thus $|E(G)| = k(n-1)$, and so $G \in \mathcal{F}(n,k)$.

(iv)⇔(v): By definition, $\tau(G) \le \overline{\tau}(G) \le \overline{\kappa'}(G)$ and $\tau(G) \le \kappa'(G) \le \overline{\kappa'}(G)$. The equivalence between (iv) and (v) now follows from these inequalities.

(v)⇒(vi): We argue by induction on $|V(G)|$. When $|V(G)| = 2$, a graph G with $\tau(G) = \overline{\tau}(G) = \kappa'(G) = \overline{\kappa'}(G) = k$ must be $K_1 *_k K_1$, and so by definition, $G \in \mathcal{G}_k$. We assume that (v)⇒(vi) holds for smaller values of $|V(G)|$. By Lemma 3.7, $G = G_1 *_k G_2$ with $\tau(G_i) = \overline{\tau}(G_i) = \kappa'(G_i) = \overline{\kappa'}(G_i) = k$ or $G_i = K_1$, for $i = 1, 2$. If $G_i \ne K_1$, then by the inductive hypothesis, $G_i \in \mathcal{G}_k$. By definition, $G \in \mathcal{G}_k$.

(vi)⇒(v): We show it by induction on $|V(G)|$. When $|V(G)| = 2$, by the definition of $\mathcal{G}_k$, $G = K_1 *_k K_1$, and then $\tau(G) = \overline{\tau}(G) = \kappa'(G) = \overline{\kappa'}(G) = k$. We assume that it holds for smaller values of $|V(G)|$. By the definition of $\mathcal{G}_k$, $G = G_1 *_k K_1$ or $G = G_1 *_k G_2$ where $G_1, G_2 \in \mathcal{G}_k$. By inductive hypothesis, $\tau(G_i) = \overline{\tau}(G_i) = \kappa'(G_i) = \overline{\kappa'}(G_i) = k$ for $i = 1, 2$, and by Lemma 3.8, $\tau(G) = \overline{\tau}(G) = \kappa'(G) = \overline{\kappa'}(G) = k$. □

4 Characterizations of Minimal Graphs with $\kappa' = \tau$

We define

$$\mathcal{F}_{k,n} = \{G : \kappa'(G) = \tau(G) = k, |V(G)| = n \text{ and } |E(G)| \text{ is minimized}\}$$

and $\mathcal{F}_k = \cup_{n>1} \mathcal{F}_{k,n}$.

In this section, we will give characterizations of graphs in $\mathcal{F}_k$. In addition, we use $\mathcal{F}_{k,n}$ to characterize graphs G with $\kappa'(G) = \tau(G)$.

Theorem 4.1 *Let G be a graph, then $G \in \mathcal{F}_k$ if and only if G satisfies*

(i) *G has an edge-cut of size k, and*
(ii) *G is uniformly dense with density k.*

Proof Suppose that $G \in \mathcal{F}_k$, then $\tau(G) = \kappa'(G) = k$. Hence G has an edge-cut of size k. Since $|E(G)|$ is minimized, we have $E_k(G) = \emptyset$. By Lemma 2.3, $d(G) = k$.

Since $\tau(G) = k$, by Theorem 2.1 and the definition of $\eta(G)$, we have $\eta(G) \geq k$. By (2), $\eta(G) \leq d(G) = k$, whence $\eta(G) = d(G) = k$. By Theorem 2.2, G is uniformly dense with density k.

On the other hand, suppose that G satisfies (i) and (ii). By (2) and Theorem 2.2, $\eta(G) = d(G) = k$. By (3), $\tau(G) = k$. Then $\kappa'(G) \geq \tau(G) = k$. But G has an edge-cut of size k, thus $\kappa'(G) = \tau(G) = k$. Since $d(G) = k$, by Lemma 2.3, $E_k(G) = \emptyset$, i.e. $|E(G)|$ is minimized. Thus $G \in \mathcal{F}_k$. □

Theorem 4.2 *A graph $G \in \mathcal{F}_k$ if and only if $G = G_1 *_k G_2$ where either $G_i = K_1$ or G_i is uniformly dense with density k for $i = 1, 2$.*

Proof Suppose that $G \in \mathcal{F}_k$. By Theorem 4.1, G has an edge-cut of size k, whence there exist graphs G_1 and G_2 such that $G = G_1 *_k G_2$. Now we will prove that G_i is uniformly dense with density k if it is not isomorphic to K_1, for $i = 1, 2$. Since $\tau(G) = k$, we have $\tau(G_i) \geq k$, and thus $d(G_i) \geq k$, for $i = 1, 2$. By (2), (3) and Theorem 2.2, it suffices to prove that $d(G_i) = k$ for $i = 1, 2$. If not, then either $d(G_1) > k$ or $d(G_2) > k$. By (1), $|E(G)| = |E(G_1)| + |E(G_2)| + k > k(|V(G_1)| - 1) + k(|V(G_2)| - 1) + k = k(|V(G)| - 1)$, and thus $d(G) = \frac{|E(G)|}{|V(G)|-1} > k$, contrary to the fact that $d(G) = k$. Hence $d(G_i) = k$, and $k \leq \tau(G_i) \leq \eta(G_i) \leq d(G_i) = k$. By Theorem 2.2, G_i is uniformly dense with density k for $i = 1, 2$. This proves the necessity.

To prove the sufficiency, first notice that G must have an edge-cut of size k, by the definition of the k-edge-join. In order to prove $G \in \mathcal{F}_k$, by Theorem 4.1, it suffices to show that G is uniformly dense with density k. Without loss of generality, we may assume that G_i is not isomorphic to K_1 for $i = 1, 2$. Then $\eta(G_i) = d(G_i) = k$ for $i = 1, 2$. By (3), $\tau(G_i) = \lfloor \eta(G_i) \rfloor = k$. Also we have $d(G_i) = \frac{|E(G_i)|}{|V(G_i)|-1} = k$ for $i = 1, 2$. Hence $E(G) = |E(G_1)| + |E(G_2)| + k = k(|V(G_1)| - 1) + k(|V(G_2)| - 1) + k = k(|V(G)| - 1)$, whence $d(G) = \frac{|E(G)|}{|V(G)|-1} = k$. Thus $k = \tau(G) \leq \eta(G) \leq d(G) = k$, i.e., $\eta(G) = d(G) = k$, and by Theorem 2.2, G is uniformly dense with density k. By Theorem 4.1, $G \in \mathcal{F}_k$. □

Theorem 4.2 has the following corollary, presenting a recursive structural characterization of graphs in $\mathcal{F}_k$.

Corollary 4.3 *Let $\mathcal{K}(k) = \{G : \kappa'(G) > \eta(G) = d(G) = k\}$. Then a graph $G \in \mathcal{F}_k$ if and only if $G = ((G_1 *_k G_2) *_k \ldots) *_k G_t$ for some integer $t \geq 2$ and $G_i \in \mathcal{K}(k) \cup \{K_1\}$ for $i = 1, 2, \ldots, t$.*

Now we can characterize all the graphs G with $\kappa'(G) = \tau(G) = k$.

Theorem 4.4 *A graph G with n vertices satisfies $\kappa'(G) = \tau(G) = k$ if and only if G has an edge-cut of size k and a spanning subgraph in $\mathcal{F}_{k,n}$.*

Proof First, suppose that G satisfies $\kappa'(G) = \tau(G) = k$. Then G must have an edge-cut C of size k since $\kappa'(G) = k$. Hence, $G = G_1 *_C G_2$ where $\tau(G_i) \geq k$ or $G_i = K_1$ for $i = 1, 2$. If $G_i = K_1$, then let $G'_i = K_1$. Otherwise, G_i must have k edge-disjoint spanning trees $T_1, T_2, \ldots, T_k$, and let G'_i be the graph with $V(G'_i) = V(G_i)$ and $E(G'_i) = \cup_{j=1}^{k} E(T_j)$. Let $G' = G'_1 *_C G'_2$. Then G' is a spanning subgraph of G with

$\kappa'(G') = k$ and $k = \tau(G') \leq \eta(G') \leq d(G') = k$. By Theorem 4.1, $G' \in \mathcal{F}_k$. Since $|V(G')| = n$, $G' \in \mathcal{F}_{k,n}$, completing the proof of necessity.

To prove the sufficiency, first notice that $\kappa'(G) \leq k$, since G has an edge-cut of size k. Graph G has a spanning subgraph $G' \in \mathcal{F}_{k,n}$, so $\tau(G') = k$, whence $\tau(G) \geq k$. Thus $k \leq \tau(G) \leq \kappa'(G) \leq k$, and we have $\kappa'(G) = \tau(G) = k$. □

5 Extensions and Restrictions with Respect to $\mathcal{F}_{k,n}$

Let G be a connected graph with n vertices and $H \in \mathcal{F}_{k,n}$. If G is a spanning subgraph of H, then H is an $\mathcal{F}_{k,n}$-*extension* of G. If H is a spanning subgraph of G, then H is an $\mathcal{F}_{k,n}$-*restriction* of G.

Theorem 5.1 *Let G be a connected graph with n vertices. Then each of the following holds.*

(i) *G has an $\mathcal{F}_{k,n}$-restriction if and only if $G = G_1 *_{k'} G_2$ for some $k' \geq k$ and graph G_i with $\eta(G_i) \geq k$ or $G_i = K_1$, for $i = 1, 2$.*
(ii) *G has an $\mathcal{F}_{k,n}$-extension if and only if $\kappa'(G) \leq k$ and $\gamma(G) \leq k$.*

Proof (i) Suppose that G has an $\mathcal{F}_{k,n}$-restriction H, by Theorem 4.2, $H = H_1 *_k H_2$ where $\tau(H_i) = \eta(H_i) = d(H_i) = k$ or $H_i = K_1$ for $i = 1, 2$. Since H is a spanning subgraph of G, we have $G = G_1 *_{k'} G_2$ for some $k' \geq k$ such that H_i is a spanning subgraph of G_i for $i = 1, 2$. If $H_i = K_1$, then $G_i = K_1$, otherwise, $\eta(G_i) \geq \tau(G_i) \geq \tau(H_i) = k$ for $i = 1, 2$, by (3).

To prove the sufficiency, it suffices to show that G has a spanning subgraph $H \in \mathcal{F}_{k,n}$. Since $G = G_1 *_{k'} G_2$, there exists an edge-cut X of size k' such that $G = G_1 *_X G_2$. Let Y be a subset of size k of X. For $i = 1, 2$, if $G_i = K_1$, then let $H_i = K_1$. Otherwise, $\eta(G_i) \geq k$, and by (3), $\tau(G_i) = \lfloor \eta(G_i) \rfloor \geq k$, and then G_i has k edge-disjoint spanning trees $T_{1,i}, T_{2,i}, \ldots, T_{k,i}$. Let H_i be the graph with $V(H_i) = V(G_i)$ and $E(H_i) = \cup_{j=1}^{k} E(T_{j,i})$, for $i = 1, 2$. Let $H = H_1 *_Y H_2$. Then H is a spanning subgraph of G and $\kappa'(H) = \tau(H) = k$. Since $d(H) = k$, by Lemma 2.3, H has the minimum number of edges with $\tau(H) = k$. Thus $H \in \mathcal{F}_{k,n}$.

(ii) If G has an $\mathcal{F}_{k,n}$-extension H, then G is a spanning subgraph of H and $\kappa'(H) = \tau(H) = k$ with minimum number of edges. Then $\kappa'(G) \leq k$. By Theorem 4.1, $d(H) = k$, i.e. $|E(H)| = k(|V(H)| - 1) = k(|V(G)| - 1)$. Thus $|E(H)| - |E(G)| = k(|V(G)| - 1) - |E(G)|$, and by Lemma 2.4, $\gamma(G) \leq k$.

To prove the sufficiency, it suffices to show that there is a graph $H \in \mathcal{F}_{k,n}$ with a spanning subgraph G. Let $\kappa'(G) = k'$, then $k' \leq k$, and G has an edge-cut X of size k'. Hence, $G = G_1 *_X G_2$. For $i = 1, 2$, if $G_i = K_1$, then let $H_i = K_1$. Otherwise, since $\gamma(G) \leq k$, by the definition of $\gamma(G)$, we have $\gamma(G_i) \leq k$. By Lemma 2.4, G_i can be reinforcing to a graph H_i which can be decomposed into k edge-disjoint spanning trees. Then $|E(H_i)| = k(|V(H_i)| - 1) = k(|V(G_i)| - 1)$, whence $d(H_i) = k$. Since $k = \tau(H_i) \leq \eta(H_i) \leq d(H_i) = k$, we have $\eta(H_i) = d(H_i) = k$, and by Theorem 2.2, H_i is uniformly dense, for $i = 1, 2$. Let $H = H_1 *_Y H_2$ where Y is an edge subset of size k with $X \subseteq Y$. Then G is a spanning subgraph of H. By Theorem 4.2, $H \in \mathcal{F}_{k,n}$, and this completes the proof of the theorem. □

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