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Realizing degree sequences as Z_3-connected graphs

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ABSTRACT

An integer-valued sequence $\pi = (d_1, \ldots, d_n)$ is *graphic* if there is a simple graph G with degree sequence of π. We say the π has a realization G. Let Z_3 be a cyclic group of order three. A graph G is Z_3-*connected* if for every mapping $b : V(G) \to Z_3$ such that $\sum_{v\in V(G)} b(v) = 0$, there is an orientation of G and a mapping $f : E(G) \to Z_3 - \{0\}$ such that for each vertex $v \in V(G)$, the sum of the values of f on all the edges leaving from v minus the sum of the values of f on the all edges coming to v is equal to $b(v)$. If an integer-valued sequence π has a realization G which is Z_3-connected, then π has a Z_3-*connected realization* G. Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence with $d_n \geq 3$. We prove in this paper that if $d_1 \geq n - 3$, then π has a Z_3-connected realization unless the sequence is $(n - 3, 3^{n-1})$ or is $(k, 3^k)$ or $(k^2, 3^{k-1})$ where $k = n - 1$ and n is even; if $d_{n-5} \geq 4$, then π has a Z_3-connected realization unless the sequence is $(5^2, 3^4)$ or $(5, 3^5)$.

1. Introduction

Graphs here are finite, and may have multiple edges without loops. We follow the notation and terminology in [2] except otherwise stated.

For a given orientation of a graph G, if an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then u is the *tail* of e and v is the *head* of e. For a vertex $v \in V(G)$, let $E^+(v)$ and $E^-(v)$ denote the sets of all edges having tail v or head v, respectively. A graph G is k-*flowable* if all the edges of G can be oriented and assigned nonzero numbers with absolute value less than k so that for every vertex $v \in V(G)$, the sum of the values on all the edges in $E^+(v)$ equals that of the values of all the edges in $E^-(v)$. If G is k-flowable we also say that G admits a nowhere-zero k-flow.

Let A be an abelian group with identity 0, and let $A^* = A - \{0\}$. Given an orientation and a mapping $f : E(G) \to A$, the *boundary* of f is a function $\partial f : V(G) \to A$ defined by, for each vertex $v \in V(G)$,

$$\partial f(v) = \sum_{e\in E^+(v)} f(e) - \sum_{e\in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

A mapping $b : V(G) \to A$ is a *zero-sum function* if $\sum_{v\in V(G)} b(v) = 0$. A graph G is A-*connected* if for every zero-sum function $b : V(G) \to A$, there exist an orientation of G and a mapping $f : E(G) \to A^*$ such that $\partial f(v) = b(v)$ for each $v \in V(G)$.

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The concept of k-flowability was first introduced by Tutte [19], and this theory provides an interesting way to investigate the coloring of planar graphs in the sense that Tutte [19] proved a classical theorem: a planar graph is k-colorable if and only if it is k-flowable. Jaeger et al. [10] successfully generalized nowhere-zero flow problems to group connectivity. The purpose of study in group connectivity is to characterize contractible configurations for integer flow problems. Let Z_3 be a cyclic group of order three. Obviously, if G is Z_3-connected, then G is 3-flowable.

An integer-valued sequence $\pi = (d_1, \ldots, d_n)$ is *graphic* if there is a simple graph G with degree sequence π. We say π has a *realization* G, and we also say G is a realization of π. If an integer-valued sequence π has a realization G which is A-connected, then we say that G is a A-connected realization of π for an abelian group A. In particular, if $A = Z_3$, then we say G is a Z_3-connected realization (or π has a Z_3-connected realization G). In this paper, we write every degree sequence $(d_1, \ldots, d_n)$ is in nonincreasing order. For simplicity, we use exponents to denote degree multiplicities, for example, we write $(6, 5, 4^4, 3)$ for $(6, 5, 4, 4, 4, 4, 3)$.

The problem of realizing degree sequences by graphs that have nowhere-zero flows or are A-connected, where A is an abelian group, has been studied. Luo et al. [17] proved that every bipartite graphic sequence with least element at least 2 has a 4-flowable realization. As a corollary, they confirmed the simultaneous edge-coloring conjecture of Cameron [3]. Fan et al. [6] proved that every degree sequence with least element at least 2 has a realization which contains a spanning Eulerian subgraph; such graphs are 4-flowable. Let A be an abelian group with $|A| = 4$. For a nonincreasing n-element graphic sequence π with least element at least 2 and sum at least $3n - 3$, Luo et al. [15] proved that π has a realization that is A-connected. Yin and Guo [20] determined the smallest degree sum that yields graphic sequences with a Z_3-connected realization. For the literature for this topic, the readers can see a survey [13]. In particular, Luo et al. [16] completely answered the question of Archdeacon [1]: Characterize all graphic sequences π realizable by a 3-flowable graph. The natural group connectivity version of Archdeacon's problem is as follows.

Problem 1.1. Characterize all graphic sequences π realizable by a Z_3-connected graph.

On this problem, Luo et al. [16] obtained the next two results.

Theorem 1.2. *Every nonincreasing graphic sequence $(d_1, \ldots, d_n)$ with $d_1 = n - 1$ and $d_n \geq 3$ has a Z_3-connected realization unless n is even and the sequence is $(k, 3^k)$ or $(k^2, 3^{k-1})$, where $k = n - 1$.*

Theorem 1.3. *Every nonincreasing graphic sequence $(d_1, \ldots, d_n)$ with $d_n \geq 3$ and $d_{n-3} \geq 4$ has a Z_3-connected realization.*

Motivated by Problem 1.1 and the results above, we present the following two theorems in this paper. These results extend the results of [16] by extending the characterizations to a large set of sequences.

Theorem 1.4. *A nonincreasing graphic sequence $(d_1, \ldots, d_n)$ with $d_1 \geq n-3$ and $d_n \geq 3$ has a Z_3-connected realization unless the sequence is $(n - 3, 3^{n-1})$ for any n or is $(k, 3^k)$ or $(k^2, 3^{k-1})$, where $k = n - 1$ and n is even.*

Theorem 1.5. *A nonincreasing graphic sequence $(d_1, \ldots, d_n)$ with $d_n \geq 3$ and $d_{n-5} \geq 4$ has a Z_3-connected realization unless the sequence is $(5^2, 3^4)$ or $(5, 3^5)$.*

We end this section with some notation and terminology. A graph is trivial if $E(G) = \emptyset$ and nontrivial otherwise. A *k-vertex* denotes a vertex of degree k. Let P_n denote the path on n vertices and we call P_n a n-path. An *n-cycle* is a cycle on n vertices. The *wheel* W_k is the graph obtained from a k-cycle by adding a new vertex, the center of the wheel, and joining it to every vertex of the k-cycle. A wheel W_k is an *odd (even)* wheel if k is odd (even). For simplicity, we say W_1 is a triangle. For a graph G and $X \subseteq V(G)$, denote by $G[X]$ the subgraph of G induced by X. For two vertex-disjoint subsets V_1, V_2 of $V(G)$, denote by $e(V_1, V_2)$ the number of edges with one endpoint in V_1 and the other endpoint in V_2.

We organize this paper as follows. In Section 2, we state some results and establish some lemmas that will be used in the following proofs. We will deal with some special degree sequences, each of which has a Z_3-connected realization in Section 3. In Sections 4 and 5, we will give the proofs of Theorems 1.4 and 1.5.

2. Lemmas

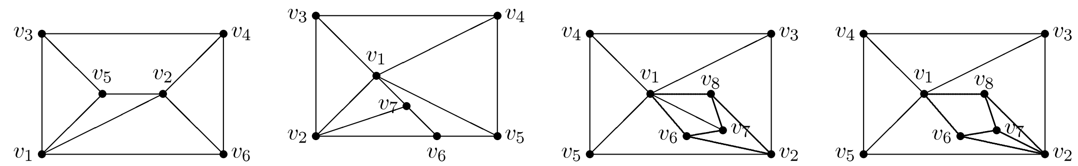
Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence with $d_1 \geq \cdots \geq d_n$. Throughout this paper, we use $\bar{\pi}$ to represent the sequence $(d_1 - 1, \ldots, d_{d_n} - 1, d_{d_n+1}, \ldots, d_{n-1})$, which is called the *residual sequence* obtained from π by deleting d_n. The following well-known result is due to Hakimi [8,9] and Kleitman and Wang [11].

Theorem 2.1. *A graphic sequence has even sum. Furthermore, a sequence π is graphic if and only if $\bar{\pi}$ is graphic.*

Some results in [4,5,7,10,12] on group connectivity are summarized as follows.

Lemma 2.2. *Let A be an abelian group with $|A| \geq 3$. The following results are known:*

(1) *K_1 is A-connected;*
(2) *K_n and K_n^- are A-connected if $n \geq 5$;*
(3) *An n-cycle is A-connected if and only if $|A| \geq n + 1$;*



(a) $\pi = (4^2, 3^4)$. (b) $\pi = (5, 4, 3^5)$. (c) $\pi = (6, 4, 3^6)$. (d) $\pi = (5^2, 3^6)$.

Fig. 1. Realizations of four degree sequences.

(4) *$K_{m,n}$ is A-connected if $m \geq n \geq 4$; neither $K_{2,t}$ $(t \geq 2)$ nor $K_{3,s}$ $(s \geq 3)$ is Z_3-connected;*
(5) *Each even wheel is Z_3-connected and each odd wheel is not;*
(6) *Let $H \subseteq G$ and H be A-connected. G is A-connected if and only if G/H is A-connected;*
(7) *If G is not A-connected, then any spanning subgraph of G is not A-connected.*
(8) *Let v be not a vertex of G. If G is A-connected and $e(v, G) \geq 2$, then $G \cup \{v\}$ is A-connected.*

Let G be a graph having an induced path with three vertices v, u, w in order. Let $G_{[uv,uw]}$ be the graph by deleting uv and uw and adding a new edge vw. The following lemma was first proved by Lai in [12] and reformulated by Chen et al. in [4].

Lemma 2.3. *Let G be a graph with $u \in V(G)$, $uv, uw \in E(G)$ and $d(u) \geq 4$, and let A be an abelian group with $|A| \geq 3$. If $G_{[uv,uw]}$ is A-connected, then so is G.*

A graph G is *triangularly connected* if for every edge $e, f \in E$ there exists a sequence of cycles $C_1, C_2, \ldots, C_k$ such that $e \in E(C_1)$, $f \in E(C_k)$, and $|E(C_i)| \leq 3$ for $1 \leq i \leq k$, and $|E(C_j) \cap E(C_{j+1})| \neq \emptyset$ for $1 \leq j \leq k-1$.

Lemma 2.4 ([5]). *A triangularly connected graph G is Z_3-connected if G has minimum degree at least 4 or has a nontrivial Z_3-connected subgraph.*

An orientation D of G is a *modular* 3-*orientation* if $|E^+(v)| - |E^-(v)| \equiv 0 \pmod 3$ for every vertex $v \in V(G)$. Steinberg and Younger [18] established the following relationship.

Lemma 2.5. *A graph G is 3-flowable if and only if G admits a modular 3-orientation.*

Let v be a 3-vertex in a graph G, and let $N(v) = \{v_1, v_2, v_3\}$. Denote by $G_{(v,v_1)}$ the graph obtained from G by deleting vertex v and adding a new edge v_2v_3. The following lemma is due to Luo et al. [14].

Lemma 2.6. *Let A be an abelian group with $|A| \geq 3$, and let $b : V(G) \mapsto A$ be a zero-sum function with $b(v) \neq 0$. If $G_{(v,v_1)}$ is Z_3-connected, then there exist an orientation D of G and a nowhere-zero mapping $f' : E(G) \mapsto A$ such that $\partial f' = b$ under the orientation D of G.*

For any odd integer k, Luo et al. [16] proved that no realization of the graphic sequence $(k, 3^k)$ and $(k^2, 3^{k-1})$ is 3-flowable. This yields the following lemma.

Lemma 2.7. *If k is odd, then neither $(k, 3^k)$ nor $(k^2, 3^{k-1})$ has a Z_3-connected realization.*

Next we provide Z_3-connected realizations for some degree sequences.

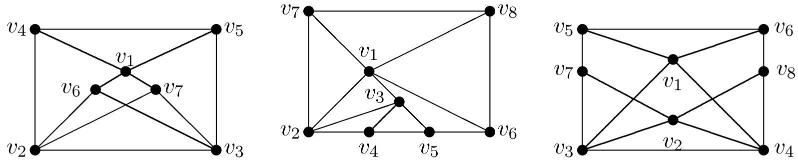
Lemma 2.8. *Each of the graphs in Fig. 1 is Z_3-connected.*

Proof. If G is the graph (a) in Fig. 1, then G is Z_3-connected by Lemma 2.2 of [14]. Thus, we may assume that G is one of the graphs (b), (c) and (d) shown in Fig. 1.

We only prove here that the graph (b) in Fig. 1 is Z_3-connected. The proofs for the graphs (c) and (d) in Fig. 1 are similar. (For more details, the readers can see http://arXiv.org.) Assume that G is the graph (b) shown in Fig. 1. Let $b : V(G) \to Z_3$ be a zero-sum function. If $b(v_3) \neq 0$, then $G_{(v_3,v_4)}$ contains a 2-cycle (v_1, v_2). Contracting this 2-cycle and repeatedly contracting all 2-cycles generated in the process, we obtain K_1. By parts (1) and (6) of Lemma 2.2, $G_{(v_3,v_4)}$ is Z_3-connected. It follows by Lemma 2.6 that there exists a nowhere-zero mapping $f : E(G) \to Z_3$ with $\partial f = b$. Thus, we may assume that $b(v_3) = 0$. Similarly, we may assume that $b(v_4) = b(v_5) = b(v_6) = b(v_7) = 0$. This means that for such b, there are only three possibilities to be considered: $(b(v_1), b(v_2)) \in \{(0, 0), (1, 2), (2, 1)\}$.

If $(b(v_1), b(v_2)) = (0, 0)$, we show that G is 3-flowable. Note that each vertex of v_3, v_4, v_5, v_6 and v_7 is of degree 3. The edges of G are oriented as follows: $|E^+(v_3)| = 3$, $|E^-(v_4)| = 3$, $|E^+(v_5)| = 3$, $|E^-(v_6)| = 3$, $|E^+(v_7)| = 3$, and v_2v_1 is oriented from v_2 to v_1. It is easy to verify that $|E^+(v)| - |E^-(v)| = 0 \pmod 3$ for each vertex $v \in V(G)$. By Lemma 2.5, G is 3-flowable. Thus, there is an $f : E(G) \to Z_3^*$ such that $\partial f(v) = b(v)$ for each $v \in V(G)$.

If $(b(v_1), b(v_2)) = (1, 2)$, note that $b(v) = 0$ for each $v \in V(G) - \{v_1, v_2\}$. The edges of G are oriented as follows: $|E^-(v_3)| = 3$, $|E^+(v_4)| = 3$, $|E^-(v_5)| = 3$, $|E^+(v_6)| = 3$, $|E^-(v_7)| = 3$ and edge v_2v_1 is oriented from v_2 to v_1. If



(a) $\pi = (4^3, 3^4)$. (b) $\pi = (5, 4^2, 3^5)$. (c) $\pi = (4^4, 3^4)$.

Fig. 2. Realizations of three degree sequences.

$(b(v_1), b(v_2)) = (2, 1)$, then the edges of G are oriented as follows: $|E^+(v_3)| = 3$, $|E^-(v_4)| = 3$, $|E^+(v_5)| = 3$, $|E^-(v_6)| = 3$, $|E^+(v_7)| = 3$ and edge v_1v_2 is oriented from v_1 to v_2. In each case, for each $e \in E(G)$, define $f(e) = 1$. It is easy to see that for $v \in \{v_3, v_4, v_5, v_6, v_7\}$, $\partial f(v) = 0 = b(v)$, $\partial f(v_1) = b(v_1)$ and $\partial f(v_2) = b(v_2)$.

Thus, for any zero-sum function b, there exist an orientation of G and a nowhere-zero mapping $f : E(G) \to Z_3$ such that $\partial f = b$. Therefore, G is Z_3-connected. ■

Lemma 2.9. *Each graph in Fig. 2 is Z_3-connected.*

Proof. We only prove here that the graph (a) in Fig. 2 is Z_3-connected. The proofs for the graphs (b) and (c) are similar (For more details, the readers can see http://arXiv.org). Denote by G the graph (a) in Fig. 2. We claim that G is 3-flowable. Assume that the edges of the graph are oriented as follows: $|E^+(v_4)| = 3$, $|E^+(v_5)| = 0$, $|E^+(v_6)| = 3$, $|E^+(v_7)| = 0$ and v_2v_3 from v_2 to v_3. Define $f(e) = 1$ for all $e \in E(G)$. It is easy to verify that $\partial f(v) = 0$ for each $v \in V(G)$. By Lemma 2.5, the graph (a) is 3-flowable.

Let $b : V(G) \to Z_3$ be a zero-sum function. If $b(v_4) \neq 0$, then $G_{(v_4,v_2)}$ contains a 2-cycle (v_1, v_5). Contracting the 2-cycle, we obtain an even wheel W_4 induced by $\{v_1, v_2, v_3, v_6, v_7\}$ with the center at v_3. By parts (3), (5) and (6) of Lemma 2.2, $G_{(v_4,v_2)}$ is Z_3-connected. By Lemma 2.6, there exists a nowhere-zero mapping $f : E(G) \to Z_3$ with $\partial f = b$. Thus, we assume $b(v_4) = 0$. By symmetry, we may assume $b(v_5) = 0$. If $b(v_6) \neq 0$, then $G_{(v_6,v_3)}$ is a graph isomorphic to Fig. 1(a) which is Z_3-connected by Lemma 2.8. By Lemma 2.6, there exists a nowhere-zero mapping $f : E(G) \to Z_3$ with $\partial f = b$. We thus assume $b(v_6) = 0$. By symmetry, we assume $b(v_7) = 0$.

So far, we may assume $b(v_4) = b(v_5) = b(v_6) = b(v_7) = 0$. We claim that $b(v_2) \neq 0$. If $b(v_2) = 0$, then denote by $G(v_2)$ the graph obtained from G by deleting v_2 and adding edges v_3v_7 and v_4v_6. Contracting all 2-cycles, we finally get an even wheel W_4 with the center at v_1. By Lemma 2.2, $G(v_2)$ is Z_3-connected. Thus, there exists a nowhere-zero mapping $f : E(G) \to Z_3$ with $\partial f = b$. By symmetry, we assume that $b(v_3) \neq 0$. Thus, we are left to discuss three cases $(b(v_1), b(v_2), b(v_3)) \in \{(1, 1, 1), (0, 1, 2), (2, 2, 2)\}$.

If $(b(v_1), b(v_2), b(v_3)) = (1, 1, 1)$, then we orient the edges of G as follows: $|E^+(v_4)| = 3$, $|E^+(v_5)| = 0$, $|E^+(v_6)| = 3$, $|E^+(v_7)| = 3$, and v_2v_3 from v_2 to v_3; if $(b(v_1), b(v_2), b(v_3)) = (0, 1, 2)$, then we orient edges of G as follows: $|E^+(v_4)| = 3$, $|E^+(v_5)| = 0$, $|E^+(v_6)| = 3$, $|E^+(v_7)| = 0$, and v_3v_2 from v_3 to v_2; if $(b(v_1), b(v_2), b(v_3)) = (2, 2, 2)$, then we orient edges of G as follows: $|E^+(v_4)| = 3$, $|E^+(v_5)| = 0$, $|E^+(v_6)| = 0$, $|E^+(v_7)| = 0$, and v_2v_3 from v_2 to v_3. In each case, for each $e \in E(G)$ define $f(e) = 1$. It is easy to verify that $\partial f(v) = b(v)$ for each $v \in V(G)$.

In each case, there exist an orientation of G and a nowhere-zero mapping $f : E(G) \to Z_3$ such that $\partial f = b$. Therefore, G is Z_3-connected. ■

3. Some special cases

Throughout this section, all sequences are graphic sequences. We provide Z_3-connected realizations for some graphic sequences.

Lemma 3.1. *Suppose that one of the following holds,*

(i) $n \geq 6$ *and* $\pi = (n-2, 4, 3^{n-2})$;
(ii) $n \geq 5$ *and* $\pi = (4^{n-4}, 3^4)$;
(iii) $n \geq 7$ *and* $\pi = (5, 4^{n-6}, 3^5)$.

Then π has a Z_3-connected realization.

Proof. (i) If $n = 6$, then by Lemma 2.8, π has a Z_3-connected realization G in Fig. 1(a). Thus, we assume that $n \geq 7$.

If $n = 7, 8$, then by Lemma 2.8, π has a Z_3-connected realization G in Fig. 1(b) (c). Thus, we assume that $n \geq 9$.

Assume that n is odd. Let W_{n-5} be an even wheel with the center at v_1 and K_4^- on vertex set $\{u_1, u_2, u_3, u_4\}$ with $d_{K_4^-}(u_1) = d_{K_4^-}(u_3) = 2$. Denote by G the graph obtained from W_{n-5} and K_4^- by adding edges u_iv_1 for each $i \in \{1, 2, 3\}$. Obviously, the graph G has a degree sequence $(n-2, 4, 3^{n-2})$. By part (5) of Lemma 2.2, W_{n-5} is Z_3-connected. The graph G/W_{n-5} is an even wheel W_4. By parts (5) and (6) of Lemma 2.2, G is Z_3-connected. This means that π has a Z_3-connected realization.

Assume that n is even. Let G_0 be the graph in Fig. 1(a) and W_{n-6} be an even wheel with the center at u_1. Denote by G the graph obtained from W_{n-6} and G_0 by identifying u_1 and v_1. Clearly, G has a degree sequence $(n-2, 4, 3^{n-2})$. Since $n \geq 10$ is even, W_{n-6} is Z_3-connected by (5) of Lemma 2.2. By Lemma 2.8, G_0 is Z_3-connected. This shows that G is Z_3-connected.

(ii) If $n = 5$, then an even wheel W_4 is a Z_3-connected realization of π; if $n = 6$, then by Lemma 2.8, π has a Z_3-connected realization G in Fig. 1(a); if $n = 7$, then by Lemma 2.9, π has a Z_3-connected realization shown in Fig. 2(a); if $n = 8$, then by Lemma 2.9, the graph (c) in Fig. 2 is Z_3-connected realization of π. If $n = 9$, then let G_1 be an even wheel W_4 induced by $\{u_0, u_1, u_2, u_3, u_4\}$ with the center at u_0 and G_2 be a K_4^- induced by $\{v_1, v_2, v_3, v_4\}$ with $d_{G_2}(v_1) = d_{G_2}(v_3) = 3$. We construct a graph G from W_4 and K_4^- by adding three edges u_1v_2, u_2v_4 and u_3v_1. Then G is a Z_3-connected realization of $(4^5, 3^4)$. Thus, we assume that $n \geq 10$.

Assume that $n = 2k$, where $k \geq 5$. By induction of hypothesis, let G_i be a Z_3-connected realization of the degree sequence $(4^{k-4}, 3^4)$ for $i \in \{1, 2\}$. Assume that $n = 2k+1$, where $k \geq 5$. By induction hypothesis, let G_1 be a Z_3-connected realization of the degree sequence $(4^{k-4}, 3^4)$ and G_2 be a Z_3-connected realization of the degree sequence $(4^{k-3}, 3^4)$. In each case, we construct a graph G from G_1 and G_2 by connecting a pair of 3-vertices of G_1 to a pair of 3-vertices of G_2 one by one. It is easy to verify that G is a Z_3-connected realization of the degree sequence $(4^{n-4}, 3^4)$.

(iii) If $n = 7$, then by Lemma 2.8, the graph (b) in Fig. 1 is a Z_3-connected realization of π; if $n = 8$, then by Lemma 2.9, the graph (b) in Fig. 2 is a Z_3-connected realization of π. If $n = 9$, then $\pi = (5, 4^3, 3^5)$. Let G_1 be an even wheel W_4 induced by $\{u_0, u_1, u_2, u_3, u_4\}$ with the center at u_0 and G_2 be a K_4^- induced by $\{v_1, v_2, v_3, v_4\}$ with $d_{G_2}(v_1) = d_{G_2}(v_3) = 3$. We construct a graph G from W_4 and K_4^- by adding three edges u_0v_2, u_1v_1, u_2v_4. We conclude that G is a Z_3-connected realization of degree sequence $(5, 4^3, 3^5)$. Thus, $n \geq 10$.

Assume that $n = 2k$, where $k \geq 5$. By (ii), let G_1 and G_2 be Z_3-connected realizations of degree sequence $(4^{k-4}, 3^4)$. Assume that $n = 2k + 1$, where $k \geq 5$. By (ii), let G_1 be a Z_3-connected realization of degree sequence $(4^{k-4}, 3^4)$ and G_2 be a Z_3-connected realization of degree sequence $(4^{k-3}, 3^4)$. In each case, choose one 4-vertex u_1 and one 3-vertex u_2 of G_1; choose two 3-vertices v_1, v_2 of G_2. We construct a graph G from G_1 and G_2 by adding u_1v_1 and u_2v_2. Thus, G is a Z_3-connected realization of degree sequence $(5, 4^{n-6}, 3^5)$. ■

Lemma 3.2. *If $\pi = (n-3, 3^{n-1})$, then π has not a Z_3-connected realization.*

Proof. Suppose otherwise that G has a Z_3-connected realization of degree sequence $(n-3, 3^{n-1})$. Let $V(G) = \{u, u_1, \dots, u_{n-3}, x_1, x_2\}$, $N_G(u) = \{u_1, \dots, u_{n-3}\}$ (N for short), and $X = \{x_1, x_2\}$. We now consider the following two cases.

Case 1. $x_1x_2 \in E(G)$.

Since G is Z_3-connected, G is 3-flowable. By Lemma 2.5 and symmetry, we may assume that $|E^+(x_1)| = 3$ and $|E^-(x_2)| = 3$. Since $d(u_i) = 3$ for $i \in \{1, \dots, n-3\}$, by Lemma 2.5, either $|E^+(u_i)| = 3$ or $|E^-(u_i)| = 3$. This implies that there exists no vertex u_i in N such that $u_ix_1, u_ix_2 \in E(G)$. Thus, $G[N]$ is the union of two paths P_1 and P_2. We relabel the vertices of N such that $P_1 = u_1 \dots u_k$ and $P_2 = u_{k+1} \dots u_{n-3}$.

Suppose first that $x_1u_1, x_1u_k \in E(G)$ and $x_2u_{k+1}, x_2u_{n-3} \in E(G)$. Since G is 3-flowable, by Lemma 2.5, P_i contains odd number of vertices for each $i \in \{1, 2\}$. Define $b(u) = b(x_1) = b(x_2) = 1$ and $b(u_i) = 0$ for each $i \in \{1, \dots, n-3\}$. It is easy to verify that there exists no $f : E(G) \to Z_3^*$ such that $\partial f(v) = b(v)$ for each $v \in V(G)$, contrary to that G is Z_3-connected.

Next, suppose that $x_1u_1, x_1u_{k+1} \in E(G)$ and $x_2u_k, x_2u_{n-3} \in E(G)$. Since G is 3-flowable, by Lemma 2.5 P_i contains even number of vertices for each $i \in \{1, 2\}$. Define $b(u) = 1$, $b(x_2) = 2$ and $b(u_i) = b(x_1) = 0$ for each $i \in \{1, \dots, n-3\}$. It is easy to verify that there exists no $f : E(G) \to Z_3^*$ such that $\partial f(v) = b(v)$ for each $v \in V(G)$, contrary to that G is Z_3-connected.

Case 2. $x_1x_2 \notin E(G)$.

Since $d(x_i) = 3$ for each $i = 1, 2, 0 \leq |N(x_1) \cap N(x_2)| \leq 3$. Assume first that $|N(x_1) \cap N(x_2)| = 3$. We assume, without loss of generality, that $u_1, u_2, u_3 \in N(x_1) \cap N(x_2)$. The subgraph induced by $\{u, x_1, x_2, u_1, u_2, u_3\}$ is $K_{3,3}$ which is not Z_3-connected by part (4) of Lemma 2.2. By part (6) of Lemma 2.2, G is not Z_3-connected, a contradiction.

Assume that $|N(x_1) \cap N(x_2)| = 2$. We assume, without loss of generality, that $u_1, u_2 \in N(x_1) \cap N(x_2)$. Since G is 3-flowable, the graph H induced by $N \setminus \{u_1, u_2\}$ consists of even cycles and a path of length even. This means that n is even. If $n = 6$, then this case cannot occur. Thus $n \geq 8$. Define $b(x_1) = 1$, $b(x_2) = 2$ and $b(u_i) = b(u) = 0$ for each $i \in \{1, \dots, n-3\}$. It is easy to verify that there exists no $f : E(G) \to Z_3^*$ such that $\partial f(v) = b(v)$ for each $v \in V(G)$, contrary to that G is Z_3-connected.

Next, assume that $|N(x_1) \cap N(x_2)| = 1$. We assume, without loss of generality, that $u_1 \in N(x_1) \cap N(x_2)$. The graph induced by $N \setminus \{u_1\}$ consists of even cycles and two paths P_1 and P_2. Since G is 3-flowable, P_i contains odd vertices for each $i \in \{1, 2\}$. Then n is even. If $n = 6, 8$, then this case cannot occur. Thus, we assume that $n \geq 10$. Define $b(x_1) = 1$, $b(x_2) = 2$ and $b(u_i) = b(u) = 0$ for each $i \in \{1, \dots, n-3\}$. In this case, there exists no $f : E(G) \to Z_3^*$ such that $\partial f(v) = b(v)$ for each $v \in V(G)$, contrary to that G is Z_3-connected.

Finally, assume that $|N(x_1) \cap N(x_2)| = 0$. Then the graph induced by the vertices of N consists of three paths P_1, P_2 and P_3, together with even cycles. We relabel the vertices of N such that $P_1 = u_1 \dots u_s$, $P_2 = u_{s+1} \dots u_t$ and $P_3 = u_{t+1} \dots u_{n-3}$. By symmetry, we consider two cases: x_1 is adjacent to both the end vertices of some P_i; x_1 is adjacent to one of each P_j for $j \in \{1, 2, 3\}$.

In the former case, we may assume that $u_1x_1, u_sx_1 \in E(G)$ and x_2u_{s+1}, x_2u_t. Since G is 3-flowable, by Lemma 2.5, both $|V(P_1)|$ and $|V(P_2)|$ are odd. If $|V(P_3)|$ is odd, then define $b(x_1) = 1$, $b(x_2) = 2$ and $b(u_i) = b(u) = 0$ for each $i \in \{1, \dots, n-3\}$. If $|V(P_3)|$ is even, then define $b(x_1) = 1$, $b(x_2) = 1$, $b(u) = 1$ and $b(u_i) = 0$ for each $i \in \{1, \dots, n-3\}$. In either case, there exists no $f : E(G) \to Z_3^*$ such that $\partial f(v) = b(v)$ for each $v \in V(G)$, contrary to that G is Z_3-connected.

In the latter case, $x_1u_1, x_1u_{s+1}, x_1u_{t+1}, x_2u_s, x_2u_t, x_2u_{n-3} \in E(G)$. It follows that $|V(P_1)|, |V(P_2)|$ and $|V(P_3)|$ have the same parity. If each of $|V(P_i)|$ for $i \in \{1, 2, 3\}$ is even, then define $b(x_1) = 1, b(x_2) = 1, b(u) = 1$ and $b(u_i) = 0$ for each $i \in \{1, \dots, n-3\}$. If each of $|V(P_i)|$ for $i \in \{1, 2, 3\}$ is odd, then define $b(x_1) = 1, b(x_2) = 2, b(u) = b(u_i) = 0$ for each $i \in \{1, \dots, n-3\}$. In either case, there exists no $f : E(G) \to Z_3^*$ such that $\partial f(v) = b(v)$ for each $v \in V(G)$, contrary to that G is Z_3-connected. ■

4. Proof of Theorem 1.4

In order to prove Theorem 1.4, we establish the following lemma.

Lemma 4.1. *Suppose that $\pi = (d_1, \dots, d_n)$ is a nonincreasing graphic sequence with $d_n \geq 3$. If $d_1 = n-2$, then π has a Z_3-connected realization.*

Proof. Suppose, to the contrary, that $\pi = (d_1, \dots, d_n)$ has no Z_3-connected realization with n minimized, where $d_1 = n-2$. By Theorem 1.3, we may assume that $d_{n-3} \leq 3$. In order to prove our lemma, we need the following claim.

Claim 1. Each of the following holds.

(i) $d_{n-3} = d_{n-2} = d_{n-1} = d_n = 3$;
(ii) $n \geq 6$.

Proof of Claim 1. (i) follows since $d_n \geq 3$.

(ii) Since $d_n = 3$, $n \geq 4$. If $n = 4$, then $d_1 = 3 = n-1$, contrary to that $d_1 = n-2$. If $n = 5$, then $\pi = (3^5)$ is not graphic by Theorem 2.1. This proves Claim 1.

If $n = 6$, then $d_1 = 4$ and $d_3 = d_4 = d_5 = d_6 = 3$. By Theorem 2.1, $d_2 = 4$. By Lemma 2.8, $\pi = (4^2, 3^4)$ has a Z_3-realization, a contradiction. Thus, we may assume that $n \geq 7$.

Claim 2. $d_3 = 3$.

Proof of Claim 2. Suppose otherwise that $d_3 \geq 4$ and G is a counterexample with $|V(G)| = n$ minimized. Then $d_2 \geq d_3 \geq 4$. Hence, $\bar{\pi} = (n-3, d_2-1, d_3-1, d_4, \dots, d_{n-1}) = (\bar{d}_1, \dots, \bar{d}_{n-1})$ with $\bar{d}_1 \geq \cdots \geq \bar{d}_{n-1}$. This implies that $\bar{d}_{n-1} \geq 3$ and $\bar{d}_1 = (n-1)-2$ or $\bar{d}_1 = d_4$. In the former case, since $\bar{d}_1 = n-3 = (n-1)-2$, by the minimality of n, $\bar{\pi}$ has a Z_3-connected realization $\bar{G}$. In the latter case, $\bar{d}_1 \neq n-3$ and hence $\bar{d}_1 = d_4 > n-3$. Since $d_1 = n-2 \geq d_4$, $d_4 = n-2$. This means that $\bar{d}_1 = d_4 = (n-1)-1$. By Theorem 1.2, either $\bar{\pi}$ has a Z_3-connected realization $\bar{G}$ or $\bar{\pi} = (k, 3^k), (k^2, 3^{k-1})$, where k is odd. If $\bar{\pi} = (k, 3^k)$, then $d_1 = k+1 = n-2$ and $d_2 = d_3 = 4$. On the other hand, $n = k+1+1 = k+2$. This contradiction proves that $\bar{\pi} \neq (k, 3^k)$. Similarly, $\bar{\pi} \neq (k^2, 3^{k-1})$. If $\bar{\pi}$ has a Z_3-connected realization $\bar{G}$, then π has a realization G of π from $\bar{G}$ by adding a new vertex v and three edges joining v to the corresponding vertices of $\bar{G}$. By part (8) of Lemma 2.2, G is Z_3-connected, a contradiction. Thus $d_3 \leq 3$. Clearly, $d_3 \geq 3$. Then $d_3 = 3$. This proves Claim 2.

By Claims 1 and 2, $\pi = (n-2, d_2, 3^{n-2})$. Since π is graphic, d_2 is even whenever n is even or odd. Moreover, $d_2 \geq 4$. Recall that $n \geq 7$. In this case, $\pi = (n-2, 4, 3^{n-2})$. By (i) of Lemma 3.1, π has a Z_3-connected realization G, a contradiction. Thus, we may assume that $d_2 \geq 6$. Since $n-2 = d_1 \geq d_2 \geq 6$, $n \geq d_2+2 \geq 8$.

Consider the case that n is even. Denote by W_{n-d_2+2} an even wheel with the center at v_1 and by S a vertex set such that $|S| = d_2-4$ and $V(W_{n-d_2+2}) \cap S = \emptyset$. Note that $|S|$ is even. We construct a graph G from W_{n-d_2+2} and S as follows: First, pick two vertices s_1, s_2 of S and add $(d_2-6)/2$ edges such that the subgraph induced by $S \setminus \{s_1, s_2\}$ is a perfect matching. Second, let v_1 connect to each vertex of S. Third, pick a vertex v_2 in W_{n-d_2+2} and let v_2 join to each vertex of S. Finally, add one new vertex x adjacent to v_2, s_1 and s_2.

We claim that G has a degree sequence $(n-2, d_2, 3^{n-2})$. Since $d_{W_{n-d_2+2}}(v_1) = n-d_2+2 \geq 4$, $d(v_1) = n-d_2+2+d_2-4 = n-2$, $d(v_2) = 3+d_2-4+1 = d_2$, each vertex of $V(G) \setminus \{v_1, v_2\}$ is a 3-vertex. Since W_{n-d_2+2} is an even wheel, by part (5) of Lemma 2.2, this wheel is Z_3-connected. By part (8) of Lemma 2.2, G is Z_3-connected, a contradiction.

Consider the case that n is odd. Denote by W_{n-d_2+1} an even wheel with the center at v_1 and by S a vertex set with $|S| = d_2-3$ and $V(W_{n-d_2+1}) \cap S = \emptyset$. We construct a graph G from W_{n-d_2+1} and S as follows: First, let v_1 connect to each vertex of S. Second, pick one vertex v_2 in W_{n-d_2+1} and let v_2 join to each vertex of S. Third, add one vertex x adjacent to three vertices of S. Finally, add $(d_2-6)/2$ edges in S so that the subgraph induced by vertices of S, each of which is not adjacent to x, is a perfect matching. We claim that G is a realization of degree sequence $(n-2, d_2, 3^{n-2})$. Since $d_{W_{n-d_2+1}}(v_1) = n-d_2+1 \geq 4$, $d(v_1) = n-d_2+1+d_2-3 = n-2$. Note that $d(v_2) = 3+d_2-3 = d_2$, each vertex of $V(G) \setminus \{v_1, v_2\}$ is a 3-vertex. Similarly, it can be verified that G is a Z_3-connected realization of π, a contradiction. ■

Proof of Theorem 1.4. Assume that $\pi = (d_1, \dots, d_n)$ is a nonincreasing graphic sequence with $d_1 \geq n-3$. If π is one of $(n-3, 3^{n-1})$, $(k, 3^k)$ and $(k^2, 3^{k-1})$, then by Lemmas 2.7 and 3.2, π has no Z_3-connected realization.

Conversely, assume that $\pi \notin \{(n-3, 3^{n-1}), (k, 3^k), (k^2, 3^{k-1})\}$. Since $d_1 \geq n-3$ and $d_n \geq 3$, $n \geq 6$. In the case that $n = 6$, by Theorem 1.2, $d_1 = 3, 4$. If $d_1 = 3$, then $\pi = (3^6)$. Since $n = 6$, $(3^6) = (n-3, 3^{n-1})$, contrary to our assumption. If $d_1 = 4$, then by (ii) of Lemma 3.1 π has a Z_3-connected realization. In the case that $n = 7$, by Theorems 1.2 and 2.1, $4 \leq d_1 \leq 5$. If $d_1 = 5$, then any realization of π contains the graph (b) of Fig. 1. By Lemma 2.8, π has a Z_3-connected realization. Assume that $d_1 = 4$. Since $n = 7$, $(4, 3^6) = (n-3, 3^{n-1})$. Thus, by our assumption, $\pi \neq (4, 3^6)$. In this case, any realization of π contains the graph (a) in Fig. 2. By Lemma 2.9, π has a Z_3-connected realization. Thus, assume that $n \geq 8$.

By Theorem 1.2 and by Lemmas 3.2 and 4.1, we are left to prove that if $d_1 = n - 3$, $d_n \geq 3$ and $d_2 \neq 3$, then π has a Z_3-connected realization. Suppose otherwise that $\pi = (d_1, \ldots, d_n)$ satisfying

$$d_1 = n - 3, \qquad d_2 \neq 3, \qquad d_n \geq 3. \tag{1}$$

Subject to (1),

$$\pi \text{ has no } Z_3\text{-realization with } n \text{ minimized.} \tag{2}$$

We establish the following claim first.

Claim 1. (i) $d_{n-3} = d_{n-2} = d_{n-1} = d_n = 3$.
(ii) $3 \leq d_3 \leq 4$.

Proof of Claim 1. By Theorem 1.3, $d_{n-3} \leq 3$. (i) follows since $d_n \geq 3$.

(ii) Suppose otherwise that subject to (1) and (2), π satisfies $d_3 \geq 5$. Since $d_2 \geq d_3$, $d_2 \geq 5$. Define $\bar{\pi} = (n-4, d_2 - 1, d_3 - 1, d_4, \ldots, d_{n-1}) = (\bar{d}_1, \ldots, \bar{d}_{n-1})$ with $\bar{d}_1 \geq \cdots \geq \bar{d}_{n-1}$. Since $d_3 \geq 5$, $d_2 - 1 \geq d_3 - 1 \geq 4$. This means that $\bar{d}_1 \geq \bar{d}_2 \geq 4$, and $\bar{d}_{n-1} \geq 3$. If $d_1 > d_4$, then $\bar{d}_1 = (n-1) - 3$. In this case, by the minimality of n, $\bar{\pi}$ has a Z_3-connected realization $\bar{G}$. If $d_1 = d_4$, then $\bar{d}_1 = d_4$. It follows that $d_4 > n - 4$. This implies that $d_1 = d_2 = d_3 = d_4 = n - 3$. Thus, $\bar{d}_1 = d_4 = n - 3 = (n-1) - 2$. By Lemma 4.1, $\bar{\pi}$ has a Z_3-connected realization $\bar{G}$. In either case, π has a realization G obtained from $\bar{G}$ by adding a new vertex v and three edges joining v to the corresponding vertices of $\bar{G}$. By part (8) of Lemma 2.2, G is Z_3-connected, a contradiction. Thus $d_3 \leq 4$. Since $d_3 \geq 3$, $3 \leq d_3 \leq 4$. This proves Claim 1.

By Claim 1, we may assume that $\pi = (n-3, d_2, d_3, \ldots, d_{n-4}, 3^4)$ with $d_3 \in \{3, 4\}$. We consider the following two cases.

Case 1. $d_3 = 3$.

In this case, $\pi = (n-3, d_2, 3^{n-2})$. Since π is graphic, d_2 is odd. Since $d_2 \neq 3$, $d_2 \geq 5$. We first assume that $d_2 = 5$. In this case, $\pi = (n-3, 5, 3^{n-2})$. If $n = 8$, by Lemma 2.8, the graph (d) in Fig. 1 is a Z_3-connected realization of $\pi = (5^2, 3^6)$. Thus, assume that $n \geq 9$.

Assume that n is odd. Denote by W_{n-5} an even wheel with the center at v_1 and by S a vertex set with $|S| = 2$. We construct graph G from W_{n-5} and S as follows: First, connect v_1 to each vertex of S. Second, choose one vertex v_2 in W_{n-5} and add two vertices x_1, x_2 such that x_i is adjacent to v_2 and each vertex of S for each $i \in \{1, 2\}$.

Since $d(v_1) = n - 5 + 2 = n - 3$, $d(v_2) = 3 + 2 = 5$ and each vertex of $V(G) \setminus \{v_1, v_2\}$ is a 3-vertex, this means that G is a realization of degree sequence $(n-3, 5, 3^{n-2})$. By part (5) of Lemma 2.2, W_{n-5} is Z_3-connected. Note that G/W_{n-5} is an even wheel W_4 which is also Z_3-connected by Lemma 2.2. It follows by part (6) of Lemma 2.2 that G is Z_3-connected, a contradiction.

Thus we may assume that n is even. Denote by W_{n-6} an even wheel with the center at v_1 and let $S = \{s_1, s_2, s_3\}$ be a vertex set. We construct graph G from W_{n-6} and S as follows: First, connect v_1 to each vertex of S. Second, choose one vertex v_2 in W_{n-6} and let v_2 be adjacent to s_1. Finally, add two vertices x_1, x_2 such that x_1 is adjacent to v_2 and s_2, s_3; x_2 is adjacent to each vertex of S.

It is easy to verify that $d(v_1) = n - 6 + 3 = n - 3$, $d(v_2) = 3 + 2 = 5$ and each vertex of $V(G) \setminus \{v_1, v_2\}$ is a 3-vertex. This means that G is a realization of degree sequence $(n-3, 5, 3^{n-2})$. By (5) of Lemma 2.2, W_{n-6} is Z_3-connected. By part (8) of Lemma 2.2, $W_{n-6} \cup \{s_1\}$ is Z_3-connected. Note that $G/\{W_{n-6} \cup \{s_1\}\}$ is an even wheel W_4 which is Z_3-connected. It follows by (6) of Lemma 2.2 that G is Z_3-connected, a contradiction.

From now on, we assume that $d_2 \geq 7$. In this case, $n \geq d_2 + 3 \geq 10$. Consider the case that n is even. Denote by W_{n-d_2+1} an even wheel with the center at v_1 and by S a vertex set with $|S| = d_2 - 4$. We construct graph G from W_{n-d_2+1} and S as follows: First, connect v_1 to each vertex of S. Second, pick one vertex s of S and let $S_1 = S \setminus \{s\}$, pick one vertex v_2 in W_{n-d_2+1} and connect v_2 to each vertex of S_1. Third, pick two vertices s_1, s_2 of S_1, and add $(d_2 - 7)/2$ edges such that the induced subgraph by $S_1 \setminus \{s_1, s_2\}$ is a perfect matching. Finally, we add two vertices x_1 and x_2 such that x_i is adjacent to v_2, x_i is adjacent to s_i for $i = 1, 2$ and s is adjacent to each of x_1 and x_2.

Since $d(v_1) = n - d_2 + 1 + d_2 - 4 = n - 3$, $d(v_2) = 3 + d_2 - 5 + 2 = d_2$ and each vertex of $V(G) - \{v_1, v_2\}$ is a 3-vertex, this implies s that G is a realization of degree sequence $(n-3, d_2, 3^{n-2})$. By (5) of Lemma 2.2, W_{n-d_2+1} is Z_3-connected. Contracting this even wheel W_{n-d_2+1} and contracting all 2-cycles generated in the process, we get K_1. By (8) of Lemma 2.2, G is Z_3-connected, a contradiction.

Consider the case that n is odd. Denote by W_{n-d_2} an even wheel with the center at v_1 and by S a vertex set with $|S| = d_2 - 3$. We construct a graph from W_{n-d_2} and S as follows. First, let v_1 be adjacent to each vertex of S. Second, pick two vertices s_3 and s_4 of S, define $S_1 = S \setminus \{s_3, s_4\}$ and pick one vertex v_2 in W so that v_2 is adjacent to each vertex of S_1. Third, add two new vertex x_1, x_2 such that x_i is adjacent to each of v_2, s_3 and s_4. Finally, add $(d_2 - 5)/2$ edges in S_1 so that the subgraph induced by vertices of $S_1 \setminus \{s_3, s_4\}$ is a perfect matching.

Since $d(v_1) = n - d_2 + d_2 - 3 = n - 3$, $d(v_2) = 3 + d_2 - 5 + 2 = d_2$, and each vertex of $V(G) \setminus \{v_1, v_2\}$ is a 3-vertex, G is a realization of degree sequence $(n-3, d_2, 3^{n-2})$. Similarly, by parts (5) and (8) of Lemma 2.2, G is Z_3-connected, a contradiction.

Case 2. $d_3 = 4$.

In this case, $d_2 \geq 4$ and $\pi = (n-3, d_2, 4, d_4, \ldots, d_{n-4}, 3^4)$. Define $\bar{\pi} = (n-4, d_2 - 1, 3, d_4, \ldots, d_{n-4}, 3^3) = (\bar{d}_1, \ldots, \bar{d}_{n-1})$ with $\bar{d}_1 \geq \cdots \geq \bar{d}_{n-1}$. If $d_1 = d_4$, then $n - 3 = 4$ and hence $n = 7$, contrary to assumption that $n \geq 8$. Thus, $d_1 > d_4$. In this case, $\bar{d}_1 = n - 4$.

Claim 2. $d_2 = 4$.

Proof of Claim 2. Suppose otherwise that $d_2 \geq 5$. Then $\bar{d}_2 \geq 4$ and $\bar{\pi}$ satisfies (1). By the minimality of n, $\bar{\pi}$ has a Z_3-connected realization $\bar{G}$. Thus, we conclude that G is a Z_3-connected realization of π obtained from $\bar{G}$ by adding a new vertex v and three edges joining v to the corresponding vertices of $\bar{G}$. This contradiction proves Claim 2.

By Claim 2, $d_2 = 4$. Assume that $i \in \{3, \ldots, n-4\}$ such that $d_i = 4$ and $d_{i+1} = 3$. Thus $\pi = (n-3, 4^{i-1}, 3^{n-i})$.

Claim 3. $i = 3$.

Proof of Claim 3. If i is even, then $n-i$ is odd (even) when n is odd (even). No matter whether n is odd or even, there are odd vertices of odd degree, a contradiction. Thus, i is odd. If $i \geq 5$, then $\bar{\pi} = (n-4, 4^{i-3}, 3^{n-i+1})$ satisfies (1). Recall that $n \geq 8$, by the minimality of n, $\bar{\pi}$ has a Z_3-connected realization $\bar{G}$. In this case, we can obtain a realization G of π from $\bar{G}$ by adding a new vertex v and three edges joining v to the corresponding vertices of $\bar{G}$. By (8) of Lemma 2.2, G is Z_3-connected, a contradiction. This proves Claim 3.

By Claim 3, $i = 3$. This leads to that $\pi = (n-3, 4^2, 3^{n-3})$. Recall that $n \geq 8$. If $n = 8$, then by Lemma 2.9, π has a Z_3-connected realization. Thus, we may assume that $n \geq 9$.

In the case that n is odd, denote by W_{n-5} the even wheel with the center at v_1 and by S a vertex set with $|S| = 2$. We construct a graph G from W_{n-5} and S as follows. First, let v_1 be adjacent to each vertex of S. Second, pick two vertices v_2, v_3 in W_{n-5} and add two vertices x_1, x_2 such that x_i is adjacent to v_{i+1} and each vertex of S for each $i \in \{1, 2\}$.

It is easy to verify that $d(v_1) = n-5+2 = n-3$, $d(v_i) = 3+1 = 4$ for each $i \in \{2, 3\}$, and each vertex of $V(G) - \{v_1, v_2, v_3\}$ is a 3-vertex. Obviously, G has a degree sequence $(n-3, 4^2, 3^{n-3})$. By (5) of Lemma 2.2, W_{n-5} is Z_3-connected. G/W_{n-5} is an even wheel W_4 which is Z_3-connected. By (6) of Lemma 2.2, G is Z_3-connected, a contradiction.

In the case that n is even, denote by W_{n-6} the even wheel with the center at v_1 and let $S = \{s_1, s_2, s_3\}$. We construct a graph G from W_{n-6} and S as follows. First, let v_1 be adjacent to each vertex of S. Second, pick two vertices v_2, v_3 in W_{n-6} so that v_2 is adjacent to s_1. Finally, we add two vertices x_1, x_2 such that x_1 is adjacent to v_3 and s_2, s_3 and such that x_2 is adjacent to each vertex of S.

It is easy to verify that $d(v_1) = n-6+3 = n-3$, $d(v_i) = 3+1 = 4$ for each $i \in \{2, 3\}$, and each vertex of $V(G) - \{v_1, v_2, v_3\}$ is a 3-vertex. Obviously, G has a degree sequence $(n-3, 4^2, 3^{n-3})$. By (5) Lemma 2.2, W_{n-6} is Z_3-connected. By (8) of Lemma 2.2, $W_{n-6} \cup \{s_1\}$ is Z_3-connected. $G/\{W_{n-6} \cup \{s_1\}\}$ is an even wheel W_4 which is Z_3-connected. By (6) of Lemma 2.2, G is Z_3-connected a contradiction. ■

5. Proof of Theorem 1.5

We first establish the following lemma which is used in the proof of Theorem 1.5.

Lemma 5.1. *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence. If $d_n \geq 3$ and $d_{n-4} \geq 4$, then either π has a Z_3-connected realization or $\pi = (5^2, 3^4)$.*

Proof. Since $d_{n-4} \geq 4$, $n \geq 5$. If $n = 5$, then by Theorem 1.3, $\pi = (4, 3^4)$. In this case, an even wheel W_4 is a Z_3-connected realization of π. If $n = 6$, then by Theorem 1.3, $\pi = (4^2, 3^4)$ or $(5^2, 3^4)$. If $\pi = (4^2, 3^4)$, then by Lemma 2.8, the graph (a) shown in Fig. 1 is a Z_3-connected realization of π. If $n = 7$, then by Theorem 1.3, $\pi = (5^2, 4, 3^4)$. Let G be the graph (b) shown in Fig. 1 which has degree sequence $\pi = (5, 4, 3^5)$. Denote by G' the graph obtained from G by adding an edge joining a vertex of degree 3 to a vertex of degree 4. By Lemma 2.8, G is a Z_3-connected realization of $(5, 4, 3^4)$ and so G' is a Z_3-connected realization of $(5^2, 4, 3^4)$. Thus, assume that $n \geq 8$.

By Theorems 1.3 and 1.4, it is sufficient to prove that if $d_{n-3} = 3$, $d_{n-4} \geq 4$ and $d_1 \leq n-4$, then π has a Z_3-connected realization. In this case, $\pi = (d_1, \ldots, d_{n-4}, 3^4)$. Suppose, to the contrary, that $\pi = (d_1, \ldots, d_n)$ satisfies

$$d_{n-3} = 3, \qquad d_{n-4} \geq 4 \quad \text{and} \quad d_1 \leq n-4. \tag{3}$$

Subject to (3),

$$\pi \text{ has no } Z_3\text{-connected realization with } n \text{ minimized.} \tag{4}$$

Assume that $d_{n-4} \geq 5$. Define $\bar{\pi} = (d_1 - 1, d_2 - 1, d_3 - 1, d_4, \ldots, d_{n-4}, 3^3) = (\bar{d}_1, \ldots, \bar{d}_{n-1})$. Since $d_1, d_2, d_3 \geq 5$ and $n \geq 8$, $\bar{d}_{n-5} \geq 4$. This implies that $\bar{\pi}$ satisfies (3), by the minimality of n, $\bar{\pi}$ has a Z_3-connected realization $\bar{G}$. Thus, we construct a realization G of π from $\bar{G}$ by adding a new vertex v and three edges joining v to the corresponding vertices of $\bar{G}$. It follows by (8) of Lemma 2.2 that G is Z_3-connected, a contradiction. Thus, we may assume that $d_{n-4} = 4$.

On the other hand, if $d_1 = 4$, then $\pi = (4^{n-4}, 3^4)$. By Lemma 3.1, π has a Z_3-connected realization, a contradiction. Thus, assume $d_1 \geq 5$. Since $d_{n-4} = 4$ and $n \geq 8$, $d_2 \geq 5$ or $d_2 = 4$.

In the former case, $\bar{d}_{n-5} \geq 4$. In this case, $\bar{\pi}$ satisfies (3), by the minimality of n, $\bar{\pi}$ has a Z_3-connected realization $\bar{G}$. Thus, we can construct a realization G of π from $\bar{G}$ by adding a new vertex v and three edges joining v to the corresponding vertices of $\bar{G}$. By (8) of Lemma 2.2, G is Z_3-connected, a contradiction.

In the latter case, $\pi = (d_1, 4^{n-5}, 3^4)$. Since π is graphic, d_1 is even. Since $d_1 \leq n-4$, $n - d_1 - 1 \geq 3$. In the case that $n - d_1 - 1 = 3$, we have $d_1 = n-4$ and $n \geq 10$ is even. Denote by W_{n-4} an even wheel with the center at v_1. We construct a graph G from W_{n-4} as follows. First, choose five vertices v_2, v_3, v_4, v_5, v_6 of W_{n-4}. Second, add three vertices x_1, x_2, x_3 and edges x_1x_2, x_2x_3. Finally, add edges $v_2x_1, v_3x_1, v_4x_2, v_5x_3, v_6x_3$. In this case, for each $i \in \{1, 2, 3\}$, x_i is a 3-vertex, and for each $j \in \{2, \ldots, 6\}$ v_j is a 4-vertex.

It is easy to see that G is a Z_3-connected realization of degree sequence $(n-4, 4^5, 3^{n-6})$. Define $S = V(W_{n-4}) \setminus \{v_2, \ldots, v_6\} = \{v_1, v_7, \ldots, v_{n-3}\}$. Then $|S| = n-4-5 = n-9 \geq 1$. If $n = 10$, then G is a realization of $(6, 4^5, 3^4)$. If $n \geq 12$, then define G' from G by adding v_jv_{n-j+4} for $7 \leq j \leq \frac{n}{2}+1$, that is, adding $(n-10)/2$ edges in S. Obviously, G' has a degree sequence $(n-4, 4^{n-5}, 3^4)$. We conclude that G' is a Z_3-connected realization of π.

In the case that $n-d_1-1 = 4$, we have $d_1 = n-5$ and $n \geq 11$ is odd. Denote by W_{n-5} an even wheel with the center at v_1. We construct a graph G from W_{n-5} as follows. First, choose six vertices $v_2, v_3, v_4, v_5, v_6, v_7$ of W_{n-5}. Second, add four vertices x_1, x_2, x_3, x_4 and edges x_1x_2, x_2x_3, x_3x_4. Third, add edges $x_1v_2, x_1v_3, x_2v_4, x_3v_5, x_4v_6, x_4v_7$. In this case, for each $i \in \{1, 2, 3, 4\}$, x_i is a 3-vertex, and for each $j \in \{2, 3, \ldots, 7\}$ v_j is a 4-vertex.

It is easy to see that G is a Z_3-connected realization of degree sequence $(n-5, 4^6, 3^{n-7})$. Define $S = V(W_{n-5}) \setminus \{v_1, v_2, \ldots, v_7\} = \{v_8, \ldots, v_{n-4}\}$. Then $|S| = n-5-6 = n-11 \geq 0$. If $n = 11$, then G is a realization of $(6, 4^6, 3^4)$. If $n \geq 13$, then define G' from G by adding edges v_jv_{n-j+4} for $8 \leq j \leq \frac{n+3}{2}$, that is, adding $(n-11)/2$ edges in S. Obviously, G' is a Z_3-connected realization of degree sequence $(n-5, 4^{n-5}, 3^4)$, a contradiction.

In the case that $n-d_1-1 \geq 5$, denote by W_{d_1} an even wheel with the center at v_0. Let $V(W_{d_1}) = \{v_0, v_1, \ldots, v_{d_1}\}$. Let $C : u_1 \ldots u_{n-d_1-1}u_1$ be a cycle of length $n-d_1-1$ and define a graph H obtained from C adding edges u_iu_{i+2} for each $i \in \{1, \ldots, n-d_1-1\}$, where the subscripts are taken modular $n-d_1$. Clearly, H is a 4-regular and is triangularly connected. Define $H' = H - \{u_2u_{n-d_1-1}\}$. Now we prove H' is Z_3-connected. Clearly, $H'_{[u_1u_2,u_1u_3]}$ is triangularly connected and contains a 2-circuit $u_2u_3u_2$. By Lemma 2.4 (a) and Lemma 2.2(3), $H'_{[u_1u_2,u_1u_3]}$ is Z_3-connected, and hence H' by Lemma 2.3. We construct a graph G from W_{d_1} and H' as follows. If $d_1 \geq 8$, then we add two edges $v_1u_{n-d_1-1}$, v_2u_2 and add edges $v_jv_{d_1-j+3}$ for $3 \leq j \leq \frac{d_1}{2}-1$, that is, add $(d_1-6)/2$ edges between vertices $\{v_3, \ldots, v_{d_1-4}\} \setminus \{v_{\frac{d_1}{2}}, v_{\frac{d_1}{2}+1}, v_{\frac{d_1}{2}+2}, v_{\frac{d_1}{2}+3}\}$ such that $d(v_i) = 4$ for each vertex of $\{v_3, \ldots, v_{d_1-4}\} \setminus \{v_{\frac{d_1}{2}}, v_{\frac{d_1}{2}+1}, v_{\frac{d_1}{2}+2}, v_{\frac{d_1}{2}+3}\}$ and the new graph is simple. If $d_1 = 6$, then we add two edges $v_1u_{n-d_1-1}$, v_2u_2. In either case, G is a Z_3-connected realization of has a degree sequence $(d_1, 4^{n-5}, 3^4)$, a contradiction. ■

Proof of Theorem 1.5. If $\pi = (5^2, 3^4)$ or $(5, 3^5)$, then by Lemma 2.7, π has no Z_3-connected realization. Thus, assume that $\pi \neq (5^2, 3^4), (5, 3^5)$.

Since $d_{n-5} \geq 4$, $n \geq 6$. If $n = 6$, then by Theorem 1.3, $\pi = (d_1, d_2, 3^4)$. By Lemma 5.1, $\pi = (d_1, 3^5)$. By our assumption that $d_{n-5} \geq 4$, $\pi = (5, 3^5)$, a contradiction. If $n = 7$, then by Theorem 1.3 and Lemma 5.1, $\pi = (d_1, d_2, 3^5)$. By our assumption that $d_{n-5} \geq 4$, $\pi = (5, 4, 3^5)$ or $(6, 5, 3^5)$. In the former, the graph (b) in Fig. 1 is a Z_3-connected realization of π. In the latter case, Theorem 1.4 shows that π has a Z_3-connected realization. Thus, we may assume that $n \geq 8$.

By Theorems 1.3 and 1.4, and Lemma 5.1, it is sufficient to prove that if $d_{n-4} = 3$, $d_{n-5} \geq 4$ and $d_1 \leq n-4$, then π has a Z_3-connected realization. Then $\pi = (d_1, \ldots, d_{n-5}, 3^5)$. Suppose to the contrary that π satisfies

$$d_{n-4} = 3, \qquad d_{n-5} \geq 4 \quad \text{and} \quad d_1 \leq n-4. \tag{5}$$

Subject to (5),

$$\pi \text{ has no } Z_3\text{-connected realization with } n \text{ minimized.} \tag{6}$$

We claim that $d_3 = 4$. Suppose otherwise that $d_3 \geq 5$. Define $\bar{\pi} = (d_1-1, d_2-1, d_3-1, d_4, \ldots, d_{n-5}, 3^4) = (\bar{d}_1, \ldots, \bar{d}_{n-1})$. Since $d_1, d_2, d_3 \geq 5$, $\bar{d}_{n-6} \geq 4$. Thus $\bar{\pi}$ satisfies (5). By the minimality of n, $\bar{\pi}$ has a Z_3-connected realization $\bar{G}$. Denote by G the graph obtained from $\bar{G}$ by adding a new vertex v and three edges joining v to the corresponding vertices. It follows by (8) of Lemma 2.2 that G is a Z_3-connected realization of π, a contradiction. Thus, $d_3 = 4$ and $\pi = (d_1, d_2, 4^{n-7}, 3^5)$.

We claim that $d_2 = 4$. Suppose otherwise that $d_2 \geq 5$. In this case, $\bar{\pi} = (d_1-1, d_2-1, 4^{n-8}, 3^6) = (\bar{d}_1, \ldots, \bar{d}_{n-1})$. Since $d_2 \geq 5$ and $d_3 = 4$, $n \geq 3+5 = 8$. This implies that $\bar{d}_{n-6} \geq 4$. Thus, $\bar{\pi}$ satisfies (5). By the minimality of n, $\bar{\pi}$ has a Z_3-connected realization $\bar{G}$. Denote by G the graph obtained from $\bar{G}$ by adding a new vertex v and three edges joining v to the corresponding vertices of $\bar{G}$. It follows from (8) of Lemma 2.2 that G is a Z_3-connected realization of π, a contradiction. Thus, $d_2 = 4$ and $\pi = (d_1, 4^{n-6}, 3^5)$.

Since π is graphic, d_1 is odd and $d_1 \geq 5$. If $d_1 = 5$, then by (iii) of Lemma 3.1, π has a Z_3-connected realization.

We are left to the case that $d_1 \geq 7$. Since $d_1 \leq n-4$, $n \geq d_1+4 \geq 11$. By (iii) of Lemma 3.1, let G' be a Z_3-connected realization of degree sequence $(5, 4^{n-6}, 3^5)$. By the construction of G' in (iii) of Lemma 3.1, G' has at least $|E(G')| - 5 - 14 = 2n-21 \geq 2(d_1+4)-21 = 2d_1-13 \geq (d_1-5)/2$ edges not incident with the any vertex of $N_{G'}(u) \cup \{u\}$, where u is a 5-vertex in G' since G' contains a pair of adjacent neighbors of u. Choose $(d_1-5)/2$ such edges, say u_iv_i for each $i \in \{1, \ldots, (d_1-5)/2\}$. Denote by the graph G from G' by deleting edges u_iv_i and adding edge uu_i, uv_i for each $i \in \{1, \ldots, (d_1-5)/2\}$. It follows by Lemma 2.3 that G is a Z_3-connected realization of degree sequence $(d_1, 4^{n-6}, 3^5)$, a contradiction. We complete our proof.

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