Cyclic base orderings in some classes of graphs

Xiaofeng Gu [*,†], Katie Horacek [‡,†], Hong-Jian Lai [§,†]

Abstract

A cyclic base ordering of a connected graph G is a cyclic ordering of $E(G)$ such that every $|V(G)-1|$ cyclically consecutive edges form a spanning tree of G. Let G be a graph with $E(G) \neq \emptyset$ and $\omega(G)$ denote the number of components in G. The invariants $d(G)$ and $\gamma(G)$ are respectively defined as $d(G) = \frac{|E(G)|}{|V(G)|-\omega(G)}$ and $\gamma(G) = \max\{d(H)\}$, where H runs over all subgraphs of G with $E(H) \neq \emptyset$. A graph G is uniformly dense if $d(G) = \gamma(G)$. Kajitani et al. [8] conjectured in 1988 that a connected graph G has a cyclic base ordering if and only if G is uniformly dense. In this paper, we show that this conjecture holds for some classes of uniformly dense graphs.

Key words: cyclic base ordering, cyclic ordering, uniformly dense graphs, uniformly dense matroids

1 Introduction

We consider finite loopless graphs with possible multiple edges, and follow [2] for undefined notations and terminology. In particular, $\omega(G)$ denotes the number of components of a graph G.

[*]Department of Mathematics, Texas State University, San Marcos, TX 78666, USA

[†]Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

[‡]Part of this research is a Capstone Project of Katie Horacek at West Virginia University, co-supervised by the other two authors

[§]College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, PRC

Let G be a nontrivial graph (that is $E(G) \neq \emptyset$). Following the terminology in [5] or [4], $d(G)$ and $\gamma(G)$ are respectively defined as

$$d(G) = \frac{|E(G)|}{|V(G)| - \omega(G)} \text{ and } \gamma(G) = \max\{d(H)\},$$

where H runs over all nontrivial subgraphs of G.

As in [5], a graph G satisfying $d(G) = \gamma(G)$ is said to be **uniformly dense**.

A **cyclic base ordering** of a connected graph G is a cyclic ordering of $E(G)$ such that every $|V(G)| - 1$ cyclically consecutive edges form a spanning tree of G.

Kajitani et al. [8] posed the following cyclic base ordering conjecture.

Conjecture 1.1. *(Kajitani et al. [8]) A connected graph G has a cyclic base ordering if and only if G is uniformly dense.*

Actually, they proved the necessity of the conjecture.

Theorem 1.1. *(Kajitani et al. [8]) For a connected graph G, if G has a cyclic base ordering, then G is uniformly dense.*

For the sufficiency, they were able to prove the following special cases.

Theorem 1.2. *(Kajitani et al. [8]) The following graphs have cyclic base orderings.*
(i) Any uniformly dense simple connected graph with at most 5 *vertices.*
(ii) Any graph consisting of two disjoint spanning trees.
(iii) Any complete graph.
(iv) Any 2-tree (See the definition in Section 5).

In this paper, we shall show that Conjecture 1.1 holds for several classes of graphs, including complete bipartite graphs, k-maximal graphs (See the definition in Section 4) and 3-trees (See the definition in Section 5). These provide with further evidence, in addition to Theorem 1.2, to support conjecture 1.1.

In next section, some properties of uniformly dense graphs will be introduced. In the subsequent sections, we will investigate cyclic base orderings in some classes of uniformly dense graphs. In the last section, we will introduce the matroid version of the cyclic base ordering conjecture and some former results for matroids.

2 Uniformly dense graphs

Let G be a nontrivial graph. Recall that $d(G) = \frac{|E(G)|}{|V(G)|-\omega(G)}$ and $\gamma(G) = \max\{d(H)\}$ where H runs over all nontrivial subgraphs of G. If $d(G) = \gamma(G)$, then G is uniformly dense. Following the terminology in [5], we further define $\eta(G) = \min \frac{|X|}{\omega(G-X)-\omega(G)}$. Let $\tau(G)$ be the maximum number of edge-disjoint spanning trees in a graph G. If $E(G) = \emptyset$, we define $\eta(G) = \infty$. A fundamental theorem of Nash-Williams [11] and Tutte [14] implies the following. (See also Catlin et al. [5])

Theorem 2.1. *(Nash-Williams [11] and Tutte [14]) For a connected graph G, $\tau(G) = \lfloor \eta(G) \rfloor$.*

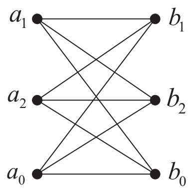
From the definition of $d(G)$, $\eta(G)$ and $\gamma(G)$, we immediately have, for any nontrivial graph G,

$$\eta(G) \leq d(G) \leq \gamma(G).$$

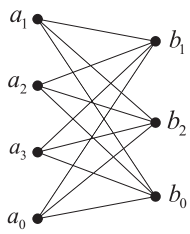
Theorem 2.2. *(Catlin et al. [5]) The following are equivalent for a nontrivial graph G.*
(i) $d(G) = \gamma(G)$.
(ii) $\eta(G) = d(G)$.
(iii) $\eta(G) = \gamma(G)$.

3 Cyclic base ordering in complete bipartite graphs

Theorem 3.1 is the main result in this section. Let G be a complete bipartite graph $K_{m,n}$ with bipartition (X, Y) such that $|X| = m$ and $|Y| = n$. We will give $E(G)$ an ordering and prove that it is a cyclic base ordering.



(a) $m = n = 3$



(b) $m = 4$ and $n = 3$

Figure 1: Examples of cyclic base ordering in complete bipartite graphs

Suppose that $X = \{a_1, a_2, \cdots, a_{m-1}, a_0\}$ and $Y = \{b_1, b_2, \cdots, b_{n-1}, b_0\}$. Let $k = \gcd(m, n) - 1$ and $l = \frac{mn}{\gcd(m,n)}$.

Let $\mathcal{O} = (e_1, e_2, \cdots, e_{mn})$ be an ordering of edges in G such that $e_i = a_s b_{t+j}$ where $s \equiv i \pmod{m}$ and $t \equiv i \pmod{n}$ for $1 \leq s \leq m-1$ and $1 \leq t \leq n-1$. when $jl+1 \leq i \leq (j+1)l$ for $j = 0, 1, 2, \cdots, k$. In particular,
(a) If $m = n$, then $k = m - 1$ and $l = m$. For example, when $m = n = 3$, $\mathcal{O} = (a_1b_1, a_2b_2, a_0b_0, a_1b_2, a_2b_0, a_0b_1, a_1b_0, a_2b_1, a_0b_2)$, as shown in Figure 1(a).
(b) If m and n are coprime, i.e., $\gcd(m, n) = 1$, then $k = 0$ and $l = mn$. Then $\mathcal{O} = (e_i)_1^{mn}$ such that $e_i = a_s b_t$. As shown in Figure 1(b), when $m = 4$ and $n = 3$,

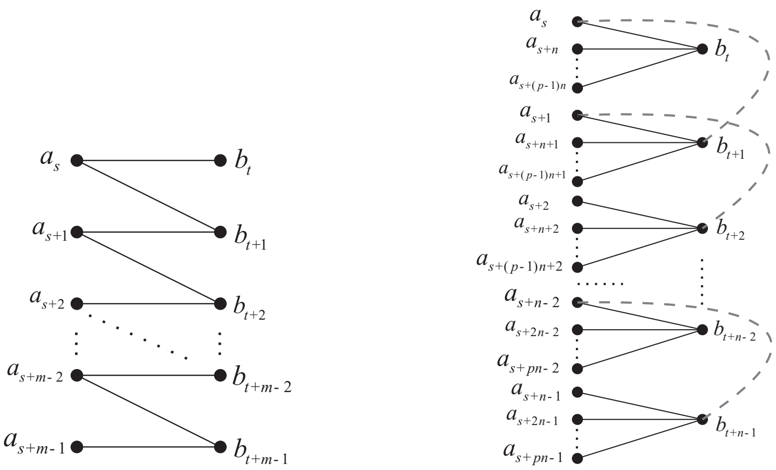
$$\mathcal{O} = (a_1b_1, a_2b_2, a_3b_0, a_0b_1, a_1b_2, a_2b_0, a_3b_1, a_0b_2, a_1b_0, a_2b_1, a_3b_2, a_0b_0).$$

We will prove that $\mathcal{O}$ is a cyclic base ordering of G.

Theorem 3.1. *Every complete bipartite graph has a cyclic base ordering. Furthermore, $\mathcal{O}$ is a cyclic base ordering of G.*

Proof: Let S be the set of any cyclically consecutive $m + n - 1$ elements of $\mathcal{O}$. We need to show that $G[S]$ is a spanning tree of G. Since $G[S]$ is a spanning subgraph with $m + n - 1$ edges and $m + n$ vertices, it suffices to show that $G[S]$ is connected.

We assume that $S = \{e_i, e_{i+1}, \cdots, e_{m+n+i-2}\} \pmod{mn}$ for some i. We use $\pmod{mn}$ for a set to mean that the subscript of each element in



(a) $m = n$ (b) $m = pn$

Figure 2: Any cyclically consecutive $m + n - 1$ elements in $\mathcal{O}$

the set is modulo mn. By the definition of $\mathcal{O}$, we suppose that

$$S = \{a_s b_t, a_{s+1} b_{t+1}, \cdots, a_{s+m-1} b_{t+m-1}, a_s b_{t+1}, \cdots, a_{s+n-2} b_{t+n-1}\}.$$

Without loss of generality, we may assume that $m \geq n$. If $m = n$, then $G[S]$ is a path $b_t a_s b_{t+1} a_{s+1} \cdots a_{s+m-1} b_{t+m-1}$ as shown in Figure 2(a), and thus is a spanning tree in G.

If m is a multiple of n, i.e., $m = pn$. Let S' be the subset consist of the first m elements of S. Then $G[S']$ has n components and each component is a star centered at b_i for $i = t, t+1, \cdots, t+n-1 \pmod{n}$, as shown in Figure 2(b). Let $G_t, G_{t+1}, \cdots, G_{t+n-1}$ denote the components. The edge $a_s b_{t+1}$ is between G_t and G_{t+1}, $a_{s+1} b_{t+2}$ is between G_{t+1} and G_{t+2}. In general, $a_{s+i} b_{t+1+i} \in S \backslash S'$ is between G_{t+i} and G_{t+1+i} for $i = 0, 1, \cdots, n-2$. Thus $G[S]$ is connected, whence is a spanning tree in G.

The last case is $m = pn + q$ where $1 \leq q < m$. Let S' be the subset containing the first m elements of S. Then $G[S']$ has n components and each component is a star centered at b_i for $i = t, t+1, \cdots, t+n-1 \pmod{n}$, which is similar to the case of $m = pn$. Let $G_t, G_{t+1}, \cdots, G_{t+n-1}$ denote the components. The edge $a_s b_{t+q}$ is between G_t and G_{t+q}, $a_{s+1} b_{t+q+1}$ is between G_{t+1} and G_{t+q+1}. In general, $a_{s+i}b_{t+q+i} \in S \backslash S'$ is between G_{t+i} and G_{t+q+i} for $i = 0, 1, \cdots, n-2$. Thus $G[S]$ is connected, whence it is a spanning tree in G. $\square$

4 Cyclic base orderings in k-maximal graphs

Throughout this section, a graph G always means a multigraph and k denotes a positive integer. Theorem 4.3 is the main result in this section.

Let G be connected and $\overline{\kappa'}(G) = \max\{\kappa'(H) : H \text{ is a subgraph of } G\}$. Mader in [10] first introduced k-maximal graphs. A graph G is **k-maximal** if $\overline{\kappa'}(G) \leq k$ but for any edge $e \notin E(G)$, $\overline{\kappa'}(G+e) \geq k+1$. We shall point out that there is a big difference between k-maximal simple graphs and k-maximal multigraphs. What we talk about here are k-maximal multigraphs, and Lemma 4.2 gives a structural characterization of a k-maximal multigraph. For the structure of a k-maximal simple graph, please refer to Mader [10] and Lai [9].

Let G_1 and G_2 be connected graphs such that $V(G_1) \cap V(G_2) = \emptyset$. Let K be a set of k edges each of which has one vertex in $V(G_1)$ and the other vertex in $V(G_2)$. The **K-edge-join** $G_1 *_K G_2$ is defined to be the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup K$. When the set K is not emphasized, we use $G_1 *_k G_2$ for $G_1 *_K G_2$, and refer $G_1 *_k G_2$ as a k-edge-join.

Let $\mathcal{G}_k$ be a family of graphs such that for any $G_1, G_2 \in \mathcal{G}_k \cup \{K_1\}$, $G_1 *_k G_2 \in \mathcal{G}_k$.

Lemma 4.1. *(Gu et al. [6]) Let G be a k-maximal graph with $|V(G)| = n$. Then $|E(G)| = k(n-1)$.*

Lemma 4.2. *(Gu et al. [6]) A connected graph G is k-maximal if and only if $G \in \mathcal{G}_k$.*

Theorem 4.3. *Any k-maximal graph G has a cyclic base ordering.*

Proof: We will show it by induction on $n = |V(G)|$. By Lemma 4.1, $|E(G)| = k(n-1)$. When $n = 2$, by Lemma 4.2, $G = kK_2$, the graph with 2 vertices and k multiple edges. Then any ordering of edges is a cyclic base ordering. Now assume that the theorem holds for smaller values of $n > 2$. By Lemma 4.2, G has an edge cut of size k denoted by $K = \{f_1, f_2, \cdots, f_k\}$ and $G = G_1 *_K G_2$. Then $G_i = K_1$ or $G_i \in \mathcal{G}_k$ for $i = 1, 2$. Since $n > 2$, at least one of G_1 and G_2 is not K_1. Without loss of generality, we may assume that $G_1 \neq K_1$.

(i) $G_2 = K_1$. By inductive hypothesis, G_1 has a cyclic base ordering, denoted by

$$\mathcal{O} = (e_1, e_2, \cdots, e_{k(n-2)}).$$

We construct an ordering of $E(G)$ from $\mathcal{O}$ by inserting f_i between $e_{i(n-2)}$ and $e_{i(n-2)+1}$ for $i = 1, 2, \cdots, k$, and get

$$\mathcal{O}' = (e_1, e_2, \cdots, e_{n-2}, f_1, e_{n-1}, \cdots, e_{2(n-2)}, f_2, e_{2(n-2)+1}, \cdots, e_{k(n-2)}, f_k).$$

Then $\mathcal{O}'$ is a cyclic base ordering of G.

(ii) $G_2 \neq K_1$. Suppose that $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$. For $i = 1, 2$, since $\tau(G_i) = k$, we have $|E(G_i)| = k(n_i - 1)$. By inductive hypothesis, G_1 has a cyclic base ordering denoted by

$$\mathcal{O}_1 = (e_1, e_2, \cdots, e_{k(n_1-1)})$$

and G_2 has a cyclic base ordering denoted by

$$\mathcal{O}_2 = (e'_1, e'_2, \cdots, e'_{k(n_2-1)}).$$

We will construct an ordering of $|E(G)|$ from $\mathcal{O}_1$ and $\mathcal{O}_2$. Let

$$S_i = (e_{(i-1)(n_1-1)+1}, \cdots, e_{i(n_1-1)}, f_i, e'_{(i-1)(n_2-1)+1}, \cdots, e'_{i(n_2-1)})$$

for $i = 1, 2, \cdots, k$. And let

$$\mathcal{O}' = (S_1, S_2, \cdots, S_k)$$

Then $\mathcal{O}'$ is a cyclic base ordering of G. This completes the proof. $\square$

By Theorem 1.1 and Theorem 2.1, we have the following corollary.

Corollary 4.4. *Every k-maximal graph is a disjoint union of k edge-disjoint spanning trees.*

5 Cyclic base orderings in 3-trees

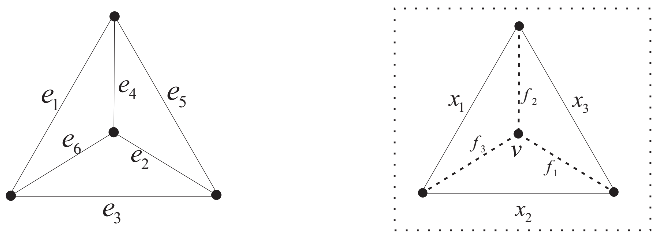
Theorem 5.2 is the main result in this section. Let k be a positive integer. A graph G is a **k-tree** if $G = K_{k+1}$ or G has a vertex v such that $G - v$ is a k-tree and such that v is adjacent to all vertices in a clique of order k. The clique is called an **adjacent clique** of v. By definition, every k-tree can be constructed by starting with a complete graph K_{k+1} and repeated adding vertices in such a way that each added vertex has exactly k adjacent vertices that form a clique. For example, 1-trees are trees. A k-tree is a simple graph. k-trees are intrinsically related to treewidth, which is an important parameter in the Robertson/Seymour theory of graph minors and in algorithmic complexity, see [1, 13].

Theorem 1.2 shows that any 2-tree has a cyclic base ordering. In this section, we will construct a cyclic base ordering inductively in a 3-tree.

Let $\mathcal{O}$ be a cyclic base ordering of a 3-tree G with n vertices. Then there are $3n - 6$ elements in $\mathcal{O}$. We divide these $3n - 6$ elements into 3 ordered groups. The first $n-2$ elements form **group 1**, the last $n-2$ elements form **group 3** and all the other $n-2$ elements form **group 2**. The three groups are denoted by S_1, S_2 and S_3, and $\mathcal{O}$ can be denoted as (S_1, S_2, S_3). Each group can be regarded as a sub-ordering of $\mathcal{O}$. Sometimes we also regard a group as a set, which can be easily seen from the context.

Lemma 5.1. *Let G be a 3-tree and C be a cycle in G. Suppose that $e \in E(C)$. Then $C - e$ contains two edges which are in a K_3.*

Proof: We show it by induction on $l = |E(C)|$. If $l = 3$, then $C = K_3$, done. Suppose that the statement holds for smaller value of $l > 3$. By definition, a 3-tree can be constructed inductively by adding a new vertex and three incident edges to a K_3 from another K_3. Thus there exists a vertex $u \in V(C)$ such that the adjacent vertices u_1, u_2 of u in C must be adjacent in G. Let edge $e_1 = uu_1$, $e_2 = uu_2$ and $e_0 = u_1u_2$. Then $C' = C - e_1 - e_2 + e_0$ is a cycle in G with $|V(C')| < l$ and e_1, e_2, e_0 form another cycle $C'' = K_3$. By inductive hypothesis, for any $e \in E(C)$, $C - e$ contains two edges which are in a K_3. □



(a) 3-tree with $n = 4$ (b) Induction for 3-tree

Figure 3: Cyclic base orderings in 3-trees

Theorem 5.2. *Any 3-tree has a cyclic base ordering.*

Proof: We will prove a stronger statement by induction on $n = |V(G)|$.
A stronger statement: any 3-tree G has a cyclic base ordering $\mathcal{O}$ such that edges in each K_3 of G are in 3 different groups of $\mathcal{O}$.

When $n = 4$, as shown in Figure 3(a), $\mathcal{O} = (e_1, e_2, e_3, e_4, e_5, e_6)$ is a cyclic base ordering in G such that edges of each K_3 are in 3 different groups. Now suppose that the statement holds for smaller value of $n > 4$. Let v be a vertex in G and x_1, x_2, x_3 are edges of the adjacent clique. The incident edges of v in G are denoted by f_1, f_2 and f_3 as shown in Figure 3(b). The edges f_1 and x_1 are called **opposite edges**. Similarly, f_2 and x_2, f_3 and x_3 are two pairs of opposite edges. Let $\mathcal{O}_{n-1} = (S_1, S_2, S_3)$ be a cyclic base ordering of $G - v$ by inductive hypothesis. By inductive hypothesis, x_1, x_2 and x_3 are in different groups. Without loss of generality, we may assume that $x_i \in S_i$ for $i = 1, 2, 3$. Let $\mathcal{O}_n = (S_1, f_1, S_2, f_2, S_3, f_3)$. Then $S_i' = (S_i, f_i)$ is the group i of $\mathcal{O}_n$ for $i = 1, 2, 3$, and edges in each K_3 of G are in 3 different groups $\mathcal{O}_n$.

In order to show that $\mathcal{O}_n$ is a cyclic base ordering of G, without loss of generality, it suffices to show that edges in (f_1, S_2, f_2) forms no cycles in G. We argue it by contradiction and suppose that some edges form a cycle C. By inductive hypothesis, C contains f_1 and f_2. Then $C - f_1 - f_2 + x_3$ is a cycle in $G - v$. Since $G - v$ is a 3-tree, by Lemma 5.1, there exist two edges in $C - f_1 - f_2 \subseteq S_2$ which are in a K_3. Then by inductive hypothesis, these two edges are in different groups in $\mathcal{O}_n$, contrary to the fact that they are in S_2, completing the proof. □

Corollary 5.3. *Every 3-tree is uniformly dense.*

6 Closing remark

The original version of Conjecture 1.1 was for matroids. We will introduce the matroid version in this section. Matroids are considered to be finite and loopless, and undefined terms can be found in Oxley [12].

Let M be a matroid with rank funcion r and ground set $E(M)$. For any $X \subseteq E(M)$ with $r(X) > 0$, the **density** of X is defined by

$$d_M(X) = \frac{|X|}{r(X)}.$$

When the matroid M is understood from the context, we often omit the subscript M. We also use $d(M)$ for $d(E(M))$. Follow the terminology in [5], the **fractional arboricity** $\gamma(M)$ is defined as

$$\gamma(M) = \max\{d(X) : r(X) > 0\}.$$

As in [5], a matroid M satisfying $d(M) = \gamma(M)$ is called **uniformly dense**.

A **cyclic base ordering** of a matroid M is a cyclic ordering of $E(M)$ such that every $r(M)$ cyclically consecutive elements form a base of M.

Kajitani et al. [8] posed the following cyclic base ordering conjecture.

Conjecture 6.1. *(Kajitani et al. [8]) A loopless matroid M has a cyclic base ordering if and only if M is uniformly dense.*

Actually, they proved the necessity of the conjecture.

Theorem 6.1. *(Kajitani et al. [8]) For a loopless matroid M, if M has a cyclic base ordering, then M is uniformly dense.*

Heuvel and Thomassé proved a special case when $|E(M)|$ and $r(M)$ are relatively prime.

Theorem 6.2. *(Heuvel and Thomassé [7]) Let M be a loopless matroid with $|E(M)|$ and $r(M)$ are coprime. Then M has a cyclic base ordering if and only if M is uniformly dense.*

A matroid is **sparse paving** if each nonspanning circuit is a hyperplane. Recently, Bonin showed that Conjecture 6.1 holds for sparse paving matroids, stated as a theorem below.

Theorem 6.3. *(Bonin [3]) Conjecture 6.1 holds for sparse paving matroids.*

We shall point out that Conjecture 6.1 is still open, and even the special case for graphs, i.e., Conjecture 1.1, remains unsolved.

References

[1] H. L. Bodlaender, A partial k-arboretum of graphs with bounded treewidth, Theoret. Comput. Sci., 209 (1998), 1-45.

[2] J. A. Bondy and U. S. R. Murty, Graph Theory, Springer, New York, 2008.

[3] J. Bonin, Basis-exchange properties of sparse paving matroids, Advances in Applied Mathematics, to appear.

[4] P. A. Catlin, J. W. Grossman, A. M. Hobbs, Graphs with uniform density, Congressus Numerentium, 65(1988), 281-286.

[5] P. A. Catlin, J. W. Grossman, A. M. Hobbs and H.-J. Lai, Fractional arboricity, strength and principal partitions in graphs and matroids, Discrete Appl. Math., 40 (1992), 285-302.

[6] X. Gu, H. -J. Lai and S. Yao, Characterizations of strength extremal graphs, Submitted.

[7] Jan van den Heuvel and Stéphan Thomassé, Cyclic orderings and cyclic arboricity of matroids, Journal of Combinatorial Theory Series B, 102 (2012), 638-646.

[8] Y. Kajitani, S. Ueno and H. Miyano, Ordering of the elements of a matroid such that its consecutive w elements are independent, Discrete math., 72 (1988), 187-194.

[9] H. -J. Lai, The size of strength-maximal graphs, Journal of Graph Theory, 14 (1990), 187-197.

[10] W. Mader, Minimale n-fach kantenzusammenhängende graphen, Math. Ann., 191 (1971), 21-28.

[11] C. St. J. A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc., 36 (1961), 445-450.

[12] J. G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.

[13] B. A. Reed, Algorithmic aspects of tree width, In: Recent Advances in Algorithms and Combinatorics, Springer, New York, 2003, 85-107.

[14] W. T. Tutte, On the problem of decomposing a graph into n connected factors, J. London Math. Soc., 36 (1961), 221-230.