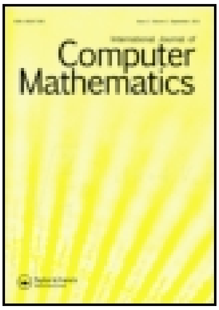

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Motivated by the Chinese Postman Problem, Boesch, Suffel, and Tindell [*The spanning subgraphs of Eulerian graphs*, J. Graph Theory 1 (1977), pp. 79–84] proposed the supereulerian graph problem which seeks the characterization of graphs with a spanning Eulerian subgraph. Pulleyblank [*A note on graphs spanned by Eulerian graphs*, J. Graph Theory 3 (1979), pp. 309–310] showed that the supereulerian problem, even within planar graphs, is NP-complete. In this paper, we settle an open problem raised by An and Xiong on characterization of supereulerian graphs with small matching numbers. A well-known theorem by Chvátal and Erdös [*A note on Hamilton circuits*, Discrete Math. 2 (1972), pp. 111–135] states that if G satisfies $\alpha(G) \leq \kappa(G)$, then G is hamiltonian. Flandrin and Li in 1989 showed that every 3-connected claw-free graph G with $\alpha(G) \leq 2\kappa(G)$ is hamiltonian. Our characterization is also applied to show that every 2-connected claw-free graph G with $\alpha(G) \leq 3$ is hamiltonian, with only one well-characterized exceptional class.

Keywords: supereulerian graphs; collapsible graphs; reductions; contraction characterizations

2010 AMS Subject Classifications: 05C45; 05C38; 05C70

1. Introduction

Graphs in this paper are finite and may have multiple edges or loops. Terms and notations not defined here are referred to [4]. In particular, for an integer $k > 0$, we use C^k to denote a cycle of length k. As in [4], a stable set of G is a vertex subset $S \subseteq V(G)$ such that no two vertices in S are joined by an edge in G; a matching of G is an edge subset $M \subseteq E(G)$ such that no two edges in M are adjacent in G. Furthermore, $\kappa(G)$, $\kappa'(G)$, $\alpha(G)$ and $\alpha'(G)$ represent the connectivity, the edge connectivity, the stability number and the matching number of a graph G, respectively. A graph is trivial if it contains no edges. The *circumference* of G, denoted by $c(G)$, is the length of a longest cycle of G. If $X \subseteq E(G)$ is an edge subset, then $V(X)$ denotes the set of vertices of G that are incident with an edge in X. For a vertex $v \in V(G)$, $E_G(v)$ denotes the set of edges incident with v in G, and $N_G(v)$ denotes the set of vertices adjacent to v in G. For a subset $W \subseteq V(G)$, define $N_G(W)$ to be the set of vertices in $V(G) - W$ that are adjacent to a vertex in W. For an

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edge subset X, $N_G(X) = N_G(V(X))$. For any integer $i \geq 1$, define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}.$$

Let $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. When $X = \{e\}$, we use G/e for $G/\{e\}$. If H is a subgraph of G, then we write G/H for $G/E(H)$.

For a graph G, $O(G)$ denotes the set of all odd degree vertices in G. A graph G is *Eulerian* if G is connected with $O(G) = \emptyset$, and G is *supereulerian* if G has a spanning Eulerian subgraph. In 1977, Boesch *et al.* [3] raised a problem to determine if a graph is supereulerian. They commented in [3] that such a problem would be difficult. In 1979, Pulleyblank [19] confirmed this remark by showing that the problem of determining if a graph is supereulerian, even within planar graphs, is NP-complete. There have been lots of researches on supereulerian graphs, as documented in Catlin's [8] excellent survey and its supplement [11].

Catlin [7] discovered collapsible graphs and the related reduction method. A graph G is collapsible if for any vertex subset W with $|W|$ even, G has a spanning connected subgraph Γ_W such that $O(\Gamma_W) = W$. A graph is *reduced* if it contains no nontrivial collapsible subgraphs. Catlin showed that every graph G has a unique collection of maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. and the contraction $G/(H_1 \cup H_2 \cup \cdots \cup H_c)$ is the *reduction* of G.

Characterizations of supereulerian graphs for certain classes of graphs have been widely investigated. See [5,9,16,18], among others. For graphs G with $\alpha'(G)$ small, the following have been proved.

THEOREM 1.1 *Let G be a graph with $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 2$. Each of the following holds.*

(i) (Lai and Yan [17]) *The graph G is supereulerian if and only if G is not contractible to a $K_{2,t}$ for some odd integer $t \geq 3$.*
(ii) (*An and Xiong* [1]) *Either G is collapsible, or G has a nontrivial collapsible subgraph H such that for some integer $t \geq 2$, $G/H \cong K_{2,t}$.*
(iii) (*An and Xiong* [1]) *If $\kappa'(G) \geq 3$ and $\alpha'(G) \leq 5$, then G is supereulerian if and only if G is not contractible to the Petersen graph.*

An and Xiong proposed a conjecture (Conjecture 12 in [1]), which can be restated as the following open problem. This research is motivated by their conjecture.

Problem 1.2 (*An and Xiong* [1]). If $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 3$, determine the collection of graphs such that G is supereulerian if and only if G is not contractible to a member in this collection.

In this paper, we have determined a graph family $\mathcal{F}'$ (see Definition 2.2 in Section 2) and prove the following main result.

THEOREM 1.3 *Let G be a graph with $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 3$. Then G is supereulerian if and only if the reduction of G is not a member in $\mathcal{F}'$.*

Theorem 1.3 has an application to hamiltonian line graphs and hamiltonian claw-free graphs. For a graph G, the *line graph* of G, denoted by $L(G)$ has vertex set $E(G)$, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. Let $K_{2,3}$ be the complete bipartite graph with vertex bipartition $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$. For integers $s_1, s_2, s_3 \geq 1$, the graph $K_{2,3}^{s_1,s_2,s_3}$ is obtained from $K_{2,3}$ by attaching s_i pendant vertices adjacent to y_i, $(1 \leq i \leq 3)$. Theorem 1.3 implies the following.

COROLLARY 1.4 *Let G be a connected simple graph. If $\kappa(L(G)) \geq 2$ and $\alpha(L(G)) \leq 3$, then $L(G)$ is hamiltonian if and only if G is not a member in $\{K_{2,3}^{s_1,s_2,s_3} : s_1 \geq s_2 \geq s_3 > 0\}$.*

A graph G is *claw-free* if it does not have an induced subgraph isomorphic to $K_{1,3}$. It has been known (Beinike [2], and Robertson, see [14, p. 74]) that every line graph is claw-free. A well-known theorem by Chvátal and Erdös [12] states that if G satisfies $\alpha(G) \le \kappa(G)$, then G is hamiltonian. Flandrin and Li showed that for 3-connected claw-free graphs, this assumption can be relaxed.

THEOREM 1.5 (Flandrin and Li [13]). *Every claw-free graph G with connectivity $\kappa(G) \ge 3$ and independence number $\alpha(G) \le 2\kappa(G)$ is hamiltonian.*

It has been a question whether Theorem 1.5 holds for 2-connected line graphs. A consequence of Theorem 1.3 answers this question.

COROLLARY 1.6 *Let G be a claw-free graph with $\kappa(G) \ge 2$ and $\alpha(G) \le 3$. Then G is hamiltonian if and only if the Ryjáček closure of G is not isomorphic to $L(H)$, for some $H \in \{K_{2,3}^{s_1,s_2,s_3} : s_1 \ge s_2 \ge s_3 > 0\}$.*

The concept of Ryjáček closure and the proofs for the corollaries will be given in the last section. In Section 2, we will present a brief introduction to Catlin's reduction method and some useful results needed for our arguments. The proof of our main theorem will be given in Section 3.

2. Preliminaries

The purpose of this section is to introduce collapsible graphs and to describe some families of reduced graphs that are useful in our proofs. Throughout the rest of this paper, $F(G)$ denotes the minimum number of additional edges that must be added to a graph G to result in a graph with two edge-disjoint spanning trees. The following theorem summarizes the useful results on collapsible graphs and reduced graphs needed in our arguments.

THEOREM 2.1 *Let G be a connected graph. Then each of the following holds.*

(i) (*Catlin, Theorem 3 of* [7]) *Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible*; *G is supereulerian if and only if G/H is supereulerian.*
(ii) (*Lemma* 2.3 *of* Catlin *et al.* [10]) *If $G \ne K_1$ is reduced, then $F(G) = 2|V(G)| - |E(G)| - 2$.*
(iii) (*Catlin et al., Theorem* 1.3 *of* [10]) *If $F(G) \le 2$, then G is collapsible if and only if the reduction of G is not isomorphic to a K_2 or to a $K_{2,t}$ for some integer $t \ge 1$.*
(iv) (*Catlin* [7]) *The reduction of G is reduced. In particular, the reduction of G is simple and contains no cycles of length* 3.
(v) (*Catlin, Lemma* 3 *of* [7]) *If G is collapsible, then any contraction of G is also collapsible.*

To answer the question in Problem 1.2, we first describe the graph families $\mathcal{F}$ and $\mathcal{F}'$, where $\mathcal{F}'$ is the excluded graph family stated in Theorem 1.3.

DEFINITION 2.2 (The families $\mathcal{F}$ and $\mathcal{F}'$). *Let $i, s_1, s_2, s_3, m, l, t$ be natural numbers with $t \ge 2$ and $i, m, l \ge 1$. Let C^i denote the cycle of length i. Let $M \cong K_{1,3}$ with centre a and ends a_1, a_2, a_3. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from M by adding s_i vertices with neighbours $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3 \pmod 3$. Define $C^6(s_1, s_2, s_3) = K_{1,3}(s_1, s_2, s_3) - a$. Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t. Let $K'_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to u' only. Hence u'' has degree 1 and u has degree t in $K'_{2,t}(u, u'')$. Let $K''_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding*

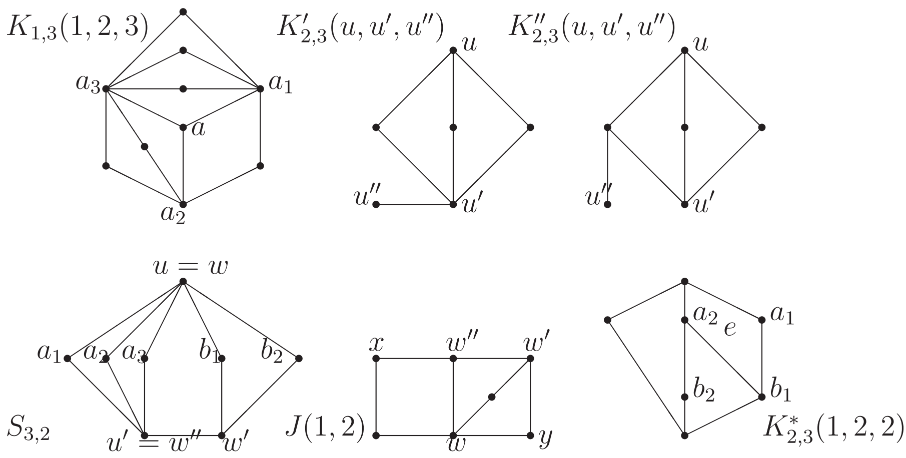

Figure 1. Some graphs in $\mathcal{F}$ with small parameters.

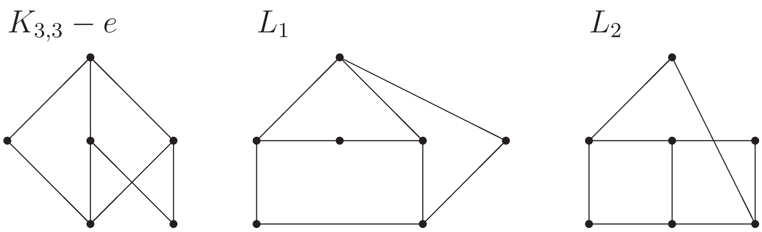

Figure 2. Some collapsible graphs: $K_{3,3} - e$, L_1 and L_2.

a new vertex u'' that joins to a vertex of degree 2 of $K_{2,t}$. Hence u'' has degree 1 and both u and u' have degree t in $K''_{2,t}(u, u'')$. We shall use $K'_{2,t}$ and $K''_{2,t}$ for a $K'_{2,t}(u, u', u'')$ and a $K''_{2,t}(u, u', u'')$, respectively. Let $S_{m,l}$ be the graph obtained from a $K_{2,m}(u, u')$ and a $K'_{2,l}(w, w', w'')$ by identifying u with w, and w'' with u'. Let $J(m, l)$ denote the graph obtained from a $K_{2,m+1}$ and a $K'_{2,l}(w, w', w'')$ by identifying w, w'' with the two ends of an edge in $K_{2,m+1}$, respectively; and $J'(m, l) = J(m, l) - ww''$. Let $K_{2,3}(1, 2, 2)$ be the union of three internally disjoint (u, w)-paths of lengths $2, 3$ and 3, respectively; and let $K^*_{2,3}(1, 2, 2)$ be obtained from $K_{2,3}(1, 2, 2)$ by adding a chord e to the 6-cycle joining two vertices of degree 2 so that no three-cycle is resulted. Let $C^7 = v_1v_2v_3v_4v_5v_6v_7v_1$ denote a cycle of length 7. Define $J^7_1 = C^7 + v_1v_4$ and $J^7_2 = J^7_1 + v_2v_5 = C^7 + \{v_1v_4, v_2v_5\}$. See Figure 1 for examples of these graphs. Let

$$\mathcal{F} = \{K_1\} \bigcup \Big(\{C^7, J^7_1, J^7_2, K_{2,3}(1, 2, 2), K^*_{2,3}(1, 2, 2)\} \bigcup \{K_{2,t} | t \geq 1\} \bigcup \{K_{1,3}(s, s', s''), C^6(s, s', s'') | s, s', s'' \geq 0\} \bigcup \{S_{m,l} : m, l \geq 1\} \Big) \bigcap \{G | \kappa(G) \geq 2\},$$

and define

$$\mathcal{F}' = \{G \in \mathcal{F} : \; G \text{ is non supereulerian.}\}$$

Define the following graphs as depicted in Figure 2, and define $L_3 = K^*_{2,3}(1, 2, 2) + \{a_1b_2\}$.

The graph $K_{3,3} - e$ is proved to be collapsible in Lemma 1 of [6]. Using the same argument in the proof of Lemma 1 of [6], it is routine to verify that L_1, L_2 and L_3 are also collapsible. We put this observation in the following lemma.

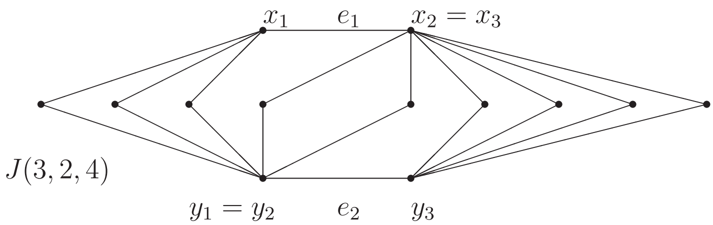

Figure 3. The graph $J(3,2,4)$.

LEMMA 2.3 *The graphs $K_{3,3} - e$, L_1, L_2 and L_3 are collapsible.*

An edge cut Y of a graph G is essential if $G - Y$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut Y with $|Y| < k$.

DEFINITION 2.4 (Families $\mathcal{F}_1$ and $\mathcal{F}_2$). *Define*

$$\mathcal{F}_1 = \{C^7, K_{2,3}(1,2,2)\} \cup \{C^6(s,s',s''), K_{1,3}(s,s',s'')|s \geq s' > 0, s'' \geq 0\} \cup \{S_{m,l}|m \geq l \geq 1\}.$$

For integers $s_1, s_2, s_3 \geq 2$, Let $K_{2,s_1}(x_1,y_1)$, $K_{2,s_2}(x_2,y_2)$, $K_{2,s_3}(x_3,y_3)$ be three disjoint graphs such that for $i \in \{1,2,3\}$, $K_{2,s_i}(x_i,y_i)$ is isomorphic to K_{2,s_i} with x_i and y_i being the two nonadjacent vertices of degree s_i. The graph $J(s_1,s_2,s_3)$ is obtained by identifying y_1 with y_2 and x_2 with x_3, and by adding new edges $e_1 = x_1x_3$ and $e_2 = y_1y_3$ (see Figure 3 for an example). Note that $J(m,0,l) = J'(m,l)$.

Define

$$\mathcal{F}_2 = \{K_1\} \cup (\mathcal{F} \cap \{\Gamma : \Gamma \text{ is essentially 4-edge-connected}\}) \cup \{J(s_1,s_2,s_3)|s_1 \geq s_3 \geq 3, s_2 \geq 2\}.$$

Lemma 2.5 below is a key lemma in the proof of Theorem 1.3. It indicates that once a certain type of subgraph appears in G, then G must be in $\mathcal{F}$. The family $\mathcal{F}_2$ will be needed in Theorem 3.2 of the next section.

LEMMA 2.5 *Let G be a reduced graph with $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 3$. If G has a subgraph $H \in \mathcal{F}_1 - \{K_1\}$, then $G \in \mathcal{F}$.*

Proof We first observe that if $H \in \mathcal{F}_1 - \{K_1, S_{1,1}\}$, then $\alpha'(H) \geq 3$. By contradiction, we assume that G is a counterexample to the lemma such that $|V(G)|$ is minimized.

CLAIM 1 $\kappa(G) \geq 2$.

By contradiction, assume that G has a cut vertex z. Then G has nontrivial connected subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{z\}$. Since $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 3$, both $\kappa'(G_1) \geq 2$ and $\kappa'(G_2) \geq 2$, and both $\alpha'(G_1) \leq 3$ and $\alpha'(G_2) \leq 3$. Since every graph in $\mathcal{F}_1$ is 2-connected, we may assume that H is a subgraph of G_1. If $H = S_{1,1}$, which is a five-cycle, then H has a matching M_1 of size 2 such that M_1 is not incident with the vertex z. Since G_2 is a 2-edge-connected nontrivial reduced graph, G_2 has a cycle of length at least 4, and so G_2 has a matching M_2 of size at least 2. As M_1 is not incident with z, it follows that $M_1 \cup M_2$ is a matching of size at least 4, contrary to $\alpha'(G) \leq 3$. This contradiction proves Claim 1. ■

Since G is a counterexample, G has a subgraph $H \in \mathcal{F}_1 - \{K_1\}$, but $G \notin \mathcal{F}$. We assume that H is maximal, in the sense that H is not properly contained in another subgraph of G in $\mathcal{F}_1$. We have the following observations.

Observation 2.6 Let H be a subgraph of G.

(i) If $H \in \{C^7, K_{2,3}(1,2,2)\} \cup \{K_{1,3}(s,s',s'')|s \geq s' \geq s'' > 0\}$, and if G has an edge with exactly one end in H, then $\alpha'(G) \geq 4$.
(ii) If $H = K_{1,3}(s,s',s'')$ with $s \geq s' \geq s'' > 0$, then adding any additional edge to join two distinct vertices in H will result in a collapsible graph. Since G is reduced, we conclude that in this case $G = K_{1,3}(s,s',s'')$.
(iii) If G is spanned by $H = K_{2,3}(1,2,2)$, then by Lemma 2.3 and by the assumption that G is reduced, G cannot have L_3 or a three-cycle as a subgraph. By inspection, $G \in \{K_{2,3}(1,2,2), K^*_{2,3}(1,2,2)\}$.
(iv) If G is spanned by C^7, then as G is reduced, C^7 can have at most two chords in G. (This is because, if C^7 with three chords, $F(G[V(C^7)]) \leq 2(7) - 10 - 2 = 2$, and so by Theorem 2.1(iii), $G[V(C^7)]$ is not reduced.) As G has no cycles of length at most 3, $G \in \{C^7, J_1^7, J_2^7\}$.

Only Observation 2.6(i) when $H = K_{1,3}(s,s',s'')$ needs an explanation. We use the notations in Figure 1. Let xy denote an edge incident with a vertex $x \in V(H)$ and $y \notin V(H)$. If x has degree 2 in H or if $x = a$, then $G[E(H) \cup \{xy\}]$ has four independent edges. Therefore, we assume that any edges in G incident with exactly one vertex in H must be incident with one in $\{a_1, a_2, a_3\}$. Since $\kappa(G) \geq 2$, and since $y \in V(G) - V(H)$, we may assume that G has a path P with $y \in V(P)$ such that $V(P) \cap V(H) = \{a_i, a_j\}$ for some $i \neq j$ and $1 \leq i,j \leq 3$. Since H is maximal, $|E(P)| \geq 3$, and so $G[E(H) \cup E(P)]$ has four independent edges. This verifies the observation.

Recall that $c(G)$ is the length of a longest cycle in G. By Observation 2.6, and since any cycle of length at least eight has four independent edges, we may assume that

$$G \text{ has no subgraph in } \{C^7, K_{2,3}(1,2,2)\} \cup \{K_{1,3}(s,s',s'') : s \geq s' \geq s'' > 0\} \text{ and } c(G) \leq 6. \quad (1)$$

By Equation (1), we only need to examine the cases when $H \in \{C^6(s,s',s'')|s \geq s' \geq s'' \geq 0\} \cup \{K_{1,3}(s,s',0)|s \geq s' > 0\} \cup \{S_{m,l}, |m \geq l \geq 1\}$. We make another observation.

Observation 2.7 Let e' be an edge in $E(G) - E(H)$ joining two distinct vertices in H. Let $H' = G[E(H) \cup \{e'\}]$ be the edge induced subgraph of G. Each of the following holds.

(i) If $H \in \{K_{1,3}(s,s',0) : s \geq s' > 0\} \cup \{S_{m,l}, |m \geq l \geq 1\}$, then H' has a K_3 or a $K_{3,3} - e$, and so G is not reduced.
(ii) If $H = C^6(s,s',s'')$ for some $s \geq s' \geq s'' \geq 0$ either with $s'' > 0$, or with $s'' = 0$ and $s' \geq 2$, then either H' has a K_3, or H' is a $K_{1,3}(t,t',t'')$ with $t \geq t' > 0$ and $t'' \geq 0$, or is an $S_{m,l}$, with $m \geq l \geq 2$, contrary to the maximality of H.

By Equation (1) and by Observation 2.7, we proceed the proof of the lemma by examining the following cases.

Case 1 $H = K_{1,3}(s,s',0)$ with $s \geq s' > 0$.

Since $G \neq H$ and by Observation 2.7(i), $V(G) - V(H)$ has a vertex z. By $\kappa(G) \geq 2$ and by $\alpha'(G) \leq 3 = \alpha'(H)$, there exist distinct vertices $u, v \in N_G(z) \cap V(H)$. Since $u, v \in N_G(z)$, the edge induced subgraph $H' = G[E(H) \cup \{uz, vz\}]$ of G either has one of $\{K_3, K_{3,3} - e, L_2\}$ as a subgraph, contrary to the assumption that G is reduced; or is a $K_{1,3}(t,t',t'')$ with $t \geq t' > 0$ and

$t'' \geq 0$ properly containing H, contrary to the maximality of H; or $\alpha'(G) \geq \alpha'(H') \geq 4$. These contradictions complete the proof for Case 1.

Case 2 $H = S_{m,l}$ for some $m \geq l \geq 1$ and with $m + l \geq 3$ maximized.

Since $G \neq H$ and by Observation 2.7(i), $V(G) - V(H)$ has a vertex z. By $\kappa(G) \geq 2$ and by $\alpha'(G) \leq 3 = \alpha'(H)$, there exist distinct vertices $u, v \in N_G(z) \cap V(H)$. Since $u, v \in N_G(z)$, the edge induced subgraph $H' = G[E(H) \cup \{uz, vz\}]$ of G either has one of $\{C^7, K_3, L_1, L_2\}$ as a subgraph, contrary to Equation (1) or the assumption that G is reduced; or is an $S_{m',l'}$ with $m \geq l' > 0$ properly containing H, contrary to the maximality of H. These contradictions complete the proof for Case 2.

Case 3 $H = C^6(s, s', s'')$ is a subgraph of G for some $s \geq s' \geq s'' \geq 0$ with either $s'' > 0$ or $s'' = 0$ and $s' \geq 2$.

Since $G \neq H$ and by Observation 2.7(i), $V(G) - V(H)$ has a vertex z. By $\kappa(G) \geq 2$ and by $\alpha'(G) \leq 3 = \alpha'(H)$, there exist distinct vertices $u, v \in N_G(z) \cap V(H)$. Since $u, v \in N_G(z)$, the edge induced subgraph $H' = G[E(H) \cup \{uz, vz\}]$ of G either has one of $\{C^7, K_{2,3}(1,2,2), K_3\}$ as a subgraph, contrary to Equation (1) or the assumption that G is reduced; or is a $C^6(t, t', t'')$ with $t \geq t' \geq t'' \geq 0$ and with either $t'' > 0$ or both $t'' = 0$ and $t' \geq 2$, properly containing H, contrary to the maximality of H; or $\alpha'(G) \geq \alpha(H') \geq 4$. These contradictions complete the proof for Case 3.

Case 4 $H = C^6(s, 1, 0)$ is a subgraph of G with either $s > 0$, and G does not have a subgraph in Cases 1, 2 or 3.

Let $P = v_1v_2v_3v_4v_5$ be a path of length 4 in H such that $d_H(v_1) = 1$, and $N_H(v_3) \cap N_H(5)$ has s vertices of degree 2 in H. Since $\kappa(G) \geq 2$, $N_G(v_1) - \{v_2\}$ has a vertex z. Since G does not have a subgraph in Cases 1, 2 or 3, and by Equation (1), $z \notin V(H)$. Since $\kappa(G) \geq 2$, the edges v_1z and v_3v_4 are in a cycle of G, and so $G - v_1$ has a path Q from z to a vertex $w \in V(H) - \{v_1, v_2\}$ such that $V(H) \cap V(Q) = \{w\}$. If $|E(Q)| \geq 3$, then G has a cycle of length at least 6, contrary to Equation (1) or to the assumption that G does not have subgraph in Cases 1, 2 or 3. Hence $|E(Q)| = 2$ and so we must have $s = 1$, $d_H(v_5) = 1$, $H = P$ and $w = v_4$. By Symmetry, there must be a vertex $z' \in V(G) - V(H)$ such that $z'v_5, z'v_2 \in E(G)$. Thus $G[V(P) \cup \{z, z'\}]$ contains a $K_{2,3}(1, 2, 2)$, contrary to Equation (1). These contradictions prove Case 4, and the proof for the lemma is done.

3. Proof of Theorem 1.3

We in this section will prove the following theorem, which, together with Theorem 2.1(i), implies Theorem 1.3.

THEOREM 3.1 *Let G be a graph with* $\kappa'(G) \geq 2$ *and* $\alpha'(G) \leq 3$. *Then the reduction of G is in* $\mathcal{F}$.

The proof of Theorem 3.1 needs a useful tool stated as Theorem 3.2 below. We shall need the graphs introduced in Definitions 2.2 and 2.4. By Definition 2.4, $\mathcal{F}_2 = \{K_1\} \cup \{K_{2,t} : t \geq 3\} \cup K_{1,3}(s, s', s'')|s \geq s' \geq 2, s'' \geq 0\} \cup \{S_{m,l}|m \geq l \geq 2\} \cup \{C^6(s, s', s'')|$ either $s \geq s' \geq 2$ and $s'' \geq 1$ or $s \geq s' \geq 3$ and $s'' = 0\} \cup \{K^*_{2,3}(1, 2, 2)\} \cup \{J(s_1, s_2, s_3)|\ s_1 \geq s_3 \geq 3, s_2 \geq 2\}$.

THEOREM 3.2 *Let G be a 2-edge-connected graph. Each of the following holds.*

(i) *Suppose that* $c(G) \leq 5$. *Then G is collapsible if and only if the reduction of G is not a member in* $\{K_{2,t}, S_{m,l}\}$, *where* $l, m \geq 1$ *and* $t \geq 2$ *are integers.*

(ii) *Suppose that G is essentially 4-edge-connected graph with $c(G) \le 6$. Then G is collapsible if and only if the reduction of G is not in $\mathcal{F}_2$.*

Proof Since any graph in $\{K_{2,t}, S_{m,l}\}$ is not collapsible, by Theorem 2.1(v), if the reduction of G is in $\{K_{2,t}, S_{m,l}\}$, then G is not collapsible.

To prove the necessity, we argue by contradiction to assume that $G \neq K_1$ is reduced, but $G \notin \{K_{2,t}, S_{m,l}\}$. By Theorem 2.1(iv), G has no cycles of length at most 3. Suppose first that $c(G) = 4$. Then G contains a $K_{2,2}$ as a subgraph. Let $t \ge 2$ be the maximum number such that G has $K_{2,t}$ as a subgraph. Since $G \neq K_{2,t}$, $V(G) - V(K_{2,t})$ has a vertex v. Since $\kappa(G) \ge 2$, G has a path P from a vertex x to a vertex y with $v \in V(P)$ such that $V(P) \cap V(K_{2,t}) = \{x, y\}$. As $v \in V(P) - V(K_{2,t})$, $|E(P)| \ge 2$. If x and y are adjacent in $K_{2,t}$, then G has a five-cycle, contrary to $c(G) = 4$. Therefore, we must have $x, y \in N_G(v)$, and x and y are of distance 2 in $K_{2,t}$. It then follows that either $c(G) \ge 5$, contrary to $c(G) = 4$; or G has a $K_{2,t+1}$, contrary to the maximality of t.

Hence $c(G) = 5$, and so G contains a $C^5 = S_{1,1}$ as a subgraph. Thus G has $S_{m,l}$ as a subgraph with $m \ge l \ge 1$ such that $m + l$ is maximized. Since $G \neq S_{m,l}$, $V(G) - V(S_{m,l})$ has a vertex v'. By $\kappa(G) \ge 2$, G has a path P' from a vertex x' to a vertex y' with $v' \in V(P')$ such that $V(P') \cap V(S_{m,l}) = \{x', y'\}$.

If $x'y' \in E(S_{m,l})$, then G has a cycle of length at least 6, contrary to $c(G) = 5$. Therefore, the distance between x' and y' in $S_{m,l}$ is 2. It follows that $G[V(S_{m,l}) \cup \{v'\}]$ either has a cycle of length at least 6, contrary to $c(G) = 5$; or is isomorphic to an $S_{m+1,l}$ or an $S_{m,l+1}$, contrary to the maximality of $m + l$. This completes the proof for Theorem 3.2(i).

To prove Theorem 3.2(ii), we argue by contradiction to assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{2}$$

By Equation (2), by the assumption that G is essentially 4-edge-connected, and by Theorem 3.2(i), we further assume that

$$G \text{ is reduced, } \kappa(G) \ge 2,\ D_2(G) \text{ is an independent set, and } c(G) = 6. \tag{3}$$

Let $C^6 = v_1v_2v_3v_4v_5v_6v_1$ be a longest cycle of G. Since G is reduced and by Lemma 2.3, G contains no K_3 or $K_{3,3} - e$ as a subgraph. Thus if C^6 has chords, then C^6 has exactly one chord, isomorphic to a $J(1,1) = K_{1,3}(1,1,0)$. Note that $C^6 = J'(1,1)$. Hence G has a subgraph $H \in \{J(m,l) | m \ge l \ge 1\} \cup \{J'(m,l) | m \ge l \ge 1\} \cup \{C^6(s,s',s'') \,|s \ge s' \ge s'' > 0\} \cup \{K_{1,3}(s,s',s'') | s \ge s' > 0, s'' \ge 0\}$. Choose such an H so that

$$|V(H)| + |E(H)| \text{ is maximized.} \tag{4}$$

If $G = H$, then as G is essentially 4-edge-connected, $G \neq J'(m,l)$ with $m \ge l \ge 1$. Since $\alpha'(G) \le 3$, $G \neq J(m,l)$ for $m \ge l \ge 2$. As $J(m,1) = K_{1,3}(m,1,0)$, we conclude that if $G = H$, then $G \in \mathcal{F}_2$, contrary to Equation (2).

Hence $G \neq H$. By Equation (4), $V(G) - V(H)$ has a vertex z. As $\kappa(G) \ge 2$, G has a path Q with $z \in V(Q)$, and $V(Q) \cap V(H) = \{u, v\}$ for some distinct u and v. Since $\alpha'(H) = 3 = \alpha'(G)$ and since G is reduced, $u, v \in N_G(z)$ and u and v are not adjacent in H. In the arguments below, we will use the notation in Figure 1.

If $H \in \{K_{1,3}(s,s',s'') | s \ge s' > 0, s'' \ge 0\} \cup \{C^6(s,s',s'') | s \ge s' \ge s'' > 0\}$, then either $u, v \in \{a_1, a_2, a_3\}$, whence Equation (4) is violated; or (by symmetry) $H = K_{1,3}(s,1,0)$, $u = a$ and $v \in D_2(H)$, whence Equation (4) is violated; or $\{u, v\} - \{a_1, a_2, a_3\} \neq \emptyset$, and $G[V(H) \cup \{z\}]$ contains a cycle of length at least 7, contrary to Equation (3).

Assume that $H \in \{J(m,l) | m \ge l \ge 1\}$. Since $J(m,1) = K_{1,3}(m,1,0)$, we assume that $m \ge l \ge 2$. If $u, v \in D_2(H)$, then $G[V(H) \cup \{z\}]$ contains a cycle of length at least 7, contrary to Equation (3). Hence we assume that $u \notin D_2(H)$. Then $G[V(H) \cup \{z\}]$ either violates Equation (4), or contains a cycle of length at least 7, contrary to Equation (3).

Finally we assume that $H \in \{J'(m,l) | m \geq l \geq 1\}$. As $J'(m,1) = C^6(m,1,1)$, we may assume $m \geq l \geq 2$. Since $J'(m,l) = J(m,0,l)$, we may assume that $H = J(s,s',s'')$ with $s \geq s'' \geq 2$ and $s' \geq 0$, and $s + s' + s''$ maximized. If $\{u,v\} \cap D_2(H) \neq \emptyset$, then $G[V(H) \cup \{z\}]$ also contains a cycle of length at least 7, contrary to Equation (3). Hence $u, v \in V(H) - D_2(H)$. It follows that $G[V(H) \cup \{z\}]$ contains a $J(t,t',t'')$ with $t + t' + t'' = s + s' + s'' + 1$, contrary to the maximality of H. This completes the proof of Theorem 3.2(ii). ∎

Proof of Theorem 3.1 By contradiction, we assume that

$$G \text{ is a counterexample to Theorem 3.1 with } |V(G)| \text{ minimized.} \quad (5)$$

By Theorem 1.1 and by Equation (5), G is reduced. By the assumption $\alpha'(G) \leq 3$, $c(G) \leq 7$. If G has a C^7 or a C^6 as a subgraph, then by Lemma 2.5, $G \in \mathcal{F}$, contrary to (5). Therefore, we must have $c(G) \leq 5$. By Theorem 3.2(i), $G \in \mathcal{F}$, contrary to Equation (5). This completes the proof. ∎

4. Proofs of Corollaries 1.4 and 1.6

To prove Corollary 1.4, we also need the following theorem of Harary and Nash-Williams, which reveals a close relationship between Eulerian subgraphs in G and Hamilton cycles in $L(G)$.

THEOREM 4.1 (Harary and Nash-Williams [15]) *Let G be a connected graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has an Eulerian subgraph H such that $E(G - V(H)) = \emptyset$.*

Let G be a graph such that $\kappa(L(G)) \geq 2$, $E_1(G)$ denote the set of pendant edges (edges incident with a vertex in $D_1(G)$) of G, and let $\Gamma = G/E_1(G)$. Let Γ' denote the reduction of Γ, and define $\Lambda(\Gamma') = \{v \in V(\Gamma')$ such that v is the contraction image of a nontrivial connected subgraph of $G\}$. Using Theorem 4.1, Shao proved the following.

PROPOSITION 4.2 (Shao, Section 1.4 of [21]). *If Γ' has an Eulerian subgraph H with $\Lambda(\Gamma') \subseteq V(H)$, then $L(G)$ is hamiltonian.*

Proof of Corollary 1.4 Let G be a graph with $\kappa(L(G)) \geq 2$ and $\alpha(L(G)) \leq 3$. Since $\kappa(L(G)) \geq 2$ and $\alpha(L(G)) \leq 3$, $\kappa'(\Gamma) \geq 2$ and $\alpha'(\Gamma) \leq 3$. Let Γ' be the reduction of Γ. If Γ' is supereulerian, then by Proposition 4.2, $L(G)$ is hamiltonian. Thus by Theorem 1.3, we may assume that $\Gamma' \in \mathcal{F}'$. By the definition of $\mathcal{F}'$, we observe that

$$\forall F \in \mathcal{F}', \text{ and } \forall v \in D_2(F), F \text{ has an Eulerian subgraph } H \text{ such that } V(F) - v \subseteq V(H). \quad (6)$$

By Equation (6) and Proposition 4.2, if $\Gamma' \in \mathcal{F}'$ such that $D_2(\Gamma') - \Lambda(\Gamma') \neq \emptyset$, then $L(G)$ is hamiltonian. Thus $L(G)$ is not hamiltonian only if $D_2(\Gamma') \subseteq \Lambda(\Gamma')$. Therefore, each vertex in $D_2(\Gamma')$ contains an edge of G, and these edges are independent. Hence $|D_2(\Gamma')| \leq \alpha'(\Gamma) \leq 3$, and so as $\Gamma' \in \mathcal{F}'$, we conclude that $L(G)$ is not hamiltonian only if $\Gamma' = K_{2,3}$ with $D_2(\Gamma') \subseteq \Lambda(\Gamma')$. Suppose that one vertex v in $D_2(\Gamma')$ is the contraction image of a nontrivial collapsible graph H. Let $A_G(H)$ denote the vertices of H that are adjacent to vertices in $V(G) - V(H)$ in G. Thus $|A_G(H)| \leq d_{\Gamma'}(v) \leq 3$. Since H is a simple collapsible graph, $|E(H)| \geq 3$, and so there must be an edge $e_1 \in E(H)$ and an edge $e_2 \in E_{\Gamma'}(v)$ such that $\{e_1, e_2\}$ is a matching in G. Let e_3, e_4 be two edges in the preimages of the two vertices of $D_2(\Gamma') - \{v\}$. Then $\{e_1, e_2, e_3, e_4\}$ would be a matching of G, contrary to $\alpha'(G) \leq 3$. With a similar argument, the two vertices of degree 3 in Γ' must be trivial, and so $G \cong K_{2,3}^{s_1,s_2,s_3}$ for some $s_1, s_2, s_3 > 0$. This proves Corollary 1.4. ■

A vertex $v \in V(G)$ is *locally connected* if $G[N_G(v)]$ is connected. Following the definition given by Ryjáček [20], a graph H is the *closure* of a claw-free graph G, denoted by $H = \mathrm{cl}(G)$, if both of the following hold.

(A) There is a sequence of graphs $G_1, \ldots, G_t$ such that $G_1 = G, G_t = H, V(G_{i+1}) = V(G_i)$ and $E(G_{i+1}) = E(G_i) \bigcup \{uv | u, v \in N_{G_i}(x_i), uv \notin E(G_i)\}$ for some $x_i \in V(G_i)$ with connected non-complete $G_i[N_{G_i}(x_i)]$, for $i = 1, \ldots, t-1$, and
(B) No vertex of H has a connected non-complete neighbourhood.

THEOREM 4.3 (Ryjáček [20]). *Let G be a claw-free graph. Then*

(i) *$cl(G)$ is uniquely determined.*
(ii) *$cl(G)$ is the line graph of a triangle-free graph.*
(iii) *G is hamiltonian if and only if $cl(G)$ is hamiltonian.*

Proof of Corollary 1.6 By Theorem 4.3, we may assume that for some simple graph H, $\mathrm{cl}(G) = L(H)$. As adding edge to a graph does not increase the independence number α and does not decrease the connectivity κ, both $\kappa(\mathrm{cl}(G)) \geq \kappa(G) \geq 2$ and $\alpha(\mathrm{cl}(G)) \leq \alpha(G) \leq 3$ hold. By Corollary 1.4, $\mathrm{cl}(G) = L(H)$ is hamiltonian if and only if $H \notin \{K_{2,3}^{s_1,s_2,s_3} : s_1 \geq s_2 \geq s_3 > 0\}$. ■

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References

[1] M. An and L. Xiong, Supereulerian graphs, collapsible graphs and matchings, *Acta Mathematicae Applicatae Sinia* (English Series), to appear.
[2] L.W. Beinike, *Derived graphs and digraphs*, H. Sachs, H. Voss, and H. Walther, eds., Beitrage zur Graphentheorie, Teubner, Leipzig, 1968, pp. 17–33.
[3] F.T. Boesch, C. Suffel, and R. Tindell, *The spanning subgraphs of Eulerian graphs*, J. Graph Theory 1 (1977), pp. 79–84.
[4] J.A. Bondy and U.S.R. Murty, *Graph Theory*, Springer, New York, 2008.
[5] H.J. Broersma and L.M. Xiong, *A note on minimum degree conditions for supereulerian graphs*, Discret. Appl. Math. 120 (2002) 35–43.
[6] P.A. Catlin, *Super-Eulerian graphs, collapsible graphs, and four-cycles*, Congr. Numer. 58 (1987), pp. 233–246.
[7] P.A. Catlin, *A reduction method to find spanning Eulerian subgraphs*, J. Graph Theory 12 (1988), pp. 29–45.
[8] P.A. Catlin, *Super-Eulerian graphs, a survey*, J. Graph Theory 16 (1992), pp. 177–196.
[9] P.A. Catlin and X.W. Li, *Supereulerian graphs of minimum degree at least 4*, J. Adv. Math. 28 (1999), pp. 65–69.
[10] P.A. Catlin, Z. Han, and H.-J. Lai, *Graphs without spanning Eulerian subgraphs*, Discrete Math. 160 (1996), pp. 81–91.
[11] Z.H. Chen and H.-J. Lai, *Reduction techniques for super-Eulerian graphs and related topics (a survey)*, Combinatorics and Graph Theory 95, Vol. 1 (Hefei), World Sci. Publishing, River Edge, NJ, 1995, pp. 53–69.
[12] V. Chvátal and P. Erdös, *A note on Hamilton circuits*, Discrete Math. 2 (1972), pp. 111–135.
[13] E. Flandrin and H. Li, *Further result on neighbourhood intersections*, Repport de Recheche, L. R. I. No. 416, Univ. Paris-Sud, Orsay, 1989.
[14] F. Harary, *Graph Theory*, Addison-Wesley Publishing, Reading, MA, 1969.
[15] F. Harary and C.St.J.A. Nash-Williams, *On Eulerian and Hamiltonian graphs and line graphs*, Canad. Math. Bull. 8 (1965), pp. 701–709.

[16] H.-J. Lai and Y. T. Liang, *Supereulerian graphs in the graph family* $C_2(6,k)$, Discret. Appl. Math. 159 (2011), pp. 467–477.

[17] H.-J. Lai and H. Y. Yan, *Supereulerian graphs and matchings*, Appl. Math. Letters 24 (2011), pp. 1867–1869.

[18] X.M. Li, D.X. Li, and H.-J. Lai, *The Supereulerian graphs in the graph family* $C(l,k)$, Discrete Math. 309 (2009), pp. 2937–2942.

[19] W.R. Pulleyblank, *A note on graphs spanned by Eulerian graphs*, J. Graph Theory 3 (1979), pp. 309–310.

[20] Z. Ryjáček, *On a closure concept in claw-free graphs*, J. Combin. Theory Ser. B 70 (1997), pp. 217–224.

[21] Y. Shao, *Claw-free graphs and line graphs*, Ph.D. diss., West Virginia University, 2005.