Linear Algebra and its Applications 458 (2014) 128–133

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Note on edge-disjoint spanning trees and eigenvalues ☆

Qinghai Liu [a,*], Yanmei Hong [b], Xiaofeng Gu [c], Hong-Jian Lai [d]

[a] *Center for Discrete Mathematics, Fuzhou University, Fuzhou, Fujian, 350002, China*
[b] *College of Mathematics and Computer Science, Fuzhou University, Fuzhou, 350108, China*
[c] *Department of Mathematics and Computer Science, University of Wisconsin–Superior, Superior, Wisconsin 54880, USA*
[d] *Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA*

ARTICLE INFO

Article history:
Received 8 December 2013
Accepted 31 May 2014
Available online xxxx
Submitted by R. Brualdi

MSC:
05C50
15A18
15A42

Keywords:
Edge disjoint spanning trees
Quotient matrix
Eigenvalue
Algebraic connectivity

ABSTRACT

Let $\tau(G)$, $\lambda_2(G)$, $\mu_{n-1}(G)$ and $\rho_2(G)$ be the maximum number of edge-disjoint spanning trees, the second largest adjacency eigenvalue, the algebraic connectivity, and the second largest signless Laplace eigenvalue of G, respectively. In this note, we prove that for any graph G with minimum degree $\delta \geq 2k$, if $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$ or $\mu_{n-1}(G) > \frac{2k-1}{\delta+1}$ or $\rho_2(G) < 2\delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$, which confirms a conjecture of Liu, Hong and Lai, and also implies a conjecture of Cioabă and Wong.

☆ This research is supported in part by NSFC (Nos. 11301086, 11326214) and also in part by Fuzhou University (Nos. 600880, 600917).

* Corresponding author.
E-mail address: qliu@fzu.edu.cn (Q. Liu).

http://dx.doi.org/10.1016/j.laa.2014.05.044
0024-3795/© 2014 Elsevier Inc. All rights reserved.

1. Introduction

In this note, we only consider finite and simple graphs. Undefined notations will follow Bondy and Murty [1]. Let G be a graph. We use $\tau(G)$ to represent the maximum number of edge-disjoint spanning trees of G. See Palmer's survey [12] for a literature review on $\tau(G)$.

Let G be a simple graph of vertex set $\{v_1, \dots, v_n\}$. The adjacency matrix of G is the $n \times n$ matrix $A(G) := (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent and otherwise $a_{ij} = 0$. As G is simple and undirected, $A(G)$ is a symmetric $(0,1)$-matrix. The eigenvalues of G are the eigenvalues of $A(G)$. We use $\lambda_i(G)$ to denote the i-th largest eigenvalue of G. Let $D(G)$ be the degree diagonal matrix of G. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the Laplacian matrix and the signless Laplacian matrix of G, respectively. We use $\mu_i(G)$ and $\rho_i(G)$ to denote the i-th largest eigenvalue of $L(G)$ and $Q(G)$, respectively. It is not difficult to see that $\mu_n(G) = 0$. Also, the second smallest eigenvalue of $L(G)$, $\mu_{n-1}(G)$, is known as the algebraic connectivity of G.

Motivated by Kirchhoff's matrix tree theorem [8] and by a problem of Seymour (see Ref. [19] of [3]), Cioabă and Wong [3] considered the following problem.

Problem 1.1. (See [3].) Let G be a connected graph. Determine the relationship between $\tau(G)$ and eigenvalues of G.

Cioabă and Wong proposed the following conjecture.

Conjecture 1.2. *(See Cioabă and Wong [3].) Let k and d be two integers with $d \geq 2k \geq 4$. If G is a d-regular graph with $\lambda_2(G) < d - \frac{2k-1}{d+1}$, then $\tau(G) \geq k$.*

A fundamental theorem of Nash-Williams and Tutte characterizes graphs with at least k edge-disjoint spanning trees. Let $(V_1, \dots, V_t)$ be a sequence of disjoint vertex subsets of $V(G)$ and $e(V_1, \dots, V_t)$ means the number of edges whose ends lie in different V_i's. For a vertex set U, we write $d(U) = e(U, V(G) \setminus U)$.

Theorem 1.3. *(See Nash-Williams [11] and Tutte [13].) Let G be a connected graph and let $k > 0$ be an integer. Then $\tau(G) \geq k$ if and only if for any partition $(V_1, \dots, V_t)$ of $V(G)$, $e(V_1, \dots, V_t) \geq k(t-1)$.*

Using this theorem, Cioabă and Wong [3] proved Conjecture 1.2 for $k = 2, 3$ and also constructed some examples to show the bound is essentially best possible. For general k, using the following result of Cioabă [4], Cioabă and Wong [3] obtained the following theorem.

Theorem 1.4. *(See Cioabă and Wang [3].) Let k and d be two integers with $d \geq 2k \geq 4$. If G is a d-regular graph with $\lambda_2(G) < d - \frac{2(2k-1)}{d+1}$, then $\tau(G) \geq k$.*

Later, Gu [6] and Li, Shi [9] and Liu, Hong, Lai [10] independently generalized this investigation into general simple graph and proposed the following conjecture.

Conjecture 1.5. *(See [6,9,10].) Let k be an integer with $k \geq 2$ and G be a graph with minimum degree $\delta \geq 2k$. If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.*

In [5,6], Gu, Lai, Li and Yao proved the conjecture holds for the cases $k = 2$ and $k = 3$ and proved that if $\lambda_2(G) < \delta - \frac{3k-1}{\delta+1}$ then $\tau(G) \geq k$. In [9], Li and Shi proved that if $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1} - O(1/n)$ then $\tau(G) \geq k$. In [10], Liu, Hong and Lai proved if $\lambda_2(G) \leq \delta - \frac{2k-2/k}{\delta+1}$ or $\lambda_2 \leq \delta - \frac{2k-1}{\delta+1}$ and $n \geq (2k-1)(\delta+1)$, then $\tau(G) \geq k$. The purpose of this note is to confirm Conjecture 1.5 which also implies Conjecture 1.2, and to prove similar results on Laplace and signless Laplace eigenvalues instead of adjacency eigenvalues. The following results are obtained.

Theorem 1.6. *Let G be a graph of minimum degree $\delta \geq 2k \geq 4$ and $\lambda_2(G)$, $\rho_2(G)$, $\mu_{n-1}(G)$ be the second largest adjacency eigenvalue, the second largest signless Laplace eigenvalue and the algebraic connectivity of G, respectively.*

(1) If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.
(2) If $\rho_2(G) < 2\delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.
(3) If $\mu_{n-1}(G) > \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.

The main tool of this paper is eigenvalue interlacing property of symmetric matrices. In Section 2, some preliminaries about eigenvalue interlacing and quotient matrices, which will be used in this paper, are displayed. In Section 3, a spectral condition on $aD + A$ for the existence of a certain number of edge-disjoint spanning trees. Theorem 1.6 will be proved in Section 4.

2. Preliminaries

Given two non-increasing real sequences $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$ with $n > m$, the second sequence is said to *interlace* the first one if $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, \ldots, m$. The following result is known as the Cauchy Interlacing Theorem. A proof of this theorem can be found on page 27 of [2].

Theorem 2.1 *(Cauchy Interlacing). Let B be a principal submatrix of a symmetric matrix A, then the eigenvalues of B interlace the eigenvalues of A.*

Let A be a symmetric matrix of order n and $V_1, \ldots, V_t$ be a partition of $\{1, \ldots, n\}$. For any $1 \leq i, j \leq t$, let b_{ij} denote the average number of neighbors in V_j of the vertices in V_i. The *quotient matrix* of this partition is the $t \times t$ matrix B whose (i, j)-th entry equals b_{ij}. Haemers [7] showed the eigenvalues of the quotient matrix B in fact interlace the eigenvalues of A.

Theorem 2.2. *(See Haemers [7].) Let A be a symmetric matrix. Then the eigenvalues of every quotient matrix of A interlace the ones of A.*

3. Eigenvalues of $aD + A$ and edge disjoint trees

For any graph G, let D be the degree diagonal matrix and A be the adjacency matrix. In this section, we consider the eigenvalues of $aD+A$ of G, where $a \in \mathbb{R}$. For convenience, we may use $\lambda_1(G,a) \geq \lambda_2(G,a) \geq \ldots \geq \lambda_n(G,a)$ to denote the eigenvalues of $aD + A$ of G.

Lemma 3.1. *Let G be a graph of minimum degree δ and A be a vertex subset of G. If $d(A) < \delta$, then $|A| \geq \delta + 1$.*

Proof. If there exists a vertex $u \in A$ such that $N(u) \subseteq A$ then $|A| \geq |N(u) \cup \{u\}| \geq \delta + 1$. So, we may assume $N(u) - A \neq \emptyset$ for any $u \in A$. Let $v \in A$ and thus $d(A) = e(\{v\}, V(G) \setminus A) + e(A - \{v\}, V(G) \setminus A) \geq e(\{v\}, V(G) \setminus A) + |A \setminus \{v\}| \geq e(v, V(G) \setminus \{v\}) = d(v) \geq \delta$, a contradiction. □

Lemma 3.2. *Let G be a graph with minimum degree δ and $a \in \mathbb{R}$ be a real number. For any two disjoint vertex sets X, Y, if $\lambda_2(G,a) \leq (a+1)\delta - \max\{\frac{d(X)}{|X|}, \frac{d(Y)}{|Y|}\}$, then $[e(X,Y)]^2 \geq ((a+1)\delta - \frac{d(X)}{|X|} - \lambda_2(G,a))((a+1)\delta - \frac{d(Y)}{|Y|} - \lambda_2(G,a))|X||Y|$.*

Proof. First, let A_{XY} be the principal submatrix of $aD + A$ of G induced by the vertices in $X \cup Y$. Then by Theorem 2.1

$$\lambda_2(A_{XY}) \leq \lambda_2(G,a).$$

Now, assume $|X| = x$, $|Y| = y$, $d(X) = p$, $d(Y) = q$ and $e(X,Y) = r$. Let $x_1 = (\sum_{u \in X}(a+1)d(u) - p)/x$ and $y_1 = (\sum_{v \in Y}(a+1)d(v) - q)/y$. Then, by the assumption on $\lambda_2(G,a)$, $x_1 \geq (a+1)\delta - \frac{p}{x} \geq \lambda_2(G,a)$ and $y_1 \geq (a+1)\delta - \frac{q}{y} \geq \lambda_2(G,a)$. The quotient matrix of the adjacency matrix of A_{XY} with respect to the partition (X,Y) is

$$A_2 = \begin{bmatrix} x_1 & \frac{r}{x} \\ \frac{r}{y} & y_1 \end{bmatrix}.$$

Direct computation yields $\lambda_2(A_2) = \frac{1}{2}(x_1 + y_1 - \sqrt{(x_1 - y_1)^2 + \frac{4r^2}{xy}})$. By Theorem 2.1, $\lambda_2(A_2) \leq \lambda_2(A_{XY}) \leq \lambda_2(G,a)$. Thus

$$\begin{aligned} r^2 &= \frac{xy}{4}\big(\big(x_1 + y_1 - 2\lambda_2(A_2)\big)^2 - (x_1 - y_1)^2\big) \\ &= xy\big(x_1 - \lambda_2(A_2)\big)\big(y_1 - \lambda_2(A_2)\big) \end{aligned}$$

$$\geq xy\big(x_1 - \lambda_2(G,a)\big)\big(y_1 - \lambda_2(G,a)\big)$$
$$\geq \left((a+1)\delta - \frac{p}{x} - \lambda_2(G,a)\right)\left((a+1)\delta - \frac{q}{y} - \lambda_2(G,a)\right)xy.$$

The proof is completed. □

Theorem 3.3. *Let k be an integer and G be a graph of order n and minimum degree $\delta \geq 2k$. If $\lambda_2(G,a) < (a+1)\delta - \frac{2k-1}{\delta+1}$ then $\tau(G) \geq k$.*

Proof. Let $V_1, V_2, \ldots, V_t$ be an arbitrary partition of $V(G)$. By Theorem 1.3, it suffices to show that $e(V_1, \ldots, V_t) \geq k(t-1)$. Without loss of generality, we may assume $d(V_1) \leq d(V_2) \leq \ldots \leq d(V_t)$. If $d(V_1) \geq 2k$, then $e(V_1, \ldots, V_t) \geq \sum_{i=1}^{t} d(V_i)/2 \geq kt > k(t-1)$. So, in the next, we may assume $d(V_1) \leq 2k-1$.

Let $s \in \{1, \ldots, t\}$ be an integer such that $d(V_s) \leq 2k-1$ and $d(V_{s+1}) \geq 2k$ (if V_{s+1} exists). Then by Lemma 3.1, $|V_i| \geq \delta+1$ for $1 \leq i \leq s$. Moreover, for each $i \in \{2, \ldots, s\}$, by Lemma 3.2, $[e(V_1, V_i)]^2 \geq ((a+1)\delta - \frac{d(V_1)}{|V_1|} - \lambda_2(G,a))((a+1)\delta - \frac{d(V_i)}{|V_i|} - \lambda_2(G,a))|V_1||V_i| > (2k-1-d(V_1))(2k-1-d(V_i)) \geq (2k-1-d(V_i))^2$. It follows that $e(V_1, V_i) > 2k-1-d(V_i)$ and thus

$$e(V_1, V_i) \geq 2k - d(V_i).$$

Hence, $d(V_1) \geq \sum_{i=2}^{s} e(V_1, V_i) \geq \sum_{i=2}^{s}(2k - d(V_i))$. So, $\sum_{i=1}^{s} d(V_i) \geq 2k(s-1)$ and thus $\sum_{i=1}^{t} d(V_i) = \sum_{i=1}^{s} d(V_i) + \sum_{i=s+1}^{t} d(V_i) \geq 2k(s-1) + 2k(t-s) = 2k(t-1)$. It follows that $e(V_1, \ldots, V_t) \geq k(t-1)$ and the proof is completed. □

4. Eigenvalues and edge-disjoint trees

For any graph G, let D and A be the degree diagonal matrix and the adjacency matrix, respectively. For any $a, b \in \mathbb{R}$, denote by $\lambda_i(G,a,b)$ the i-th largest eigenvalue of $aD+bA$. Then $\lambda_i(G,a,1) = \lambda_i(G,a)$. In this section, we generalize the result of Theorem 3.3 as follows.

Theorem 4.1. *Let k be an integer and G be a graph of order n and minimum degree $\delta \geq 2k$. If $b > 0$ and $\lambda_2(G,a,b) < (a+1)\delta - \frac{b(2k-1)}{\delta+1}$, or $b < 0$ and $\lambda_{n-1}(G,a,b) > (1-a)\delta - \frac{b(2k-1)}{\delta+1}$, then $\tau(G) \geq k$.*

Proof. Noting that $aD + bA = b(\frac{a}{b}D + A)$, if $b > 0$ then $\lambda_i(G,a,b) = b\lambda_i(G, \frac{a}{b})$; if $b < 0$ $\lambda_{n-i}(G,a,b) = b\lambda_i(G, \frac{a}{b})$. Thus, the result follows from Theorem 3.3 clearly. □

Corollary 4.2. *Let $k \geq 2$ be an integer and G be a graph with minimum degree $\delta \geq 2k$ and of order n. If one of the following holds then $\tau(G) \geq k$.*

(1) The second largest adjacency eigenvalue $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$.
(2) The second largest signless-Laplace eigenvalue $\rho_2(G) < 2\delta - \frac{2k-1}{\delta+1}$.
(3) The algebraic connectivity $\mu_{n-1}(G) > \frac{2k-1}{\delta+1}$.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[2] A.E. Brouwer, W.H. Haemers, Spectra of Graphs, Springer Universitext, 2012, 250 pp., monograph, available online at http://www.win.tue.nl/~aeb/2WF02/spectra.pdf.
[3] S.M. Cioabă, W. Wong, Edge-disjoint spanning trees and eigenvalues of regular graphs, Linear Algebra Appl. 437 (2012) 630–647.
[4] S.M. Cioabă, Eigenvalues and edge-connectivity of regular graphs, Linear Algebra Appl. 432 (2010) 458–470.
[5] X. Gu, H. Lai, P. Li, S. Yao, Edge-disjoint spanning trees, edge connectivity and eigenvalues in graphs, submitted for publication.
[6] X. Gu, Connectivity and spanning trees of graphs, PhD dissertation, West Virginia University, 2013.
[7] W.H. Haemers, Interlacing eigenvalues and graphs, Linear Algebra Appl. 226 (228) (1995) 593–616.
[8] G. Kirchhoff, Über die Auflöung der Gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer Ströme geführt wird, Ann. Phys. Chem. 72 (1847) 497–508.
[9] G. Li, L. Shi, Edge-disjoint spanning trees and eigenvalues of graphs, Linear Algebra Appl. 439 (2013) 2784–2789.
[10] Q. Liu, Y. Hong, H. Lai, Edge-disjoint spanning trees and eigenvalues, Linear Algebra Appl. 444 (2014) 146–151.
[11] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. Lond. Math. Soc. 36 (1961) 445–450.
[12] E.M. Palmer, On the spanning tree packing number of a graph: a survey, Discrete Math. 230 (2001) 13–21.
[13] W.T. Tutte, On the problem of decomposing a graph into n factors, J. Lond. Math. Soc. 36 (1961) 221–230.